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ITERATING THE BAR CONSTRUCTION

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ABSTRACT. For a 1-connected space X Adams's bar construction $B(C^*(X))$ describes $H^*(\Omega X)$ only as a graded module and gives no information about the multiplicative structure. Thus it is not possible to iterate the bar construction in order to determine the cohomology of iterated loop spaces $\Omega^i X$. In this paper for an n-connected pointed space X a sequence of $A(\infty)$-algebra structures $\{m_i^{(k)}\}, k = 1, 2, \dots, n$, is constructed, such that for each $k \leq n$ there exists an isomorphism of *graded algebras*

$$H^*(\Omega^k X) \cong$$
$$\cong (H(B(\cdots(B(B(C^*(X); \{m_i^{(1)}\}); \{m_i^{(2)}\}); \cdots); \{m_i^{(k-1)}\})); m_2^{(k)*}).$$

INTRODUCTION

For a 1-connected pointed space X Adams [1] found a natural isomorphism of graded modules

$$H(B(C^*(X)) \cong H^*(\Omega X),$$

where $B(C^*(X))$ is the bar construction of DG-algebra $C^*(X)$. The method cannot be extended directly for iterated loop spaces $\Omega^k X$ for $k \geq 2$, since the bar construction $B(A)$ of a DG-algebra A is just a DG-coalgebra, and it does not carry the structure of a DG-algebra in order to produce a double bar construction $B(B(A))$.

However, for $A = C^*(X)$ Baues [2] has constructed an associative product

$$\mu : B(C^*(X)) \otimes B(C^*(X)) \to B(C^*(X)),$$

which turns $B(C^*(X))$ into a DG-algebra and which is *geometric*: for 1-connected X there exists an isomorphism of *graded algebras*

$$H(B(C^*(X)) \cong H^*(\Omega X)$$

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and for a 2-connceted X there exists an isomorphism of *graded modules*

$$H(B(B(C^*(X))) \cong H^*(\Omega^2 X).$$

But again, as mentioned in [2], this method cannot be extended for $\Omega^3 X$ either, since "it is impossible to construct a 'nice' product on $B(B(C^*(X))$."

We remark here that in order to produce the bar construction $B(M)$ it is not necessary to have, on a DG-module M, a *strict associative* product

$$\mu : M \otimes M \to M;$$

it suffices to have a *strong homotopy associative* product, or, which is the same, to have an $A(\infty)$-*algebra* structure on M. This notion was introduced by Stasheff in [3]. An $A(\infty)$-algebra $(M, \{m_i\})$ is a graded module M equipped with a sequence of operations

$$\{m_i : M \otimes \cdots (i\text{-}times) \cdots \otimes M \to M,\ i = 1, 2, 3, \dots;\ \deg m_i = 2 - i\},$$

which satisfies the suitable associativity conditions (see below). Such a sequence defines on $B(M)$ the correct differential

$$d_m : B(M) \to B(M),$$

which is a coderivation with respect to the standard coproduct. This DG-coalgebra $(B(M); d_m)$ is denoted by $B(M, \{m_i\})$ and called the bar construction of $A(\infty)$-algebra $(M, \{m_i\})$.

In particular, an $A(\infty)$-algebra of the type

$$(M, \{m_1, m_2, 0, 0, \dots\})$$

is just a DG-algebra with a differential m_1 and a strict associative product m_2 (up to signs). For such an $A(\infty)$-algebra, $B(M, \{m_i\})$ coincides with the usual bar construction. For a general $A(\infty)$-algebra $(M, \{m_i\})$ the first operation $m_1 : M \to M$ is a differential, which is a derivation with respect to the second operation $m_2 : M \otimes M \to M$; this operation is not neccesarily associative, but is homotopy associative (the operation m_3 is a suitable homotopy). Thus we can consider homology of DG-module (M, m_1). Then the product m_2 induces, on $H(M, m_1)$, the *strict associative product* m_2^*.

Now we can formulate the main result of this paper.

Theorem A. *Let X be an n-connected pointed space. Then there exists a sequence of $A(\infty)$-algebra structures $\{m_i^{(k)}\}$, $k = 1, 2, \dots, n$, such that for each $k \le n$ there exists an isomorphism of* graded algebras

$$H^*(\Omega^k X) \cong$$
$$\cong (H(B(\cdots(B(B(C^*(X); \{m_i^{(1)}\}); \{m_i^{(2)}\}); \cdots); \{m_i^{(k-1)}\})); m_2^{(k)*}).$$

Remark 0.1. The latter $A(\infty)$-algebra structure $\{m_i^{(n)}\}$ allows one to produce the next bar construction

$$(B(\cdots(B(B(C^*(X);\{m_i^{(1)});\{m_i^{(2)});\cdots);\{m_i^{(n)}\})),$$

but it is not clear whether it is geometric, i.e., whether homology of this bar construction is isomorphic to $H^*(\Omega^{n+1}X)$ if X is not $(n+1)$-connected.

The proof is based on

Theorem B. *Let $(M,\{m_i\})$ be an $A(\infty)$-algebra, (C,d) be a DG-module, and*

$$f:(M,m_1)\to(C,d)$$

be a weak equivalence of DG-modules. Assume further that M and C are connected and free as graded modules. Then there exist
(1) *an $A(\infty)$-algebra structure $\{m_i'\}$ on C with $m_1'=d$;*
(2) *a morphism of $A(\infty)$-algebras*

$$\{f_i\}:(M,\{m_i\})\to(C,\{m_i'\})$$

with $f_1=f$.

Let us mention some results from the literature dedicated to the problem of iterating the bar construction.

In [4] Khelaia has constructed, on $C^*(X)$, an aditional structure which, in particular, contains Steenrod's $\cup_1$ product, and which is used to introduce, in the bar construction $BC^*(X)$, a homotopy associative product which is geometric: there exists an isomorphism of graded algebras

$$H^*(B(C^*(X))\cong H^*(\Omega X).$$

One can show that this product is strong homotopy associative. Thus there is the possibility to produce the next bar construction $B(B(C^*(X)))$, but the aditional structure itself is lost in $B(C^*(X))$ and hence this structure is not enough to determine the product in $B(B(C^*(X)))$.

Later Smirnov [5], using the technique of operands, introduced a more powerful aditional structure – the E_∞-structure which can be transferred on $B(C^*(X))$. Hence there is the possibility of an iteration.

The structure of an m-algebra introduced by Jastin Smith [6] is also transferable on the bar constuction and, as mentioned in [6], is smaller and has computational advantages against Smirnov's one.

Seemingly, the structure introduced in this paper, i.e., the sequence of $A(\infty)$-algebra structures $\{m_i^{(k)}\}$, should be the smallest one because the $A(\infty)$-algebra structure is a minimal structure required to produce the bar construction.

The disadvantage of our structure is that it cannot be considered as a sequence of operations from $\mathrm{Hom}(\otimes^k C^*(X), C^*(X))$ as in [5] and [6]. In the forthcoming publication we are going to make up for this disadvantage.

The proof of Theorem B is based on the perturbation lemma (see [7,8]), extended for chain equivalences in [9].

In Section 1 Stasheff's notion of an $A(\infty)$-algebra is given. Section 2 is dedicated to the perturbation lemma and in Section 3 Theorem A is proved.

1. $A(\infty)$-Algebras

The notion of an $A(\infty)$-algebra was introduced by J. Stasheff in [3].

An $A(\infty)$-algebra is a graded module $M = \sum\limits_{i=0}^{\infty} M^i$ with a given sequence of operations

$$\{m_i : \otimes^i M \to M;\ \ i = 1, 2, 3, \dots\}$$

which satisfies the following conditions:

(1) $\deg\ m_i = 2 - i$;

(2) $\sum\limits_{k=1}^{n-1} \sum\limits_{j=1}^{n-k+1} m_{n-j+1}(a_1 \otimes \cdots \otimes a_k \otimes m_j(a_{k+1} \otimes \cdots \otimes a_{k+j}) \otimes \cdots \otimes a_n) = 0$

for each $a_i \in M$ and $n > 0$.

The sequence of operations $\{m_i\}$ defines on the tensor coalgebra

$$T'(s^{-1}M) = R + s^{-1}M + s^{-1}M \otimes s^{-1}M + \cdots = \sum_{i=0}^{\infty} \otimes^i s^{-1}M$$

(here $s^{-1}M$ is the desuspension of M) a differential $d_m : T(s^{-1}M) \to T(s^{-1}M)$, given by

$$d_m(s^{-1}a_1 \otimes \cdots \otimes s^{-1}a_n) =$$
$$= \sum_{k=1}^{n-1} \sum_{j=1}^{n-k+1} s^{-1}a_1 \otimes \cdots \otimes s^{-1}a_k \otimes s^{-1}m_j(a_{k+1} \otimes \cdots \otimes a_{k+j}) \otimes \cdots \otimes s^{-1}a_n.$$

This differential turns $(T'(s^{-1}M), d_m)$ into a differential graded coalgebra called the *bar-construction* of an $A(\infty)$-algebra $(M, \{m_i\})$ and denoted by $B(M, \{m_i\})$. Conversely, using the cofreeness of the tensor coalgebra one can show that any differential $d : T'(s^{-1}M) \to T'(s^{-1}M)$ which is a coderivation at the same time coincides with d_m for some $A(\infty)$-algebra structure $\{m_i\}$ (see [10] for details).

For an $A(\infty)$-algebra of the type $(M; \{m_1, m_2, 0, 0, \dots\})$, i.e., for a differential algebra with the differential m_1 and the multiplication m_2 the bar-construction coincides with the usual one.

A morphism of $A(\infty)$-algebras

$$\{f_i\} : (M; \{m_i\}) \to (N; \{n_i\})$$

is defined as a sequence of homomorphisms

$$\{f_i : \otimes^i M \to N;\ \ i = 1, 2, 3, \dots\}$$

which satisfies the following conditions:
(1) deg $f_i = 1 - i$;
(2) $\sum\limits_{k=1}^{n-1} \sum\limits_{j=1}^{n-k+1} f_{n-j+1}(a_1 \otimes \cdots \otimes a_k \otimes m_j(a_{k+1} \otimes \cdots \otimes a_{k+j}) \otimes a_{k+j+1} \otimes \cdots \otimes a_n) =$

$$= \sum_{t=1}^{n} \sum_{k_1+\cdots+k_t=n} n_t(f_{k_1}(a_1 \otimes \cdots \otimes a_{k_1}) \otimes \cdots \otimes f_{k_t}(a_{n-k_t+1} \otimes \cdots \otimes a_n))$$

for each $a_i \in M$ and $n > 0$.

Each $A(\infty)$-algebra morphism induces a DG-coalgebra morphism

$$B(\{f_i\}) : B(M, \{m_i\}) \to B(N, \{n_i\})$$

by

$$B(\{f_i\})(s^{-1}a_1 \otimes \cdots \otimes s^{-1}a_n) = \sum_{t=1}^{n} \sum_{k_1+\cdots+k_t=n} s^{-1}f_{k_1}(a_1 \otimes \cdots \otimes a_{k_1}) \otimes$$
$$\otimes \cdots \otimes s^{-1}f_{k_t}(a_{n-k_t+1} \otimes \cdots \otimes a_n).$$

Conversely, as above, because of the cofreeness of a tensor coalgebra, each DG-colagebra morphism

$$F : B(M, \{m_i\}) \to B(N, \{n_i\})$$

coincides with $B(\{f_i\})$ for a suitable $A(\infty)$-algebra morphism $\{f_i\}$.

Remark 1.1. The first component $f_1 : (M; m_1) \to (N; n_1)$, which is a chain map, is homotopy multiplicative with respect to the homotopy associative products m_2 and n_2. Therefore it induces a map of graded algebras

$$f_1^* : (H(M, m_1), m_2*) \to (H(M, m_1), m_2*).$$

We call an $A(\infty)$-algebra morphism $\{f_i\}$ a *weak equivalence* if f_1^* is an isomorphism.

2. Perturbation Lemma

The general perturbation lemma [7,8] deals with the following problem.

Let (M, d_M) and (N, d_N) be cochain complexes, and suppose they are *equivalent* in some sense, which will be discussed. A *perturbation* of the differential d_N is a homomorphism $t : N \to N$ which satisfies

$$d_N t + t d_N + tt = 0$$

so that $d_t = d_N + t$ is a new differential on N. Then, according to the perturbation lemma, in some suitable circumstances, there exists a perturbation $t' : M \to M$ so that the complexes $(M, d_{t'})$ and (N, d_t) remain "equivalent." There are (co)algebraic versions of the perturbation lemma (see [11,9]): if M and N are DG-(co)algebras and t is a (co)derivation, then, in suitable circumstances, t' is a (co)derivation too.

Using the so-called "tensor trick" (see [11]), the perturbation lemma allows one to transfer not only differentials, but certain algebraic structures too. Suppose, for example, that N is equipped with a DG-algebra structure. Applying the functor $T's^{-1}$ we get new complexes $T'(s^{-1}M)$ and $T'(s^{-1}N)$ with differentials induced by d_M and d_N, respectively. The product operation of N determines the perturbation t so that

$$(T'(s^{-1}N), d_t) = B(N).$$

Then, according to the coalgebra perturbation lemma, there appears a new differential on $T'(s^{-1}N)$, which is a coderivation too and therefore can be interpreted as an $A(\infty)$-algebra structure on M.

Let us specify what an "equivalence" means.

The basic perturbation lemma [7,8] requires of M and N to form a filtered SDR (strong defomation retraction)

$$((M, d_M) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} (N, d_N), \nu)$$

which consists of the following data:
(1) the cochain complexes (M, d_M) and (N, d_N), both filtered with complete filtrations;
(2) the filtration preserving chain maps α and β such that $\beta\alpha = id_M$;
(3) the filtration preserving chain homotopy which is a homomorphism $\nu : N \to N$ of degree -1 such that

$$\alpha\beta - id_N = d_N \nu + \nu d_N.$$

We can now formulate the following

Perturbation Lemma. *Let*

$$((M, d_M) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} (N, d_N), \nu)$$

be a filtered SDR and $t : N \to N$ be a perturbation of d_N which increases filtration. Then there are formulas for $t', \alpha', \beta', \nu'$ such that

$$((M, d_{t'}) \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} (N, d_t), \nu')$$

is a filtered SDR and $t', \alpha - \alpha', \beta - \beta', \nu - \nu'$ increase filtrations.

Remark 2.1. In [11,9] the (co)algebra version of this lemma is shown: if the initial SDR is (co)algebraic, i.e., M and N are filtered DG-(co)algebras, α and β are multiplicative, ν is a (co)derivation homotopy, and if t is a (co)derivation, then the resulting SDR is (co)algebraic too.

In [9] the perturbation lemma is extended for chain equivalences in the following sense: a filtered chain equivalence

$$(\mu, (M, d_M) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} (N, d_N), \nu)$$

consists of
– the filtered cochain complexes M and N;
– the filtration preserving chain maps α and β;
– the filtration preserving chain homotopies $\mu : M \to$ and $\nu : N \to N$ such that

$$\beta\alpha - id_M = d_M\mu + \mu d_M, \quad \alpha\beta - id_N = d_N\nu + \nu d_N.$$

Extended Perturbation Lemma. *Let*

$$(\mu, (M, d_M) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} (N, d_N), \nu)$$

be a filtered chain equivalence and $t : N \to N$ be a perturbation of d_N, which increases filtration. Then there are formulas for $t', \alpha', \beta', \mu', \nu'$ such that

$$(\mu', (M, d_{t'}) \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} (N, d_t), \nu')$$

is a filtered chain equivalence and $t', \alpha - \alpha', \beta - \beta', \mu - \mu', \nu - \nu'$ increase filtrations.

Although there is no (co)algebraic version of this extended perturbation lemma in a general setting, we are going to use in this paper the following result, dual to Theorem (2.3*) from [9].

Proposition 1. *Let*

$$(\mu, X \underset{\beta}{\overset{\alpha}{\rightleftarrows}} Y, \nu)$$

be a chain equivalence, let

$$(T'\mu, T'(X) \underset{T'\beta}{\overset{T'\alpha}{\rightleftarrows}} (Y), T'\nu)$$

be the corresponding filtered chain equivalence of DG-coalgebras, and let t be a comultiplicative perturbation of the differential on $T'(Y)$ (i.e., it increases the augmentation filtration of $T'(Y)$ and is a coderivation). Then there exist a comultiplicative perturbation t' of the differential on $T'(X)$ and a filtered chain equivalence of DG-coalgebras

$$(T'_t\mu, T'_t(X) \underset{T'_t\beta}{\overset{T'_t\alpha}{\rightleftarrows}} T'_t(Y), T'_t\nu),$$

where $T'_t(X)$ and $T'_t(Y)$ refer to new chain complexes.

This proposition (a weak form of the extended coalgebra perturbation lemma) will be used to prove

Theorem B. *Let $(M, \{m_i\})$ be an $A(\infty)$-algebra, (C, d) be a DG-module, and*

$$f : (M, m_1) \to (C, d)$$

be a weak equivalence of DG-modules. Assume further that M and C are connected and free as graded modules. Then there exist
(1) an $A(\infty)$-algebra structure $\{m'_i\}$ on C with $m'_1 = d$;
(2) a morphism of $A(\infty)$-algebras

$$\{f_i\} : (M, \{m_i\}) \to (C, \{m'_i\})$$

with $f_1 = f$.

Proof. Since M and C are free, it is possible to construct a chain equivalence

$$(\mu, (C, d) \underset{f}{\overset{g}{\rightleftarrows}} (M, m_1), \nu).$$

Using the desuspension functor s^{-1}we get the chain equivalence

$$(s^{-1}\mu s, (s^{-1}C, s^{-1}ds) \underset{s^{-1}fs}{\overset{s^{-1}gs}{\rightleftarrows}} (s^{-1}M, s^{-1}m_1 s), s^{-1}\nu s).$$

Applying the functor T', we get the filtered chain equivalence

$$(T's^{-1}\mu s, (T'(s^{-1}C), T'(s^{-1}ds))) \underset{T'(s^{-1}fs)}{\overset{T'(s^{-1}gs)}{\rightleftarrows}} ((T'(s^{-1}M), T'(s^{-1}m_1 s)), T'(s^{-1}\nu s)).$$

Let t be the perturbation of $T'(s^{-1}m_1 s))$ given by

$$t = d_B - T'(s^{-1}m_1 s)),$$

where d_B is the differential of the bar construction $B(M, \{m_i\})$. Clearly, t is a coderivation and increases the standard filtration of $T'(s^{-1}M)$. Then, by virtue of the proposition there exist a perturbation t' and a filtered chain equivalence of DG-algebras

$$(T'_t s^{-1}\mu s, (T'_t(s^{-1}C), d' = T'(s^{-1}ds) + t') \underset{T'_t(s^{-1}fs)}{\overset{T'_t(s^{-1}gs)}{\rightleftarrows}} (T'_t((s^{-1}M), d_B = T'(s^{-1}m_1 s) + t), T'_t s^{-1}\nu s),$$

The differential d' can be interpreted as the $A(\infty)$-algebra structure $\{m'_i\}$ and the map $T'_t(s^{-1}fs)$ as the morphism of $A(\infty)$-algebras

$$\{f_i\} : (M, \{m_i\}) \to (C, \{m'_i\}),$$

which completes the proof. $\square$

3. Proof of Theorem A

For an n-connected space X and $0 \le k \le n$ we are going to construct a sequences of $A(\infty)$-algebra structures $\{m_i^{(k)}(X)\}$ and weak equivalences of $A(\infty)$-algebras

$$\{f_i^{(k)}(X)\} : C^*(\Omega^k X) \to$$
$$((B(\cdots(B(B(C^*(X); \{m_i^{(1)}(X)\}); \{m_i^{(2)}(X)\}); \cdots); \{m_i^{(k)}(X)\}).$$

Then

$$f_1^{(k)*}(X) : H^*(\Omega^k X) \to$$
$$(H(B(\cdots(B(B(C^*(X); \{m_i^{(1)}(X)\}); \cdots); \{m_i^{(k-1)}(X)\})); m*_2^{(k)(X)})$$

will be the required isomorphism from Theorem A.

According to Adams and Hilton [1] (see also Brown [12]), for a 1-connected space X there exists a weak equivalence of DG-coalgebras

$$f : C^*(\Omega X) \to BC^*(X).$$

Furthermore, the cup product

$$\mu : C^*(\Omega X) \otimes C^*(\Omega X) \to C^*(\Omega X)$$

turns $(C^*(\Omega X), d, \mu)$ into a DG-algebra, i.e., it can be considered as an $A(\infty)$-algebra

$$(C^*(\Omega X), \{m_1 = d,\ m_2 = \mu,\ m_{>2} = 0\}).$$

Now by Theorem B there exists, on the bar construction $BC^*(X)$, a structure of $A(\infty)$-algebra $(BC^*(X), \{m_i^{(1)}(X)\})$ and a weak equivalence of $A(\infty)$-algebras

$$\{f_i^{(1)}(X)\} : (C^*(\Omega X), \{m_1 = d, m_2 = \mu, m_{>2} = 0\}) \to (BC^*(X), \{m_i^{(1)}(X)\}).$$

For the next inductive step, assume that X is 2-connected. Then, if instead of X we consider the loop space ΩX (which is 1-connected), by virtue of the preceding we have a weak equivalence of $A(\infty)$-algebras

$$\{f_i^{(1)}(\Omega X)\} : C^*(\Omega^2 X) \to (BC^*(\Omega X), \{m_i^{(1)}(\Omega X)\}).$$

Moreover, the $A(\infty)$-morphism $\{f_i^{(1)}(X)\}$ induces a weak equivalence of DG-coalgebras

$$B(\{f_i^{(1)}(X)\}) : B(C^*(\Omega X) \to B(BC^*(X), \{m_i^{(1)}(X)\}).$$

By Theorem B this weak equivalence transfers the $A(\infty)$-algebra structure $\{m_i^{(1)}(\Omega X)\}$ of $B(C^*(\Omega X)$ to $B(BC^*(\Omega X), \{m_i^{(1)}(\Omega X)\})$ and we get the $A(\infty)$-algebra structure $\{m_i^{(2)}(X)\}$ and the weak equivalence of $A(\infty)$-algebras

$$\{\bar{f}_i^{(2)}(X)\} : (BC^*(\Omega X), \{m_i^{(1)}(\Omega X)\}) \to \\ (B(BC^*(X), \{m_i^{(1)}(X)\}), \{m_i^{(2)}(X)\}).$$

Now we can define the $A(\infty)$-morphism $\{f_i^{(2)}(X)\}$ as the composition

$$\{\bar{f}_i^{(2)}(X)\} \circ \{f_i^{(1)}(\Omega X)\} : C^*(\Omega^2 X) \to (BC^*(\Omega X), m_i^{(1)}(\Omega X)) \to \\ (B(BC^*(X), \{m_i^{(1)}(X)\}), \{m_1^{(2)}(X)\}).$$

Suppose now that $\{m_i^{(k-1)}(X)\}$ and $\{f_i^{(k-1)}(X)\}$ have already been constructed for $k \leq n$. Then, since $\Omega^{k-1}X$ is at least 1-connected, there exists a weak equivalence of $A(\infty)$-algebras

$$\{f_i^{(1)}(\Omega^{k-1}X)\} : C^*(\Omega^k X) \to (BC^*(\Omega^{k-1}X), \{m_i^{(1)}(\Omega^{k-1}X)\}).$$

Moreover, we also have a weak equivalence of DG-coalgebras

$$B(\{f_i^{(k-1)}(X)\}) : B(C^*(\Omega^{k-1}X) \to$$
$$B(\cdots(BC^*(X),\{m_i^{(1)}(X)\}),\cdots),\{m_i^{(k-1)}(X)\}).$$

By Theorem B this weak equivalence transfers the $A(\infty)$-algebra structure $(\{m_i^{(1)}(\Omega^{k-1}X)\}$ of $B(C^*(\Omega^{k-1}X))$ to

$$B(\cdots(BC^*(X),\{m_i^{(1)}(X)\}),\cdots),\{m_i^{(k-1)}(X)\})$$

and we get the $A(\infty)$-algebra structure $\{m_i^{(k)}(X)\}$ and the morphism of $A(\infty)$-algebras

$$\{\bar{f}_i^{(k)}(X)\} : (B(C^*(\Omega^{k-1}X),\{m_i^{(1)}(\Omega^{k-1}X)\}) \to$$
$$(B(\cdots(BC^*(X),\{m_i^{(1)}(X)\}),\cdots),\{m_i^{(k-1)}(X)\}),\{m_i^{(k-1)}(X)\}).$$

Now we can define the $A(\infty)$-morphism $\{f_i^{(k)}(X)\}$ as the composition

$$\{\bar{f}_i^{(k)}(X)\}\circ\{f_i^{(1)}(\Omega^{k-1}X)\}:C^*(\Omega^kX)\to(B(C^*(\Omega^{k-1}X),\{m_i^{(1)}(\Omega^{k-1}X)\})$$
$$\to(B(\cdots(BC^*(X),\{m_i^{(1)}(X)\}),\cdots),\{m_i^{(k-1)}(X)\}),\{m_i^{(k-1)}(X)\}).$$

This completes the proof.

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