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COVERINGS AND RING-GROUPOIDS

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ABSTRACT. We prove that the set of homotopy classes of the paths in a topological ring is a ring object (called ring groupoid). Using this concept we show that the ring structure of a topological ring lifts to a simply connected covering space.

INTRODUCTION

Let X be a connected topological space, $\widetilde{X}$ a connected and simply connected topological space, and let $p\colon \widetilde{X} \to X$ be a covering map. We call such a covering *simply connected.* It is well known that if X is a topological group, e is the identity element of X, and $\widetilde{e} \in \widetilde{X}$ such that $p(\widetilde{e}) = e$, then $\widetilde{X}$ becomes a topological group such that $p\colon \widetilde{X} \to X$ is a morphism of topological groups. In that case we say that the group structure of X *lifts* to $\widetilde{X}$. This can be proved by the lifting property of the maps on covering spaces (see, for example, [1]).

In the non-connected case the situation is completely different and was studied by R. L. Taylor [2] for the first time. Taylor obtained an obstruction class k_X from the topological space X and proved that the vanishing of k_X is a necessary and sufficient condition for the lifting of the group structure of X to $\widetilde{X}$ as described above. In [3] this result was generalized in terms of group-groupoids, i.e., group objects in the category of groupoids, and crossed modules, and then written in a revised version in [4].

In this paper we give a similar result: Let X and $\widetilde{X}$ be connected topological spaces and $p\colon \widetilde{X} \to X$ a simply connected covering. If X is a topological ring with identity element e, and $\widetilde{e} \in \widetilde{X}$ such that $p(\widetilde{e}) = e$, then the ring structure of X lifts to $\widetilde{X}$. That is, $\widetilde{X}$ becomes a topological ring with identity $\widetilde{e} \in \widetilde{X}$ such that $p\colon \widetilde{X} \to X$ is a morphism of topological rings. For this the following helps us:

In [5] Brown and Spencer defined the notion of a group-groupoid. They also proved that if X is a topological group, then the fundamental groupoid

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$\pi_1 X$, which is the set of all rel end points homotopy classes of paths in the topological space X, becomes a group-groupoid.

We introduce here the notion of a ring-groupoid, which is a ring object in the category of groupoids.

On the other hand, in [6] it was proved by Brown that if X is a semilocally simply connected topological space, i.e., each component has a simply connected covering, then the category TCov/X of topological coverings of X is equivalent to the category GpdCov/$\pi_1 X$ of groupoid coverings of the fundamental groupoid $\pi_1 X$.

In addition to this, in [3] it was proved that if X is a topological group whose underlying space is semilocally simply connected, then the category TGCov/X of topological group coverings of X is equivalent to the category GpGpdCov/π_1X of group-groupoid coverings of $\pi_1 X$.

Here we prove that if X is a topological ring, whose underlying space is semilocally simply connected, then the category TRCov/X of topological ring coverings of X is equivalent to the category RGpdCov/$\pi_1 X$ of ring-groupoid coverings of $\pi_1 X$.

1. Ring-Groupoids

A *groupoid* G is a small category in which each morphism is an isomorphism. Thus G has a set of morphisms, which we call *elements* of G, a set O_G of *objects* together with functions $\alpha, \beta \colon G \to O_G$, $\epsilon \colon O_G \to G$ such that $\alpha\epsilon = \beta\epsilon = 1$. The functions α, β are called *initial* and *final* maps respectively. If $a, b \in G$ and $\beta a = \alpha b$, then the *product* or *composite* ba exists such that $\alpha(ba) = \alpha(a)$ and $\beta(ba) = \beta(b)$. Further, this composite is associative, for $x \in O_G$ the element ϵx denoted by 1_x acts as the identity, and each element a has an inverse a^{-1} such that $\alpha(a^{-1}) = \beta a$, $\beta(a^{-1}) = \alpha(a)$, $a^{-1}a = \epsilon\alpha a$, $aa^{-1} = \epsilon\beta a$

In a groupoid G, for $x, y \in O_G$ we write $G(x, y)$ for the set of all morphisms with initial point x and final point y. We say G is *transitive* if for all $x, y \in O_G$, $G(x, y)$ is not empty. For $x \in O_G$ we denote the star $\{a \in G \colon \alpha a = x\ \}$ of x by G^x. The *object group* at x is $G(x) = G(x, x)$. Let G be a groupoid. The *transitive component* of $x \in O_G$ denoted by $C(G)_x$ is the full subgroupoid of G on those objects $y \in O_G$ such that $G(x, y)$ is not empty.

A *morphism* of groupoids $\widetilde{G}$ and G is a functor, i.e., it consists of a pair of functions $f \colon \widetilde{G} \to G$, $O_{\widetilde{f}} \colon O_{\widetilde{G}} \to O_G$ preserving all the structure.

Covering morphisms and universal covering groupoids of a groupoid are defined in [6] as follows:

Let $f \colon \widetilde{G} \to G$ be a morphism of groupoids. Then f is called a *covering morphism* if for each $\widetilde{x} \in O_{\widetilde{G}}$, the restriction $\widetilde{G^{x}} \to G^{\widetilde{fx}}$ of f is bijective.

A covering morphism $f\colon \widetilde{G} \to G$ of transitive groupoids is called *universal* if $\widetilde{G}$ covers every covering of G, i.e., if for every covering morphism $a\colon A \to G$ there is a unique morphism of groupoids $a'\colon \widetilde{G} \to A$ such that $aa' = f$ (and hence a' is also a covering morphism). This is equivalent to saying that for $\widetilde{x}, \widetilde{y} \in O_{\widetilde{G}}$ the set $\widetilde{G}(\widetilde{x}, \widetilde{y})$ has one element at most.

We now give

Definition 1.1. A ring-groupoid G is a groupoid endowed with a ring structure such that the following maps are the morphisms of groupoids:

(i) $m\colon G \times G \to G, (a,b) \mapsto a+b$, group multiplication,

(ii) $u\colon G \to G, a \mapsto -a$, group inverse map,

(iii) $e\colon (\star) \to G$, where $(\star)$ is a singleton,

(iv) $n\colon G \times G \to G, (a,b) \mapsto ab$, ring multiplication.

So by (iii) if e is the identity element of O_G then 1_e is that of G.

In a ring-groupoid G for $a, b \in G$ the groupoid composite is denoted by $b \circ a$ when $\alpha(b) = \beta(a)$, the group multiplication by $a+b$, and the ring multiplication by ab.

Let $\widetilde{G}$ and G be two ring-groupoids. A morphism $f\colon \widetilde{G} \to G$ from $\widetilde{G}$ to G is a morphism of underlying groupoids preserving the ring structure. A morphism $f\colon \widetilde{G} \to G$ of ring-groupoids is called a *covering* (resp. *a universal covering*) if it is a covering morphism (resp. a universal covering) on the underlying groupoids.

Proposition 1.2. *In a ring-groupoid G, we have*

(i) $(c \circ a) + (d \circ b) = (c+d) \circ (a+b)$ *and*

(ii) $(c \circ a)(d \circ b) = (cd) \circ (ab)$.

Proof. Since m is a morphism of groupoids,

$$\begin{aligned}(c \circ a) + (d \circ b) &= m[c \circ a, d \circ b] = m[(c,d) \circ (a,b)] = \\ &= m(c,d) \circ m(a,b) = (c+d) \circ (a+b).\end{aligned}$$

Similarly, since n is a morphism of groupoids we have

$$\begin{aligned}(c \circ a)(d \circ b) &= n[c \circ a, d \circ b] = n[(c,d) \circ (a,b)] \\ &= n(c,d) \circ n(a,b) = (cd) \circ (ab). \qquad \square\end{aligned}$$

We know from [5] that if X is a topological group, then the fundamental groupoid $\pi_1 X$ is a group-groupoid. We will now give a similar result.

Proposition 1.3. *If X is a topological ring, then the fundamental groupoid $\pi_1 X$ is a ring-groupoid.*

Proof. Let X be a topological ring with the structure maps

$$m\colon X \times X \to X, \quad (a,b) \mapsto a+b,$$
$$n\colon X \times X \to X, \quad (a,b) \mapsto ab$$

and the inverse map

$$u\colon X \to X, \quad a \mapsto -a.$$

Then these maps give the following induced maps:

$$\pi_1 m\colon \pi_1 X \times \pi_1 X \to \pi_1 X, \quad ([a],[b]) \mapsto [b+a]$$
$$\pi_1 n\colon \pi_1 X \times \pi_1 X \to \pi_1 X, \quad ([a],[b]) \mapsto [ba]$$
$$\pi_1 u\colon \pi_1 X \times \pi_1 X \to \pi_1 X, \quad [a] \mapsto [-a] = -[a].$$

It is known from [5] that $\pi_1 X$ is a group groupoid. So to prove that $\pi_1 X$ is a ring-groupoid we have to show the distributive law: since for $a, b \in G$ $a(b+c) = ab + ac$ we have

$$[a]([b]+[c]) = [a]([b+c]) = [a(b+c)] = [ab+ac] = [ab]+[ac] \qquad \square$$

Proposition 1.4. *Let G be a ring-groupoid, e the identity of O_G. Then the transitive component $C(G)_e$ of e is a ring-groupoid.*

Proof. In [3] it was proved that $C(G)_e$ is a group-groupoid. Further it can be checked easily that the ring structure on G makes $C(G)_e$ a ring. $\square$

Proposition 1.5. *Let G be a ring-groupoid and e the identity of O_G. Then the star $G^e = \{a \in G \colon \alpha(a) = e\}$ of e becomes a ring.*

The proof is left to the reader.

2. Coverings

Let X be a topological space. Then we have a category denoted by TCov/X whose objects are covering maps $p\colon \widetilde{X} \to X$ and a morphism from $p\colon \widetilde{X} \to X$ to $q\colon \widetilde{Y} \to X$ is a map $f\colon \widetilde{X} \to \widetilde{Y}$ (hence f is a covering map) such that $p = qf$. Further for X we have a groupoid called a *fundamental groupoid* (see [6], Ch. 9) and have a category denoted by $\mathrm{GpdCov}/\pi_1 X$ whose objects are the groupoid coverings $p\colon \widetilde{G} \to \pi_1 X$ of $\pi_1 X$ and a morphism from $p\colon \widetilde{G} \to \pi_1 X$ to $q\colon \widetilde{H} \to \pi_1 X$ is a morphism $f\colon \widetilde{G} \to \widetilde{H}$ of groupoids (hence f is a covering morphism) such that $p = qf$.

We recall the following result from Brown [6].

Proposition 2.1. *Let X be a semilocally simply connected topological space. Then the category TCov/X of topological coverings of X is equivalent to the category* $\mathrm{GpdCov}/\pi_1 X$ *of covering groupoids of the fundamental groupoid $\pi_1 X$.*

Let X and $\widetilde{X}$ be topological groups. A map $p\colon \widetilde{X} \to X$ is called a *covering morphism* of topological groups if p is a morphism of groups and p is a covering map on the underlying spaces. For a topological group X, we have a category denoted by TGCov/X whose objects are topological group coverings $p\colon \widetilde{X} \to X$ and a morphism from $p\colon \widetilde{X} \to X$ to $q\colon \widetilde{Y} \to X$ is a map $f\colon \widetilde{X} \to \widetilde{Y}$ such that $p = qf$. For a topological group X, the fundamental groupoid $\pi_1 X$ is a group-groupoid and so we have a category denoted by GpGpdCov/$\pi_1 X$ whose objects are group-groupoid coverings $p\colon \widetilde{G} \to \pi_1 X$ of $\pi_1 X$ and a morphism from $p\colon \widetilde{G} \to \pi_1 X$ to $q\colon \widetilde{H} \to \pi_1 X$ is a morphism $f\colon \widetilde{G} \to \widetilde{H}$ of group-groupoids such that $p = qf$.

Then the following result is given in [4].

Proposition 2.2. *Let X be a topological group whose underlying space is semilocally simply connected. Then the category TGCov/X of topological group coverings of X is equivalent to the category* GpGpdCov/$\pi_1 X$ *of covering groupoids of the group-groupoid* $\pi_1 X$.

In addition to these results, we here prove Proposition 2.3.

Let X and $\widetilde{X}$ be topological rings. A map $p\colon \widetilde{X} \to X$ is called a *covering morphism* of topological rings if p is a morphism of rings and p is a covering map on the underlying spaces. So for a topological ring X, we have a category denoted by TRCov/X, whose objects are topological ring coverings $p\colon \widetilde{X} \to X$ and a morphism from $p\colon \widetilde{X} \to X$ to $q\colon \widetilde{Y} \to X$ is a map $f\colon \widetilde{X} \to \widetilde{Y}$ such that $p = qf$. Similarly, for a topological ring X, we have a category denoted by RGpdCov/$\pi_1 X$ whose objects are ring-groupoid coverings $p\colon \widetilde{G} \to \pi_1 X$ of $\pi_1 X$ and a morphism from $p\colon \widetilde{G} \to \pi_1 X$ to $q\colon \widetilde{H} \to \pi_1 X$ is a morphism $f\colon \widetilde{G} \to \widetilde{H}$ of ring-groupoids such that $p = qf$.

Let X be a topological ring whose underlying space is semilocally simply connected. Then we prove the following result.

Proposition 2.3. *The categories TRCov/X and RGpdCov/$\pi_1 X$ are equivalent.*

Proof. Define a functor

$$\pi_1\colon \text{ TRCov/X}\to\text{RGpdCov}/\pi_1 X$$

as follows: Let $p\colon \widetilde{X} \to X$ be a covering morphism of topological rings. Then the induced morphism $\pi_1 p\colon \pi_1\widetilde{X} \to \pi_1 X$ is a covering morphism of group-groupoids (see [3]), i.e., it is a morphism of group-groupoids and coverings on the underlying groupoids. Further the morphism $\pi_1 p$ preserves the ring structure as follows:

$$(\pi_1 p)[ab] = [p(ab)] = [p(a)p(b)] = [p(a)][p(b)] = (\pi_1 p)[a](\pi_1 p)[a].$$

So $\pi_1 p\colon \pi_1\widetilde{X} \to \pi_1 X$ becomes a covering morphism of ring-groupoids.

We now define a functor

$$\eta\colon \mathrm{RGpdCov}/\pi_1 X \to \mathrm{TRCov/X}$$

as follows: If $q\colon \widetilde{G} \to \pi_1 X$ is a covering morphism of ring groupoids, then we have a covering map $p\colon \widetilde{X} \to X$, where $p = O_q$ and $\widetilde{X} = O_{\widetilde{G}}$. Further p is a morphism of topological groups (see [3]). Further we will prove that the ring multiplication

$$\widetilde{n}\colon \widetilde{X} \times \widetilde{X} \to \widetilde{X}, \quad (a, b) \mapsto ab$$

is continuous.

By assuming that X is semilocally simply connected, we can choose a cover U of simply connected subsets of X. Since the topology $\widetilde{X}$ is the lifted topology (see [6], Ch. 9) the set consisting of all liftings of the sets in U forms a basis for the topology on $\widetilde{X}$. Let $\widetilde{U}$ be an open neighborhood of $\widetilde{e}$ and a lifting of U in U. Since the multiplication

$$n\colon X \times X \to X, \quad (a, b) \mapsto ab$$

is continuous, there is a neighborhood V of e in X such that $n(V \times V) \subseteq U$. Using the condition on X and choosing V small enough we can assume that V is simply connected. Let $\widetilde{V}$ be the lifting of V. Then $p\widetilde{n}(\widetilde{V} \times \widetilde{V}) = n(V \times V) \subseteq U$ and so we have $\widetilde{n}(\widetilde{V} \times \widetilde{V}) \subseteq \widetilde{U}$. Hence

$$\widetilde{n}\colon \widetilde{X} \times \widetilde{X} \to \widetilde{X}, \quad (a, b) \mapsto ab$$

becomes continuous. Since by Proposition 2.2 the category of topological group coverings is equivalent to the category of group-groupoid coverings, the proof is completed by the following diagram:

$$\begin{array}{ccc} TRCov/X & \xrightarrow{\pi_1} & RGpdCov/\pi_1 X \\ \downarrow & & \downarrow \\ TGCov/X & \xrightarrow{\pi_1} & GpGpdCov/\pi_1 X. \end{array} \qquad \square$$

Before giving the main theorem we adopt the following definition:

Definition 2.4. Let $p\colon \widetilde{G} \to G$ be a covering morphism of groupoids and $q\colon H \to G$ a morphism of groupoids. If there exists a unique morphism $\widetilde{q}\colon H \to \widetilde{G}$ such that $p = q\widetilde{q}$ we say q lifts to $\widetilde{q}$ by p.

We recall the following theorem from [6] which is an important result to have the lifting maps on covering groupoids.

Theorem 2.5. *Let $p\colon \widetilde{G} \to G$ be a covering morphism of groupoids, $x \in O_G$ and $\widetilde{x} \in O_{\widetilde{G}}$ such that $p(\widetilde{x}) = x$. Let $q\colon H \to G$ be a morphism of groupoids such that H is transitive and $\widetilde{y} \in O_H$ such that $q(\widetilde{y}) = x$. Then the morphism $q\colon H \to G$ uniquely lifts to a morphism $\widetilde{q}\colon H \to \widetilde{G}$ such that $\widetilde{q}(\widetilde{y}) = \widetilde{x}$ if and only if $q[H(\widetilde{y})] \subseteq p[\widetilde{G}(\widetilde{x})]$, where $H(\widetilde{y})$ and $\widetilde{G}(\widetilde{x})$ are the object groups.*

Let G be a ring groupoid, e the identity of O_G, and let $\widetilde{G}$ be just a groupoid, $\widetilde{e} \in O_{\widetilde{G}}$ such that $p(\widetilde{e}) = e$. Let $p\colon \widetilde{G} \to G$ be a covering morphism of groupoids. We say the ring structure of G *lifts* to $\widetilde{G}$ if there exists a ring structure on $\widetilde{G}$ with the identity element $\widetilde{e} \in O_{\widetilde{G}}$ such that $\widetilde{G}$ is a group-groupoid and $p\colon \widetilde{G} \to G$ is a morphism of ring-groupoids.

Theorem 2.6. *Let $\widetilde{G}$ be a groupoid and G a ring-groupoid. Let $p\colon \widetilde{G} \to G$ be a universal covering on the underlying groupoids such that both groupoids $\widetilde{G}$ and G are transitive. Let e be the identity element of O_G and $\widetilde{e} \in O_{\widetilde{G}}$ such that $p(\widetilde{e}) = e$. Then the ring structure of G lifts to $\widetilde{G}$ with identity $\widetilde{e}$.*

Proof. Since G is a ring-groupoid as in Definition 1.1 it has the following maps:

$$\begin{aligned} m\colon G \times G \to G, &\quad (a,b) \mapsto a+b,\\ u\colon G \to G, &\quad a \mapsto -a,\\ n\colon G \times G \to G, &\quad (a,b) \mapsto ab. \end{aligned}$$

Since $\widetilde{G}$ is a universal covering, the object group $\widetilde{G}(\widetilde{e})$ has one element at most. So by Theorem 2.5 these maps respectively lift to the maps

$$\begin{aligned} \widetilde{m}\colon \widetilde{G} \times \widetilde{G} \to \widetilde{G}, &\quad (\widetilde{a},\widetilde{b}) \mapsto \widetilde{a}+\widetilde{b},\\ \widetilde{u}\colon \widetilde{G} \to \widetilde{G}, &\quad \widetilde{a} \mapsto -\widetilde{a},\\ \widetilde{n}\colon \widetilde{G} \times \widetilde{G} \to \widetilde{G}, &\quad (\widetilde{a},\widetilde{b}) \mapsto \widetilde{a}\widetilde{b} \end{aligned}$$

by $p\colon \widetilde{G} \to G$ such that

$$\begin{aligned} p(\widetilde{a}+\widetilde{b}) &= p(\widetilde{a}) + p(\widetilde{b}),\\ p(\widetilde{a}\widetilde{b}) &= p(\widetilde{a})p(\widetilde{b}),\\ p(\widetilde{u}(\widetilde{a})) &= -(p\widetilde{a}). \end{aligned}$$

Since the multiplication $m\colon G \times G \to G \mapsto a+b$ is associative, we have $m(m \times 1) = m(1 \times m)$, where 1 denotes the identity map. Then again by Theorem 2.5 these maps $m(m \times 1)$ and $m(1 \times m)$ respectively lift to

$$\widetilde{m}(\widetilde{m} \times 1), \widetilde{m}(1 \times \widetilde{m})\colon \widetilde{G} \times \widetilde{G} \times \widetilde{G} \to \widetilde{G}$$

which coincide on $(\widetilde{e},\widetilde{e},\widetilde{e})$. By the uniqueness of the lifting we have $\widetilde{m}(\widetilde{m} \times 1) = \widetilde{m}(1 \times \widetilde{m})$, i.e., $\widetilde{m}$ is associative. Similarly, $\widetilde{n}$ is associative. Further the distributive law is satisfied as follows:

Let $p_1, p_2\colon G \times G \times G \to G$ be the morphisms defined by

$$p_1(a,b,c) = ab, \quad p_2(a,b,c) = bc$$

and

$$(p_1, p_2)\colon G \times G \times G \to G \times G, \quad (a, b, c) \mapsto (ab, ac)$$

for $a, b, c \in G$. Since the distribution law is satisfied in G, we have $n(1\times m) = m(p_1, p_2)$. The maps $n(1 \times m)$ and $m(p_1, p_2)$ respectively lift to the maps

$$\widetilde{n}(1 \times \widetilde{m}),\ \widetilde{m}(\widetilde{p_1}, \widetilde{p_2})\colon \widetilde{G} \times \widetilde{G} \times \widetilde{G} \to \widetilde{G}$$

coinciding at $(\widetilde{e}, \widetilde{e}, \widetilde{e})$. So by Theorem 2.5 we have $\widetilde{n}(1 \times \widetilde{m}) = \widetilde{m}(\widetilde{p_1}, \widetilde{p_2})$. That means the distribution law on $\widetilde{G}$ is satisfied. The rest of the proof is straightforward. □

From Theorem 2.6 we obtain

Corollary 2.7. *Let X and $\widetilde{X}$ be path connected topological spaces and $p\colon \widetilde{X} \to X$ be a simply connected covering, i.e., $\widetilde{X}$ is simply connected. Suppose that X is a topological ring, and e is the identity element of the group structure on X. If $\widetilde{e} \in \widetilde{X}$ with $p(\widetilde{e}) = e$, then $\widetilde{X}$ becomes a topological ring with identity $\widetilde{e}$ such that p is a morphism of topological rings.*

Proof. Since $p\colon \widetilde{X} \to X$ is a simply connected covering, the induced morphism $\pi_1 p\colon \pi_1\widetilde{X} \to \pi_1 X$ is a universal covering morphism of groupoids. Since X is a topological ring by Proposition 1, $\pi_1 X$ is a ring-groupoid. By Theorem 2.6 $\pi_1\widetilde{X}$ becomes a ring-groupoid and again by Proposition 2.3 $\widetilde{X}$ becomes a topological ring as required. □

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