GEORGIAN MATHEMATICAL JOURNAL: Vol. 5, No. 5, 1998, 453-468

HOMOTOPY CLASSES OF ELLIPTIC TRANSMISSION PROBLEMS OVER C^*-ALGEBRAS

G. KHIMSHIASHVILI

Abstract. The topological aspects of B.Bojarski's approach to Riemann–Hilbert problems are developed in terms of infinite-dimensional grassmanians and generalized to the case of transmission problems over C^*-algebras. In particular, the homotopy groups of certain grassmanians related to elliptic transmission problems are expressed through K-groups of the basic algebra. Also, it is shown that the considered grassmanians are homogeneous spaces of appropriate operator groups. Several specific applications of the obtained results to singular operators are given, and further perspectives of our approach are outlined.

1. Introduction. The aim of the paper is to compute the homotopy groups of certain geometric objects over C^*-algebras which are relevant to the homotopy classification of abstract elliptic transmission problems introduced by B. Bojarski [1] as natural generalizations of the classical Riemann–Hilbert problems for analytic vector-functions [2]. As was later realized by the author [3], the whole issue fits nicely into Fredholm structures theory [4] or, more precisely, into the homotopy theory of operator groups started by R. Palais [5] and developed by M. Rieffel [6] and K. Thomsen [7].

Similar geometric objects appear in loop groups theory, K-theory, and the geometric aspects of operator algebras, and have recently gained considerable attention [8], [9], [10], [11]. This circumstance, and especially discussions with B. Bojarski about certain aspects of the Fredholm theory for transmission problems, encouraged the author to reexamine the geometric approach to transmission problems in the spirit of recent developments in the K-theory of operator algebras.

Recall that in 1979 B. Bojarski formulated a topological problem which appeared important in his investigation of the so-called Riemann–Hilbert

1991 *Mathematics Subject Classification.* 47C15, 58B15.

Key words and phrases. Hilbert C^*-modules, Kuiper's theorem, transmission problems, restricted linear group, abstract singular operators, homotopy groups, K-groups.

transmission problems [1]. This problem was later solved independently in [12] and [13] (cf. also [8]). Moreover, these results were used in studying several related topics of global analysis and operator theory [10], [14], [15].

An important advantage of the geometric formulation of elliptic transmission problems in terms of Fredholm pairs of subspaces of a Hilbert space given in [1] was that it permitted various modifications and generalizations. Thus it became meaningful to consider similar problems in more general contexts [12]. Along these lines, the author was able to develop the main part of Fredholm structures theory in the context of Hilbert C^*-modules [3], which led to some progress in the theory of generalized transmission problems [15], [16].

This approach will also enable us to investigate below the case of transmission problems over an arbitrary C^*-algebra. Clearly, it gives a wide generalization of the above-mentioned results, since they correspond to the case in which the considered algebra is taken to be the field of complex numbers $\mathbf{C}$. Moreover, it enables one to investigate further elliptic problems associated with abstract singular and bisingular operators over C^*-algebras.

It should be noted that our generalized setting may be regarded as the investigation of the homotopy classes of families of elliptic transmission problems parameterized by a (locally) compact topological space X. In fact, this corresponds to considering transmission problems over the algebra of continuous functions on the parameter space $C(X)$, which is also a very special case of our results.

To make the presentation concise, we freely use the terms and constructions from several previous papers on related topics ([1], [3], [9], [11]). An exhaustive description of the background and necessary topological notions is contained in [1], [8] and [17].

2. Transmission problems and grassmanians. We pass now to the precise definitions needed to present B. Bojarski's geometric approach to transmission problems [1]. Basically, we use the same concepts as in [1], but sometimes in a slightly different form adjusted for the case of Hilbert C^*-modules.

Let A be a unital C^*-algebra. Denote by H_A the standard Hilbert module over A, i.e.,

$$H_A = \left\{ \{a_i\}, a_i \in A, i = 1, 2, \dots \ : \ \sum_{i=1}^{\infty} a_i a_i^* \in A \right\}. \tag{1}$$

Since there exists a natural A-valued scalar product on H_A possessing usual properties [18], one can introduce direct sum decompositions and consider various types of bounded linear operators on H_A. Denote by $B(H_A)$ the collection of all A-bounded linear operators having A-bounded adjoints.

This algebra is one of the most fundamental objects in Hilbert C^*-modules theory [9], [18], [19].

As is well known, $B(H_A)$ is a Banach algebra and it is useful to consider also its group of units $GB = GB(H_A)$ and the subgroup of unitaries $U = U(H_A)$. For our purpose it is important to have adjoints, which, as shown in [19], is not the case for an arbitrary bounded operator on the Hilbert A-module H_A. In particular, for this algebra we have an analog of the polar decomposition [19], which implies that $GB(H_A)$ is retractable to $U(H_A)$. Thus these two operator groups are homotopy equivalent, which is important for our consideration.

Compact linear operators on H_A are defined to be A-norm limits of finite rank linear operators [19]. Their collection is denoted by $K(H_A)$.

Recall that the main object of B. Bojarski's approach in [1] is a special group of operators associated with a fixed direct sum decomposition of a given complex Hilbert space. With this in mind, we fix a direct sum decomposition of Hilbert A-modules of the form $H_A = H_+ + H_-$, where H_+ and H_- are both isomorphic to H_A as A-modules. As is well known, any operator on H_A can be written as a (2×2)-matrix of operators with respect to this decomposition (see formula (5)). We denote by π_+ and π_- the natural orthogonal projections defined by this decomposition.

Introduce now the subgroup $GB_r = GB_r(H_A)$ of $GB(H_A)$ consisting of operators whose off-diagonal terms belong to $K(H_A)$. Let $U_r = U_r(H_A)$ denote the subgroup of its unitary elements. To relate this to transmission problems, we must have an analog of the so-called special grassmanian introduced in [8]. In fact, this is nearly equivalent to working with Fredholm pairs of subspaces which were first used in this context by B. Bojarski [1]. To do the same in our generalized setting, some technical preliminaries are needed.

Recall that there is a well-defined notion of a finite rank A-submodule of a Hilbert A-module [18]. This enabled A. Mishchenko and A. Fomenko to introduce the notion of a Fredholm operator in a Hilbert A-module by requiring that its kernel and image be finite-rank A-submodules [18]. It turns out that many important properties of usual Fredholm operators remain valid in this context, too. Thus, if the collection of all Fredholm operators on H_A is denoted by $F(H_A)$, then there exists a canonical homomorphism $\mathrm{ind} = \mathrm{ind}_A : F(H_A) \longrightarrow K_0(A)$, where $K_0(A)$ is the usual topological K-group of the basic algebra A [17].

This means simply that Fredholm operators over C^*-algebras have indices obeying the usual additivity law. In the sequel, we will freely refer to the detailed exposition of these results in [9].

Granted the above technicalities, we can now introduce a special grassmanian $Gr_+ = Gr_+(H_A)$ associated with the given decomposition. It consists of all A-submodules V of H_A such that the projection π_+ restricted on V

is Fredholm while the projection π_- restricted on V is compact. Using the analogs of the local coordinate systems for $Gr_+(H_{\mathbf{C}})$ constructed in [8], we can verify that $Gr_+(H_A)$ is a Banach manifold modelled on the Banach space $K(H_A)$. For our purpose it suffices to consider Gr_+ as a metrizable topological space with the topology induced by the standard one on the infinite grassmanian $Gr^\infty(A)$ (see formula (4)).

Now the problem that we are interested in is to investigate the topology of $Gr_+(H_A)$ and $GB_r(H_A)$. Note that for $A = \mathbf{C}$ this is the problem formulated by B. Bojarski in [1].

Our main results are as follows.

Theorem 1. *The group $GB_r(H_A)$ acts transitively on $Gr_+(H_A)$ with contractible isotropy subgroups.*

Theorem 2. *All even-dimensional homotopy groups of $Gr_+(H_A)$ are isomorphic to the index group $K_0(A)$ while its odd-dimensional homotopy groups are isomorphic to the Milnor group $K_1(A)$.*

Of course, the same statements hold for the homotopy groups of $GB_r(H_A)$, since by Theorem 1 these two spaces are homotopy equivalent. We formulate the result for $Gr_+(H_A)$ because it is the space of interest for transmission problems theory.

The homotopy groups of $GB_r(H_A)$ were first computed by the author in 1987 [3] without mentioning grassmanians. Later, similar results were obtained by S. Zhang [11] in the framework of K-theory. The contractibility of isotropy subgroups involved in Theorem 1 was previously established only for $A = \mathbf{C}$ [8].

In proving Theorem 1, we will obtain more precise information on the structure of isotropy subgroups. It should also be noted that the contractibility of isotropy subgroups follows from a fundamental result on C^*-modules called the generalization of Kuiper's theorem for Hilbert C^*-modules, which was obtained independently by E. Troitsky [20] and J. Mingo [9]. Particular cases of Theorem 2 for various commutative C^*-algebras A may be useful to construct classifying spaces for K-theory.

The solution of B. Bojarski's original problem is now immediate (cf. [9], [12], [13]).

Corollary 1. *Even-dimensional homotopy groups of the collection of classical Riemann–Hilbert problems are trivial while odd-dimensional ones are isomorphic to additive group of integers* $\mathbf{Z}$.

Note that the nontriviality of these groups can be interpreted in terms of the so-called spectral flow of order zero pseudo-differential operators, which has recently led to some interesting developments by B. Booss and K. Wojciechowsky [10] sheding new light on the Atiyah-Singer index formulas in the odd-dimensional case.

Similar results hold for abstract singular operators over A (for the definition of abstract singular operators see [21] and [3]).

Corollary 2. *Homotopy groups of invertible singular operators over a unital C^*-algebra A are expressed by the relations (where n is natural and arbitrary)*

$$\begin{gathered}\pi_0 \cong K_0(A), \quad \pi_1 \cong \mathbf{Z} \oplus \mathbf{Z} \oplus K_1(A); \\ \pi_{2n} \cong K_0(A), \quad \pi_{2n+1} \cong K_1(A).\end{gathered} \tag{2}$$

Specifying this result for the algebras of continuous functions one can, in particular, compute the homotopy classes of invertible classical singular integral operators on arbitrary regular closed curves in the complex plane $\mathbf{C}$ (see [21], [3] for the precise definitions).

Corollary 3. *If $K \in \mathbf{C}$ is a smooth closed curve with k components, then homotopy groups of invertible classical singular integral operators on K are expressed by the relations (where n is natural and arbitrary):*

$$\pi_0 \cong \mathbf{Z}, \quad \pi_1 \cong \mathbf{Z}^{2k+1}; \quad \pi_{2n} = 0, \quad \pi_{2n+1} \cong \mathbf{Z}. \tag{3}$$

As shown in [15], this information also enables one to find homotopy classes and index formulas for the so-called bisingular operators. The latter can be defined by purely algebraic means, starting from the algebra of abstract singular operators. One is thus led to the notion of a bisingular operator over a C^*-algebra and to the description of homotopy classes of elliptic bisingular operators. The notion was introduced in [15] and the description of index ranges follows from the results of this paper.

Corollary 4. *Abstract elliptic bisingular operators over a C^*-algebra A are homotopically classified by their indices taking values in $K_0(A)$. The index homomorphism is an epimorphism onto $K_0(A)$.*

As is well known, the usual bisingular operators correspond to certain pseudo-differential operators on the two-torus $\mathbf{T}^2$ [22]. In a similar manner, one may recover some of the known results on homotopy groups of invertible pseudodifferential operators over other two-surfaces [16].

One can also obtain an index formula for abstract bisingular operators in terms of homotopy classes of their operator-valued symbols which can be described by Theorem 2. For brevity, the results concerning the index formulas for bisingular operators will not be presented here.

Theorem 1 will be proved in the next section, when developing the necessary geometric constructions over C^*-algebras. The outlines of proofs of

Theorem 2 and the corollaries are given in separate sections. In the conclusion, we give some remarks on related results and further applications.

3. More on operators in Hilbert C^*-modules. It is standard in C^*-algebras theory to identify subspaces with projections. Thus direct sum decompositions of the type described in Section 2 correspond to the so-called infinite grassmanian over A which can be written as

$$Gr^{\infty}(A) = \Big\{ p \in B(H_A) : \ \ p = p^2 = p^* \ \text{ and } \ p \sim Id \sim Id - p \Big\}, \qquad (4)$$

where "$\sim$" denotes the Murray-von Neumann equivalence between projections [17].

Fixing such a decomposition is equivalent to fixing a projection with image and kernel being A-modules of infinite rank. Having fixed such a projection p which will play the role of the projection π_+ introduced above, one can readily verify the useful characterization of GB_r.

Lemma 1. $GB_r(H_A) = \{x \in B(H_A) : xp - px \in A \otimes K(H)\}$, *where* $K(H)$ *stands for the ideal of compact operators in the usual separable complex Hilbert space* H*.*

The above-mentioned (2×2)-matrix representation of $x \in B(H_A)$ can be rewritten as

$$x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \qquad (5)$$

where $x_{11} = pxp$, $x_{12} = px(1-p)$, $x_{21} = (1-p)xp$, $x_{22} = (1-p)x(1-p)$.

It is obvious now that $GB_r(H_A)$ is $*$-isomorphic to the group of units of the C^*-algebra consisting of (2×2)-matrices over $B(H_A)$ whose off-diagonal entries are the elements of $A \otimes K(H)$. Further, from the existence, additivity, and stability properties of the Fredholm index (see diagram (6) below) it follows that, for $x \in GB_r(H_A)$, both x_{11} and x_{22} should be Fredholm operators with the opposite indices, which is important for the sequel.

Using simple algebraic identities for such (2×2)-matrices (explicitly written in [8] for matrices over $B(H)$), and the fact that $K(H_A)$ is an ideal in $B(H_A)$, it is easy to verify that if such a (2×2)-matrix is applied to an element V of $Gr_+(H_A)$, then the restriction to V of the first projection π_+ is transformed into $x_{11}\pi_+ + x_{12}$ and thus remains Fredholm, while the restriction to V of the second projection gives $x_{22}\pi_- + x_{21}$ and remains compact. This means that xV is again in $Gr_+(H_A)$ and we have proved

Lemma 2. *The restricted linear group* $GB_r(H_A)$ *acts on the special grassmanian* $Gr_+(H_A)$*.*

Now, it is evident that to determine the isomorphy class of stability subgroups it is sufficient to identify it for a “coordinate submodule” H_+ in $GB_r(H_A)$. It readily follows from the existence of polar decompositions that the latter subgroup is homotopy equivalent to the isotropy subgroup of H_+ in the restricted unitary group $U_r(H_A)$ (which acts on $Gr_+(H_A)$ as a subgroup of $GB_r(H_A)$).

Analyzing the description of a similar isotropy subgroup in the case of the usual Hilbert space given in [8], one easily finds that in view of the above technical results for Hilbert C^*-modules the same conclusion holds in our case, too.

Lemma 3. *The stability subgroup of H_+ in $U_r(H_A)$ is isomorphic to $U(H_+) \times U(H_-)$.*

Recall that the latter group is contractible according to the result of E. Troitsky and J. Mingo [9], [20].

To prove Theorem 1 it remains to check the transitivity, which is the most delicate part of the proof. We will use the method of proof from [8] adapted to our situation. Note that Fredholm operators with vanishing indices can be transformed into invertible ones by a compact perturbation. The corresponding statement for Hilbert C^*-modules is contained in the so-called fundamental commutative diagram of Fredholm structures theory. In our case it has the form

$$\begin{array}{ccccccc} GB(H_A) & \longrightarrow & F_0(H_A) & \longrightarrow & F(H_A) & \longrightarrow & B(H_A) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ GB/GK & \longrightarrow & G_0 & \longrightarrow & G & \longrightarrow & B(H_A)/K(H_A) \end{array} \tag{6}$$

Here $F(H_A)$ and $F_0(H_A)$ stand for semigroups of all Fredholm operators and those with the zero index, respectively. G denotes the group of units of the factor-algebra $B(H_A)/K(H_A)$ and G_0 its identity component. The right vertical arrow is the Calkin factorization and the left one is the factor-homomorphism on the factor-group below. The upper arrows are inclusions.

The commutativity of this diagram is well known to experts and follows from the facts established in [18] and [19] (cf. also [9]). Also, it is a standard verification that the left lower corner horizontal arrow is a homeomorphism. In topological terms, the latter fact means that G_0 is the classifying space for the K-functor [17] and its homotopy groups are isomorphic to the corresponding K-groups of the basic algebra. This conclusion is explained in full detail in [9].

Let us now return to our situation and take an A-submodule V belonging to the special grassmanian $Gr_+(H_A)$. By definition, there exists a Fredholm operator $T \in B(H_+, H_-)$ such that V is its graph, i.e., is the set of points (x, Tx) with respect to the given decomposition of H_A.

To prove the transitivity of action, it is sufficient to obtain a (2×2)-matrix $M \in GB_r(H_A)$ of the form described above such that $M(H_+) = V$. For this, we consider first the diagonal matrix $\operatorname{diag}(T, T^*)$, where T^* is the adjoint operator of T. From the additivity property of the Fredholm index it follows that this matrix has the zero index when considered as an element of $B(H_A)$. Considering its class in the space of (2×2)-matrix over the Calkin algebra B/K, one sees that it is invertible. Thus our diagram shows that this matrix can be turned into the invertible one by a compact perturbation. In other words, there exist compact off-diagonal terms x_{21}, x_{12} such that our diagonal matrix completed with such entries becomes invertible as an operator on H_A. This already implies the existence of the desired matrix M. One could also finish the proof arguing as in [8], Ch. 6.

4. Proof of Theorem 2. The simplest way to verify Theorem 2 is as follows. One notices that Proposition 6.2.4 of [8] suggests that $GB_r(H_A)$ should be homotopy equivalent to $F(H_+)$. This would already prove Theorem 2 because from diagram (6) it follows that homotopy groups of $F(H_A)$ are isomorphic with K-theory of A. In fact, it may be actually proved that $GB_r(H_A)$ is homotopy equivalent to $F(H_+)$, using a suitable modification of the argument from [8], Ch. 6. However, in order to make this argument rigorous one needs to develop a substantial portion of Hilbert modules theory. For the sake of brevity, here we prefer another way, more algebraic in spirit, which closely follows the lines of [11]. In doing so we will borrow freely the concepts and results from [11] and [17].

Throughout this section we will use the identification of direct sum decompositions with projections and fix a $p \in B(H_A)$ with $p = p^2 = p^*$. Below we will omit some tedious details which are standard in the theory of operator algebras and K-theory.

As was explained, it suffices to compute the homotopy groups of the restricted linear group $GB_r(H_A)$. Denote by $GB_r^0(H_A)$ its identity component. As is well known, in dealing with K-theory invariants, it is useful to consider the conjugations by unitary operators. With this in mind, we introduce the notation $UpU^* = \{vpv^* : v \in U_r(H_A)\} = \{vpv^* : v \in GB_r(H_A),\ vv^* = v^*v = Id\}$. The following simple proposition is verified using the standard techniques of K-theory (cf. [17]).

Lemma 4. *$U(upu^*)U^*$ is the path component of UpU^* containing upu^*.*

One also has an equivalent description of the K_0-functor which was already used in [19] and [9].

Lemma 5. *For any such $p \in Gr^\infty(A)$ the group $K_0(A)$ is isomorphic to the fundamental group $\pi_1(UpU^*)$.*

Indeed, later we will produce an explicit isomorphism between these two groups in terms of some partial isometries associated with elements of $GB_r(H_A)$, which plays an important role in the argument.

Following [11], a unitary operator $x \in U_r(H_A)$ will be called p-adapted if both off-diagonal terms of the corresponding (2×2)-matrix (see formula (6)) are some partial isometries in $A \otimes K(H)$.

It is easy to calculate some associated projections needed in the sequel.

Lemma 6 ([11]). *If x is a p-adapted unitary, then $p - x_{11}x_{11}^*$, $p - x_{11}^*x_{11}$, $(Id - p) - x_{22}x_{22}^*$, $(Id - p) - x_{22}^*x_{22}$ are projections in $A \otimes K(H)$.*

The following results from [11] amount to a partial isometry description of the K-functor. Equivalent statements can be found in [19] and [17]. A similar factorization for the case $A = C$ was also used in [8].

Proposition 1. *Any $X \in GB_r(H_A)$ can be represented as*

$$x = (Id + k) \cdot \mathrm{diag}(z_1, z_2) \cdot u, \tag{7}$$

where $k \in A \otimes K(H)$, the second factor is invertible, and u is a p-adapted unitary.

Recall that according to one of the basic constructions any partial isometry $b \in A \otimes K(H)$ defines a class $[bb^*] \in K_0(A)$ [17]. The following proposition follows from this construction and the equivalence relation in $K_0(A)$.

Proposition 2. *The class*

$$[u_{12}u_{12}^*] - [u_{21}u_{21}^*] \in K_0(A) \tag{8}$$

is independent of a p-adapted unitary u entering into a representation of a given $x \in GB_r(H_A)$ in form (7).

Now we are able to define the mappings giving the desired group isomorphisms. Our strategy is to consider the group GB_r as a fibration over its homogeneous space GB_r/GK and, next, to compute the homotopy groups of GB_r/GK, since the homotopy groups of the fibre $GK(H_A)$, being the standard participants in K-theory, are well known.

Observe first that representation (7) implies the equality of cosets $x \cdot GK(H_A) = u \cdot GK(H_A)$ of the elements x and u with respect to the subgroup $GK(H_A)$. By Lemma 5, for such u we have the following direct sum of projections:

$$(p - u_{12}u_{12}^*) \oplus (u_{12}u_{12}^*). \tag{9}$$

As is well known, direct sums do not have any influence on the stable equivalence relation involved in the definition of $K_0(A)$. In other words, it is meaningful to assign to element (9) the class

$$[u_{12}u_{12}*]-[u_{21}u_{21}^*]\in K_0(A) \tag{10}$$

A connection between the considered basic topological spaces is established by

Lemma 7. *Element* (9) *belongs to the subset* UpU^*.

For the proof it suffices to observe that this statement follows from Proposition 3.1 in [9] by which for any two projections $r_1,r_2\in A\otimes K(H)$ there exists unitary $w\in GK(H_A)$ such that $wpw^*=(p-r_1)\oplus r_2$.

By virtue of these lemmas we arrive at the basic correspondence giving the desired isomorphism at the level of fundamental groups. Below it is assumed that the base point of GB_r is the identity, and that of UPU^* is p.

Proposition 3. *The maps defined by the relations*

$$\begin{aligned} u\cdot GB_r(H_A)\mapsto [(p-u_{12}u_{12}^*)\oplus u_{21}u_{21}^*]_{UpU^*}\mapsto \\ \mapsto [u_{21}u_{21}^*]-[u_{12}u_{12}^*]\in K_0(A) \end{aligned} \tag{11}$$

are the bijections inducing the isomorphisms

$$\pi_0(GB_r)(=GB_r(H_A)/GK(H_A))\cong\pi_0(UpU^*)\cong K_0(A). \tag{12}$$

Now the results concerning the computation of higher homotopy groups can be formulated as follows (cf. [11]).

Proposition 4. *For any natural* n *one has the isomorphisms*

$$\pi_{2n+1}(GB_r(H_A))\cong\pi_{2n+1}(UpU^*)\cong K_1(A), \tag{13}$$
$$\pi_{2n+2}(GB_r(H_A))\cong\pi_{2n+2}(UpU^*)\cong K_0(A). \tag{14}$$

These isomorphisms can be verified by means of the long exact sequence of homotopy groups associated with a natural operator fibration over UpU^* with the contractible total space $U_\infty(A)$ which is, as above, the group of unitaries in the unitization of $A\otimes K(H)$.

To this end, we consider the map defined by $u\mapsto upu^*$. Clearly, its fibers are all isomorphic with the commutant of p in U, i.e., $(p')_U=\{u\in U_\infty(A): up=pu\}$. It is also simple to check that this map is a submersion and, according to an infinite-dimensional generalization of Ehresmann's theorem [4], defines a locally trivial fibration with the fiber p'.

The long exact homotopy sequence of this fibration breaks, as usual, into short exact sequences:

$$0\to\pi_{k+1}(UpU^*)\to\pi_k(p')\to\pi_k(U_\infty(A))\to 0. \tag{15}$$

Since the homotopy groups of the stabilized unitary group $U_\infty(A)$ are isomorphic to the K-groups of A, these exact sequences immediately imply that $\pi_{2n+2}(UpU^*) \cong K_0(A)$ and $\pi_{2n+1} \cong K_1(A)$. Recalling that UpU^* is weakly homotopy equivalent to $GB_r(H_A)$, we obtain the desired conclusion.

Now Theorem 2 becomes an immediate consequence of Propositions 3 and 4.

5. Proofs of corollaries. We will make a few comments on the formulations and proofs of the corollaries.

Corollary 1 is simply a special case of Theorem 2, where $A = C(S^1)$ is the algebra of continuous functions on the unit circle, which is clear from the interpretation of Riemann–Hilbert problems given in [14]. By a similar reasoning, Corollary 3 follows from Corollary 2.

Corollary 2 can be proved by Theorem 2 using the scheme of [12], where the same result for classical singular integral operators on closed contours was derived from the solution of B. Bojarski's original problem. To do this, we need first to clarify which one of several possible definitions of abstract singular operators (cf. [1], [21]) is actually appropriate in our setting.

We will use a modification of the approach of [21]. Fix an invertible operator $U \in GB(H_A)$ with the properties:

1. Both operators U and U^{-1} have spectral radii equal to 1;
2. There exists a projector $p \in GB(H_A), p \sim Id \sim Id - p$, such that

$$Up = pUp,\ Up \neq pU,\ pU^{-1} = pU^{-1}p\ ; \tag{16}$$

3. $\operatorname{coker}(U \mid im\ p)$ is an A-module of finite rank.

There are many such operators. For example, one may take the right shift in the $\mathbf{Z}$-graded Hilbert A-module and the projector on the "positive halfspace" (these are the abstract counterparts of multiplication by the independent variable and the Hardy projector from the theory of classical singular integral operators [21]). Denote by $R(U)$ the C^*-subalgebra generated by U and U^{-1}. It is trivial to verify that for any $T \in R(U)$ the commutator $[T,p] = Tp - pT$ is compact, i.e., $[T,p] \in K(H_A)$.

Moreover, the information about A-Fredholm operators contained in diagram (6) enables one to apply the arguments from [21] and obtain a description of invertible elements in $R = R(U)$.

Proposition 5. *Invertible operators are dense in $R(U)$ and characterized by the condition that at least one of their restrictions on $im\ p$ or $im\ (Id - p)$ is a semi-Fredholm operator.*

Following [21], any operator of the form

$$T = Lp + Mq + C, \tag{17}$$

where $q = 1 - p$, L, $M \in R(U)$, $C \in K(H_A)$, is called an abstract singular operator over A (associated with the pair (U,p)). Their collection will be denoted by $S(U)$.

This is a true generalization of the usual singular operators which are obtained when $A = \mathbf{C}$, U is the unitary operator of multiplication by an independent variable in $H = L_2(S^1)$, and p is the Hardy projector (for details see [21]).

A standard application of the Gelfand spectrum theory provides symbols of singular operators which are functions on the spectrum of U. Assuming U to be unitary, it follows that with any operator T of form (17) one may naturally associate a pair of continuous functions $h(T) = (h(L), h(M))$ on the unit circle. A symbol is called nondegenerate if both its components are nowhere vanishing on S^1. As usual, the index ind $h(T)$ of such a nondegenerate symbol is defined as the difference of argument increments of its components along S^1. Thus we can now formulate the key characterization of elliptic singular operators.

Proposition 6. *An operator $T \in S(U)$ of form (17) is Fredholm if and only if its symbol is nondegenerate, i.e., both its coefficients L, M are invertible operators.*

After the above preparations, the proof runs in complete analogy with that from [21]. To compute $\pi_*(S(U))$ over A one has only to compute the homotopy groups of pairs of invertible operators in $GR(U)$. The latter group being homotopy equivalent to $GB_r(H_A)$ with $\pi_+ = p$, the answer is provided by Theorem 2. Adding the groups from the latter theorem to the homotopy groups of nondegenerate symbols computed in [15], one obtains Corollary 2.

Finally, Corollary 4 can be obtained from Corollary 2 using the scheme of [15], where this was done for the classical counterparts of our results. However, this requires a lot of technical preparation. In particular, one needs to generalize the tensor product construction of conventional bisingular operators from the algebra of pseudodifferential operators on the unit circle (see [22]). These technicalities are rather tedious and require a separate presentation.

6. Concluding remarks. Note that in the geometry of Hilbert C^*-modules there are some related topics which admit a nice presentation in terms of special grassmanians and transmission problems. Part of these results has already been indicated in [3].

Here we will discuss only one topic most closely related to the geometric study of elliptic transmission problems [1], [11]. The point is that our Theorem 2 suggests that there should exist a finer geometric structure of the grassmanian $Gr_+(H_A)$ expressed in terms of a stratification similar to the

Birkhoff stratification by partial indices of invertible matrix-functions on the unit circle [8], [23] which plays a prominent role in the classical theory of transmission problems [14], [23].

Such a stratification can be constructed using the geometric language developed in this paper. To this end, let us fix a path component Gr_γ of the grassmanian $Gr_+(H_A)$ corresponding to a certain element $\gamma \in K_0(A)$. By Proposition 3 it is clear that γ is essentially the Fredholm index of the projection π_+ restricted to any element V of this component.

Since $K_0(A)$ is a group, it is reasonable to consider all pairs $(\alpha, \beta) \in K_0 \times K_0$, where $\alpha - \beta = \gamma$. For any pair denote by $B_{\alpha,\beta}$ the subset of all V such that the following relations hold for classes in $K_0(A)$ (recall that any projective A-module generates a class in $K_0(A)$):

$$[\ker \pi_+ \mid V] = \alpha, \quad [\operatorname{coker} \pi_+ \mid V] = \beta. \tag{18}$$

Evidently, such a collection is a subset of the given component and one has

$$Gr_\gamma = \bigcup B_{\alpha,\beta}. \tag{19}$$

The path component Gr_γ being arbitrary, we obtain a natural decomposition of the special grassmanian Gr_+ which is similar to the classical Birkhoff stratification [14], [23] (in fact, our decomposition is cruder, which can be seen in the case of classical transmission problems with respect to the unit circle). Of course, it is tempting to verify which properties of the Birkhoff stratification are still valid in our generalized setting and to generalize some of the results on its geometric structure obtained in the classical case [14], [23]. We are now at the beginning of such an investigation but certain results are already available of which we present only two.

Proposition 7. *All $B_{\alpha,\beta}$ are Banach analytic subspaces of $Gr_+(H_A)$ in the sense of A. Douady* [24].

Proposition 8. *Decomposition* (19) *is a complex analytic stratification of $Gr_+(H_A)$* [24].

These results are of technical nature and require a big portion of the Banach analytic geometry in the spirit of [24], which is completely irrelevant to the present exposition. We give them only to indicate more connections with nontrivial geometric problems one of which will be formulated below.

Note that a less precise version of Proposition 7 was obtained in the classical case ($A = \mathbf{C}$) by S. Disney [23]. The classical counterpart of Proposition 8 was implicitly used by B. Bojarski [14] in his investigation of the stability properties of partial indices.

We conclude with a purely geometric problem suggested by our constructions, which leads to highly nontrivial homological computations even in the classical case [23], [16]. Recall that a complex analytic subset of a complex

Banach manifold has a well-defined cohomological fundamental class in the cohomology of the ambient manifold [23]. A discussion of the orientation classes for K-theory in [23] shows that the same is valid for extraordinary cohomological theories like K-theory. Hence fundamental classes of $B_{\alpha,\beta}$ are well defined and there arises a problem of computing them in terms of K-theory. As was mentioned, some results for the classical case were obtained in [16], but our knowledge of these fundamental classes is still very poor.

An intriguing open problem is to construct a finer analytic stratification of the special grassmanian $Gr_+(H_A)$ similar to that in [8] to obtain more topological invariants for transmission problems. There is some evidence that this should be possible for commutative A.

Our constructions and results find a rather natural interpretation in terms of Fredholm structures over A. Granted diagram (6), the basic notions of this theory can be introduced as in [4]. Several important results of Fredholm structures theory have direct analogs for structures over A. In particular, a family of A-Fredholm operators parametrized by points of a manifold M defines an A-Fredholm structure on M.

Applying this result to our restricted grassmanian $Gr_+(H_A)$, one obtains an A-Fredholm structure on it. Moreover, the Birkhoff strata $B_{\alpha,\beta}$ are the Fredholm submanifolds with respect to this structure, and following [16] one can introduce their Chern classes and express them as pull-backs of universal classes carried by the classifying bundle for A-Fredholm structures. An example of the results in this direction is given in [16].

Acknowledgments

The author acknowledges useful discussions with B. Bojarski, who encouraged him to continue work on these topics in the autumn of 1994, during a Banach Center Semester on functional analysis and partial differential equations.

A substantial part of this research was carried out during the author's long-term visit to the University of Lodz. It is the author's pleasure to express cordial thanks to all members of the Department of Theoretical Physics for the warm hospitality and assistance during the whole period of stay.

During the final period of work on this paper the author was supported by Grant 1.8 of the Georgian Academy of Sciences.

References

1. B. Bojarski, Abstract linear conjugation problems and Fredholm pairs of subspaces. (Russian) *Differential and Integral equations. Boundary value*

problems. (*Russian*) *Collection of papers dedicated to the memory of Academician I. Vekua,* 45–60, *Tbilisi University Press, Tbilisi*, 1979.

2. N. I. Muskhelishvili, Singular integral equations. (Russian) 3*d ed., Nauka, Moscow*, 1968.

3. G. Khimshiashvili, On the homotopy structure of invertible singular operators. *Complex Analysis and Applications*' 97. *Proc. of Intern. Conf.*, 230–234, *Publishing House of the Bulgarian Academy of Sciences, Sofia*, 1989.

4. J. Eells, Fredholm structures. *Proc. Symp. Pure Math.* **18**(1970), 62–85.

5. R. Palais, On the homotopy type of certain groups of operators. *Topology* **3**(1965), 271–279.

6. M. Rieffel, Dimension and stable rank in the K-theory of C^*-algebras. *Proc. Lond. Math. Soc.* **46**(1983), 301–333.

7. K. Thomsen, Non-stable K-theory for operator algebras. *K-theory* **3**(1990), 1–25.

8. A. Pressley and G. Segal, Loop Groups. *Clarendon Press, Oxford*, 1986.

9. J. Mingo, K-theory and multipliers of stable C^*-algebras. *Trans. Amer. Math. Soc.* **299**(1987), 397–411.

10. B. Booss and K. Wojciechowsky, Elliptic boundary problems for Dirac operators. *Birkhäuser, Basel*, 1992.

11. S. Zhang, Factorizations of invertible operators and K-theory of C^*-algebras. *Bull. Amer. Math. Soc.* **28**(1993), 75–83.

12. G. Khimshiashvili, On the homotopy structure of invertible singular integral operators. *Bull. Acad. Sci. Georgian SSR* **108**(1982), 273–276.

13. K. Wojciechowsky, A note on the space of pseudodifferential projections with the same principal symbol. *J. Operator Theory* **15**(1985), 201–216.

14. B. Bojarski, Some analytical and geometrical aspects of the Riemann–Hilbert transmission problems. *Complex Analysis: Methods, Trends, Applications,* 97–110, *Akademie-Verlag, Berlin*, 1983.

15. G. Khimshiashvili, Polysingular Operators and Topology of Invertible Singular Operators. *Z. Anal. Anwendungen* **5**(1986), No. 1, 139–145.

16. G. Khimshiashvili, Lie groups and transmission problems. *Contemporary Mathematics* **131**(1992), No. 2, 121–132.

17. B. Blackadar, K-theory for operator algebras. *Springer-Verlag*, 1986.

18. A. Mishchenko and A. Fomenko, The index of elliptic operators over C^*-algebras. (Russian) *Izv. Akad. Nauk SSSR, Ser. Mat.* **43**(1979), 831–859.

19. G. Kasparov, Hilbert C^*-modules: Theorems of Stinespring and Voiculescu. *J. Operator Theory* **4**(1980), 133–150.

20. E. Troitsky, On the contractibility of general linear groups of a Hilbert C^*-module. (Russian) *Funktsional. Anal. i Prilozhen.* **20**(1986), 58–64.

21. Z. Prössdorf, Einige Klassen singulärer Gleichungen. *Akademie-Verlag, Berlin*, 1974.

22. R. Douglas, Banach algebra techniques in Toeplitz operators theory. *Amer. Math. Soc., Providence, R.I.*, 1973.

23. S. Disney, The exponents of loops on the complex general linear group. *Topology* **12**(1973), 297–315.

24. A. Douady, Le probleme des modules pour les sous-espaces analytique compacts. *Ann. Inst. Fourier, Grenoble* **1**(1966), 1–95.

(Received 30.10.1996)

Author's address:
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 380093
Georgia