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THE DEPENDENCE OF SOLUTION UNIQUENESS CLASSES OF BOUNDARY VALUE PROBLEMS FOR GENERAL PARABOLIC SYSTEMS ON THE GEOMETRY OF AN UNBOUNDED DOMAIN

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Abstract. General boundary value problems are considered for general parabolic (in the Douglas–Nirenberg–Solonnikov sense) systems. The dependence of solution uniqueness classes of these problems on the geometry of a nonbounded domain is established.

The dependence of solution uniqueness classes of the first boundary value problem for a second-order parabolic equation in an unbounded domain on the domain geometry was considered in [2]. It was established there that the uniqueness class could be wider than the solution uniqueness class of the Cauchy problem for the above-mentioned equation. Analogous results were obtained in [2] by the method of barrier functions.

In [3] O. Oleinik constructed examples of second-order parabolic equations in the exponentially narrowing domains for which the solution uniqueness class for the second boundary value problem is the same as the solution uniqueness class of the Cauchy problem although the solution uniqueness class in this domain is wider. Later, in [4] E. Landis showed that if the domain narrows with a sufficient quickness at $|x| \to \infty$, then the solution uniqueness class for the second boundary value problem can be wider than that of the Cauchy problem. In [5] the author considered degenerating parabolic equations of second order and obtained analogous uniqueness theorems for general boundary value problems.

This paper deals with general boundary value problems for general parabolic systems in unbounded domains. Such problems were studied in [6], where solvability conditions similar to the Shapiro-Lopatinski conditions for

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elliptic boundary value problems were found. The solvability conditions for analogous problems in unbounded domains were obtained in [7], [8].

Let ω be an unbounded domain in $\mathbb{R}^{n+1}$ contained between the planes $\{t=0\}$ and $\{t=T=\text{const}>0\}$ and the surface γ lying in-between these planes. Let us consider, in this domain, a linear system of differential equations with complex-valued coefficients of the form

$$\sum_{j=1}^{N}\sum_{|\alpha|+2b\beta\leq s_k+t_j} a_{kj}^{\alpha\beta}(x,t)D_x^\alpha\frac{\partial^\beta}{\partial t^\beta}u_j(x,t)=0, \tag{1}$$

where b, N are positive integers, $s_1,s_2,\dots,s_N$, $t_1,t_2,\dots,t_N$, β are integers, $s_j\leq 0$ and $t_j\geq 0$ for all $j=1,2,\dots,N$, $a_{kj}^{\alpha\beta}(x,t)\equiv 0$ if $s_k+t_j<0$; $\sum\limits_{j=1}^{N}(s_j+t_j)=2bm$, m is a positive integer, $\alpha=(\alpha_1,\dots,\alpha_n)$ is a multi-index; $|\alpha|=\alpha_1+\alpha_2+\dots+\alpha_n$; $D_x^\alpha=D_{x_1}^{\alpha_1}\cdot D_{x_2}^{\alpha_2}\cdots D_{x_n}^{\alpha_n}$, $D_{x_j}=-i\frac{\partial}{\partial x_j}$ (i is the imaginary unit).

The matrix $L_0(x,t,\xi,\sigma)$ with elements

$$\sum_{|\alpha|+2b\beta\leq s_k+t_j} a_{kj}^{\alpha\beta}(x,t)\xi^\alpha\sigma^\beta \quad (k,j=1,2,\dots,N),$$

where $\xi\in\mathbb{R}^n$, σ is a complex-valued number, $\xi^\alpha=\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}$, will be called the principal part of the symbol of system (1). It is assumed that system (1) is uniformly parabolic in the domain ω.

Following [6]–[9], we shall say that system (1) is parabolic in the domain ω if there exists a positive constant λ_0 called a parabolicity constant such that for any $(x,t)\in\omega$ and $\xi\in\mathbb{R}^n$ the roots $\sigma_1,\dots,\sigma_m$ of the polynomial $P(x,t,\xi,\sigma)=\det L_0(x,t,\xi,\sigma)$ with respect to σ satisfy the inequality

$$\operatorname{Re}\sigma_s(x,t,\xi)\leq -\lambda_0|\xi|^{2b} \quad (s=1,2,\dots,m), \tag{2}$$

where $|\xi|^2=\sum\limits_{j=1}^{n}\xi_i^2$.

Denote by $\widehat{L}_0(x,t,\xi,\sigma)$ the matrix of algebraic complements to the elements of the matrix $L_0(x,t,\xi,\sigma)$. Then $\widehat{L}_0=p\cdot L_0^{-1}$.

On the surface γ, we give general boundary conditions of the form

$$\sum_{j=1}^{N}\sum_{|\alpha|+2b\beta\leq q_\nu+t_j} b_{\nu_j}^{\alpha\beta}(x,t)D_x^\alpha\frac{\partial^\beta}{\partial t^\beta}u_j(x,t)\Big|_\gamma=0 \quad (\nu=1,2,\dots,bm), \tag{3}$$

where q_ν are some integers.

Let $(\widehat{x},\widehat{t}) \in \gamma$. Consider the matrix $B_0(\widehat{x},\widehat{t},\xi,\sigma)$ with

$$\sum_{|\alpha|+2b\beta=q_\nu+t_j} b^{\alpha\beta}_{\nu_j}(\widehat{x},\widehat{t})\xi^\alpha\sigma^\beta.$$

Let $\nu_0 = (\nu_1,\dots,\nu_n)$ be the unit vector of the external normal to the surface $s_{\widehat{t}} = \gamma \cap \{x,t : t=\widehat{t}\}$ at the point $(\widehat{x},\widehat{t})$, and $\eta(\widehat{x},\widehat{t})$ be any tangential vector to $s_{\widehat{t}}$ at the same point. It follows from the parabolicity condition (see [6] and [10]) that the polynomial $P(\widehat{x},\widehat{t},\eta(\widehat{x},\widehat{t})+\tau\nu_0(\widehat{x},\widehat{t}),\sigma)$ with respect to τ has bm roots, $\tau^s_+(\widehat{x},\widehat{t},\eta,\sigma)$, $(s=1,2,\dots,bm)$, with positive imaginary parts if there exists a constant λ_1 such that $0<\lambda_1<\lambda_0$ and the inequalities

$$\operatorname{Re}\sigma \geq -\lambda_1|\eta(\widehat{x},\widehat{t})|^{2b}, \quad |\sigma|^2+|\eta(\widehat{x},\widehat{t})|^{4b}>0$$

are fulfilled.

We set $M^+(\widehat{x},\widehat{t},\eta,\sigma,\tau) = \prod\limits_{s=1}^{bm}(\tau-\tau^s_+(\widehat{x},\widehat{t},\eta,\sigma))$.

As to the boundary conditions (3) and system (1), it is assumed that the conditions for being complementary are fulfilled for any point $(\widehat{x},\widehat{t}) \in \gamma$. The condition for being complementary for system (1) and the boundary conditions (3) is fulfilled at the point $(\widehat{x},\widehat{t})$ if the rows of the matrix

$$A(\widehat{x},\widehat{t},\eta(\widehat{x},\widehat{t})+\tau\nu_0(\widehat{x},\widehat{t}),\sigma) \equiv B_0\big(\widehat{x},\widehat{t},\eta(\widehat{x},\widehat{t})+\tau\nu_0(\widehat{x},\widehat{t}),\sigma\big)\times$$
$$\times\widehat{L}_0\big(\widehat{x},\widehat{t},\eta(\widehat{x},\widehat{t})+\tau\nu_0(\widehat{x},\widehat{t}),\sigma\big)$$

are linearly independent modulo the polynomial $M^+(\widehat{x},\widehat{t},\eta,\sigma,\tau)$ with respect to τ provided that $\operatorname{Re}\sigma \geq -\lambda_1|\eta(\widehat{x},\widehat{t})|^{2b}$, $|\sigma|^2+|\eta(\widehat{x},\widehat{t})|^{4b}>0$.

We set

$$\Big(\sum_{s=0}^{bm-1} q^s_{hj}(\widehat{x},\widehat{t},\eta,\sigma)\tau^s\Big) \equiv A\big(\widehat{x},\widehat{t},\eta(\widehat{x},\widehat{t})+\tau\nu_0(\widehat{x},\widehat{t}),\sigma\big)\times$$
$$\times\big(\operatorname{mod} M^+(\widehat{x},\widehat{t},\eta,\sigma,\tau)\big).$$

Consider the matrix $Q(\widehat{x},\widehat{t},\eta,\sigma)$ with elements $q^s_{b_j}(\widehat{x},\widehat{t},\eta,\sigma)$, which has bm rows and bmN columns ($h=1,2,\dots,bm$; $s=0,1,\dots,bm-1$; $j=1,2,\dots,N$).

Let $\Delta_k(\widehat{x},\widehat{t},\eta,\sigma)$, $(k=1,2,\dots,N_1)$ be the minors of order bm of the matrix Q and let $\Delta(\widehat{x},\widehat{t},\eta,\sigma) = \max_k|\Delta_k(\widehat{x},\widehat{t},\eta,\sigma)|$.

According to [6] the condition for being complementary is fulfilled uniformly on the surface γ if

$$\Delta_\gamma \equiv \inf_{(\widehat{x},\widehat{t})\in\gamma} \Delta\big(\widehat{x},\widehat{t},\eta(\widehat{x},\widehat{t}),\sigma\big) > 0, \tag{4}$$
$$\operatorname{Re}\sigma \geq -\lambda|\eta(\widehat{x},\widehat{t})|^{2b}, \quad |\sigma|^2+|\eta(\widehat{x},\widehat{t})|^{4b}=1.$$

On $\omega_0 = \overline{\omega} \cap \{x, t : t = 0\}$ we give initial conditions of the form

$$\sum_{j=1}^{N} \sum_{|\alpha|+2b\beta \le r_h+t_j} c_{h_j}^{\alpha\beta}(x,t) D_x^\alpha \frac{\partial^\beta}{\partial t^\beta} u_j(x,t)\Big|_{\omega_0} = 0 \ \ (h = 1, 2, \dots, m), \quad (5)$$

where r_h are some negative integer numbers and the coefficients $c_{h_j}^{\alpha\beta}(x,t) \equiv 0$, when $r_h + t_j < 0$. Consider the matrix $C_0(x, t, \xi, \sigma)$ with elements

$$\sum_{|\alpha|+2b\beta=r_h+t_j} c_{h_j}^{\alpha\beta}(x,t)\xi^\alpha \sigma^\beta.$$

For system (1) and the boundary conditions (5) the condition for being complementary is fulfilled at the point $(\widehat{x}, \widehat{t}) \in \overline{\omega}$ if the rows of the matrix $H(\widehat{x}, \widehat{t}, \sigma) = C_0(\widehat{x}, \widehat{t}, 0, \sigma) \cdot \widetilde{L}_0(\widehat{x}, \widehat{t}, 0, \sigma)$ are linearly independent modulo σ^m.

Let $\Big(\sum\limits_{s=0}^{m-1} d_{h_j}^s(\widehat{x}, \widehat{t})\sigma^s\Big) \equiv H(\widehat{x}, \widehat{t}, \sigma) (\mathrm{mod}\ \sigma^m)$.

Consider the matrix $H(\widehat{x}, \widehat{t})$ with elements $d_{h_j}^s(\widehat{x}, \widehat{t})$, which has m rows and mN columns ($s = 0, 1, \dots, m-1$; $j = 1, 2, \dots, N$). Let $E_k(\widehat{x}, \widehat{t})$, $(k = 1, 2, \dots, L_1)$ be the minors of order m of the matrix $H(\widehat{x}, \widehat{t})$ and let

$$\Delta(\widehat{x}, \widehat{t}) = \max_k |E_k(\widehat{x}, \widehat{t})|.$$

According to [6] the condition for being complementary is fulfilled uniformly in $\overline{\omega}$ if

$$\Delta_0 \equiv \inf_{\overline{\omega}} \Delta(\widehat{x}, \widehat{t}) > 0. \quad (6)$$

We introduce a space of functions, where a solution of problem (1), (3), (5) will be considered.

Let $G \subset \mathbb{R}_x^n \times (0, T)$ be a finite domain. Denote by $\langle u \rangle_{p,0}^G$ the norm of the function u in the space $L_p(G)$, $\langle u \rangle_{p,0}^G = \Big(\int_G |u|^p \, dx \, dt\Big)^{1/p}$.

We define the norm $\|u\|_{p,2bs}^G = \sum\limits_{|\alpha|+2b\beta \le 2bs} \langle D_x^\alpha \frac{\partial^\beta}{\partial t^\beta} u \rangle_{p,0}^G$, where s is some positive integer.

Denote by $W_p^{2bs,s}(G)$ the space of functions obtained by completing with respect to the norm $\|u\|_{p,2bs}^G$ the set of smooth functions in $\overline{G}$.

As to the surface γ, it is assumed that it satisfies the conditions of uniform local unbending. By [9] this means the following:

In the space $\mathbb{R}^{n+1}_{x,t}$ we consider the sets

$$H(\widehat{x},\widehat{t};\rho)=\{x,t:|x_j-\widehat{x}_j|<\rho,\ \ j=1,2,\dots,n;\ \ -p^{2b}<t-\widehat{t}<\rho^{2b}\};$$
$$H^-(\widehat{x},\widehat{t};\rho)=\{x,t:|x_j-\widehat{x}_j|<\rho,\ \ j=1,2,\dots,n;\ \ -p^{2b}<t-\widehat{t}\le 0\};$$
$$H^+(\widehat{x},\widehat{t};\rho)=\{x,t:|x_j-\widehat{x}_j|<\rho,\ \ j=1,2,\dots,n;\ \ 0\le t-\widehat{t}<\rho^{2b}\};$$
$$H^+(\widehat{x},\widehat{t};\rho_1,\rho_2)=\{x,t:|x_j-\widehat{x}_j|<\rho_1,\ \ j=1,2,\dots,n;\ \ 0\le t-\widehat{t}<\rho_2^{2b}\}.$$

Analogous sets in the space $\mathbb{R}^{n+1}_{y,\tau}$ are denoted by $s(\widehat{y},\widehat{\tau},\rho)$, $s^-(\widehat{y},\widehat{\tau},\rho)$, $s^+(\widehat{y},\widehat{\tau},\rho)$, and $s^+(\widehat{y},\widehat{\tau},\rho_1,\rho_2)$, respectively.

The surface γ will be said to satisfy the condition of uniform local unbending with the constants $d(\omega)$, $M(\omega)$ and $\gamma\in C^{s^*+t^*,b}$, where $t^*=\max\{t_j\}$ and $s^*>q=\max(0,q_1,\dots,q_{bm})$, if the following conditions are fulfilled:

1. For any point $(\widehat{x},0)\in\partial\omega_0$ there exists a neighborhood $O_{\widehat{x},0}$ such that the set $\overline{\omega}\cap O_{\widehat{x},0}$ is homeomorphic under some nondegenerate transformation of the coordinates $\Psi_{\widehat{x},0}=\{y=f(x,t),\ \tau=t: f(\widehat{x},0)=\widehat{x}\}$ to the set $s^+(0,0;\varkappa_1)\cap\{y_n\ge 0\}$, $0<\varkappa\le 1$, and the set $\overline{\gamma}\cap O_{\widehat{x},0}$ is homeomorphic to $s^+(0,0;\varkappa_1)\cap\{y_n=0\}$. It is assumed that the number $\varkappa_1$ does not depend on the point $(\widehat{x},0)\in\partial\omega_0$. Let

$$M_0=\bigcup_{(\widehat{x},0)\in\partial\omega_0}\Psi^{-1}_{\widehat{x},0}\Big(s^+\Big(0,0;\frac{\varkappa_1}{2}\Big)\cap\{y_n\ge 0\}\Big).$$

2. For any point $(\widehat{x},T)\in\partial\omega_T$, where $\omega_T=\overline{\omega}\cap\{t=T\}$, there exists a neighborhood $O_{\widehat{x},T}$ such that the set $\overline{\omega}\cap O_{\widehat{x},T}$ is homeomorphic under some nondegenerate transformation of the coordinates $\Psi_{\widehat{x},T}=\{y=f(x,t),\ \tau=t-T;\ f(\widehat{x},T)=0\}$ to the sets $s^-(0,0;\varkappa_2)\cap\{y_n\ge 0\}$, and the set $\gamma\cap O_{\widehat{x},T}$ is homeomorphic to $s^-(0,0;\varkappa_2)\cap\{y_n=0\}$, $0<\varkappa_2\le 1$. It is assumed that the number $\varkappa_2$ does not depend on the point $(\widehat{x},T)\in\partial\omega_T$. Let

$$M_T=\bigcup_{(\widehat{x},0)\in\partial\omega_T}\Psi^{-1}_{\widehat{x},T}\Big(s^-\Big(0,0;\frac{\varkappa_2}{2}\Big)\cap\{y_n\ge 0\}\Big).$$

3. The transformations $\Psi_{\widehat{x},0}$, $\Psi_{\widehat{x},T}$, $\Psi^{-1}_{\widehat{x},0}$, $\Psi^{-1}_{\widehat{x},T}$ are given by the functions whose norms in the space $C^{s^*+t^*,t}$ are bounded by the variable $K_1(\gamma)\ge 1$.

4. For any point $(\widehat{x},\widehat{t})\in\gamma_1$, where

$$\gamma_1=\gamma\cap\Big\{x,t:\frac{1}{2}\Big(\frac{\varkappa_1}{2}\Big)^{2b}(k_1(\gamma)(n+1))^{-1}\le t\le$$
$$\le T-\frac{1}{2}\Big(\frac{\varkappa_2}{2}\Big)^{2b}(k_1(\gamma)(n+1))^{-1}\Big\}$$

there exists a neighborhood $O_{\widehat{x},\widehat{t}}$ such that the set $O_{\widehat{x},\widehat{t}}\cap\overline{\omega}$ is homeomorphic under some nondegenerate transformation of the coordinates $\Psi_{\widehat{x},\widehat{t}}=\{y=$

$f(x,t)$, $\tau = t - \widehat{t}$; $f(\widehat{x},\widehat{t}) = 0\}$ to the set $s(0,0;\varkappa_3) \cap \{y_n \geq 0\}$, $0 < \varkappa_3 \leq 1$, and the set $O_{\widehat{x},\widehat{t}} \cap \gamma$ is homeomorphic to $s(0,0;\varkappa_3) \cap \{y_n = 0\}$. Assume that $\varkappa_3$ does not depend on the point $(\widehat{x},\widehat{t}) \in \gamma_1$. Let

$$M_{\gamma_1} = \bigcup_{(\widehat{x},wht) \in \gamma_1} \Psi^{-1}_{\widehat{x},\widehat{t}}\Big(s\Big(0,0;\frac{\varkappa_3}{2}\Big) \cap \{y_n \geq 0\}\Big).$$

5. The transformations $\psi_{\widehat{x},\widehat{t}}$ and $\psi^{-1}_{\widehat{x},\widehat{t}}$ are given by the functions whose norms in the space $C^{s^*+t^*,t}$ are bounded by the variable $K_2(\gamma) \geq 1$.

Let $K(\gamma) = \max\{K_1(\gamma), K_2(\gamma)\}$. Since the transformations $\psi^{-1}_{\widehat{x},0}$, $\psi^{-1}_{\widehat{x},T}$ and $\psi^{-1}_{\widehat{x},\widehat{t}}$ shorten the distance between the points $(n+1)K(\gamma)$ times at most, the distance from any point of the set $\omega_1 = \omega \backslash \{M_0 \cup M_T \cup M_{\gamma_1}\}$ to $\overline{\gamma}$ is at least

$$d_1(\gamma) = \min\Big\{\frac{1}{2}\Big(\frac{\varkappa_1}{2}\Big)^{2b}(K(j)(n+1))^{-1},\ \frac{1}{2}\Big(\frac{\varkappa_2}{2}\Big)^{2b}(K(j)(n+1))^{-1},\\ \Big(\frac{\varkappa_3}{2}\Big)^{2b}(K(j)(n+1))^{-1}\Big\}.$$

We set $d(\omega) = d_1(\gamma)(n+1)^{-\frac{1}{2}}$, $M(\omega) = K(\gamma)(n+1)$. It is easy to check that $d(\omega) < 1$, and for any point $(x_0,t_0) \in \omega_1$ the parallelepiped $H(x_0,t_0;\rho) \cap \{0 < t < T\})$ belongs to ω when $\rho < d(\omega)$.

The following statement is proved in [9].

Lemma 1. *Let system* (1) *be uniformly parabolic in ω with the parabolicity constant λ_0. Let system* (1)*, and the boundary and initial conditions* (3) *and* (5) *satisfy the conditions for being complementary to* (4) *and* (6)*, respectively. It is assumed that the surface γ satisfies the condition of a uniform local unbending with the constants $d(\omega)$, $M(\omega)$, and $\gamma \in C^{s^*+t^*,b}$, where $t^* = \max(t_1,\dots,t_N)$, $s^* > q_* = \max(0,q_{b1},\dots,q_{bn})$. Let the integer numbers t_j, s_k, q_ν, and r_h be divisible by $2b$, and for the coefficients of problem* (1), (3), (5) *the condition*

$$\big\|a^{\alpha\beta}_{kj}; C^{s^*-s_k,b}(\omega)\big\| + \big\|b^{\alpha'\beta'}_{\nu j}; C^{s^*-q_\nu,b}(\gamma)\big\| +\\ + \sup_{\tau\in[0,T]} \big\|c^{\alpha''\beta''}_{hj};\ C^{s^*-r_h}(\overline{\omega} \cap \{t = \tau\})\big\| \leq M,$$

be fulfilled, where $M = \text{const} > 0$, $k,j = 1,2,\dots,N$, $\nu = 1,2,\dots,bm$, $h = 1,2,\dots,m$, $|\alpha|+2b\beta \leq s_k+t_j$, $|\alpha'|+2b\beta' \leq q_\nu+t_j$, $|\alpha''|+2b\beta'' \leq r_h+t_j$.

Let $l > 0$, $0 < \tau \leq T$, $1 \leq p < \infty$, and $0 < \rho \leq \min(1, d(\omega), \tau^{1/2b})$. Then for any solution u of the homogeneous problem (1), (3), (5) *such that*

$$u_j \in W^{s^*+t_j,(s^*+t_j)/2b}_p\big(\omega(l+M(\omega)\rho;0,\tau)\big) \quad (j = 1,2,\dots,N),$$

we have an estimate

$$\sum_{j=1}^{N}\rho^{s^*+t^*}\left\|u_j;W_p^{s^*+t_j,(s^*+t_j)/2b}(\omega(l;0,\tau))\right\|\le$$
$$\le C_0\sum_{j=1}^{N}\rho^{s^*-t_j}\left\|u_j;L_p(\omega(l+M(\omega)\rho;0,\tau))\right\|, \tag{7}$$

where $\omega(l;0,\tau)=\omega\cap\{x,t:|x_j|<l,\ j=1,2,\dots,n;\ 0<t<\tau\}$*; the constant* C_0 *depends on the set of constants* $\{\mathcal{K}\}=\{\lambda_0,n,s^*,m,b,t_1,\dots,t_N,s_1,\dots,s_N,M,M(\omega),d(\omega)\}$.

Let us introduce an additional independent variable x_0 and assume that the domain $\Omega=\omega\times\mathbb{R}^1_{x_0}$. In the domain Ω we consider an additional system of the form

$$\sum_{j=1}^{N}\sum_{|\alpha|+2b\beta\le s_k+t_j}a_{kj}^{\alpha\beta}(x,t)D_x^\alpha\Big(\frac{\partial^\beta}{\partial t^\beta}+p_1D_{x_0}^{2b}\Big)\nu_j(x,x_0,t)=0 \tag{8}$$
$$(k=1,2,\dots,N)$$

with the boundary conditions on $\Gamma=\gamma\times\mathbb{R}^1_{x_0}$

$$\sum_{j=1}^{N}\sum_{|\alpha|+2b\beta\le q_\nu+t_j}b_{\nu j}^{\alpha\beta}(x,t)D_x^\alpha\Big(\frac{\partial^\beta}{\partial t^\beta}+p_1D_{x_0}^{2b}\Big)\nu_j(x,x_0,t)=0 \tag{9}$$
$$(\nu=1,2,\dots,bm),$$

and the initial conditions on $\Omega_0=\omega\times\mathbb{R}^1_{x_0}$

$$\sum_{j=1}^{N}\sum_{|\alpha|+2b\beta\le r_h+t_j}c_{hj}^{\alpha\beta}(x,t)D_x^\alpha\Big(\frac{\partial^\beta}{\partial t^\beta}+p_1D_{x_0}^{2b}\Big)\nu_j(x,x_0,t)=0 \tag{10}$$
$$(h=1,2,\dots,m),$$

where p_1 is a positive integer.

The following statement is also proved in [9].

Lemma 2.

(a) *System* (8) *is uniformly parabolic in the domain* Ω *with the parabolicity constant* $\lambda(p_1)$*;* $\lambda(p_1)\to\lambda_0$ *for* $p_1\to\infty$*, where* λ_0 *is the parabolicity constant of system* (1).

(b) *For system* (8) *and the initial conditions* (10)*, the conditions for being complementary are fulfilled uniformly in* Ω *with the constant* Δ_0 *defined by condition* (6).

(c) *There are positive constants* A_0 *and* p_0 *such that, for* $p_1 \geq p_0$, *system* (8) *and the boundary conditions* (9) *satisfy the conditions for being complementary uniformly on* Γ *with the constant* $\Delta_\Gamma \geq A_0 \Delta_\gamma$, *where* Δ_γ *is defined by condition* (4).

(d) *The coefficients of problem* (8)–(10) *satisfy the conditions of Lemma* 1 *with the constant* $M(p_1) > 0$.

(e) *The function* $v(x, x_0, t) = \exp\{i\mu x_0 - p_1 \mu^{2b} t\} u(x,t)$ *is a solution of problem* (8)–(10) *for any real parameter* μ *if only* u *is a solution of problem* (1), (3), (5).

Let u be a solution of problem (1), (3), (5). Consider two additional functions $w(x,t) = \exp\{-p_1 \mu^{2b} t\} u(x,t)$ and $v(x, x_0, t) = \exp\{i\mu x_0\} w(x,t)$.

By Lemma 2 the function v is a solution of problem (8)–(10) and thus this problem satisfies the conditions of Lemma 1. Then for v the inequality

$$\sum_{j=1}^{N} \rho^{s^*+t^*} \big\| v_j; W^{s^*+t_j,(s^*+t_j)/2b}(\Omega(l;0,\tau)) \big\| \leq$$
$$\leq C_0 \sum_{j=1}^{N} \rho^{s^*-t_j} \big\| v_j; L_p(\Omega(l + M(\omega)\rho; 0, \tau)) \big\| \tag{11}$$

holds, where $\Omega(l;0,\tau) = \omega(l;0,\tau) \times \{|x_0| < l\}$ and the constant C_0 depends on the set of constant $\{K_{p_1}\} = \{\lambda(p_1), n, s^*, t_1, \dots, t_N, s_1, \dots, s_N, b, m, M(p_1), s(\omega), M(\omega)\}$.

Lemma 3. *For the functions* w *and* v, *for any* $l > 0$ *and* $0 < \tau \leq T$ *the estimate*

$$\big\| v_j; L_p(\Omega(l;0,\tau)) \big\| \leq C_1^j \big\| w_j; L_p(\omega(l;0,\tau)) \big\| \tag{12}$$

holds, where the constant C_1^j *depends on* l *and* p.

Proof. It is easy to see that

$$\big\| v_j; L_p(\Omega(l;0,\tau)) \big\| = \big\| \exp\{i\mu x_0\} w_j; L_p(\Omega(l;0,\tau)) \big\| =$$
$$= \bigg(\int\limits_{\Omega(l;0,\tau)} |w_j|^p dx\, dx_0\, dt \bigg)^{1/p} = e^{1/p} \bigg(\int\limits_{\omega(l;0,\tau)} |w_j|^p dx\, dt \bigg)^{1/p}.$$

Hence follows inequality (12). □

Lemma 4. *For the functions* v *and* w, *for any* $l > 0$, $\mu > 0$, $0 < \tau \leq T$ *the inequality*

$$C_2^j(\mu, l) \big\| w_j; L_p(\omega(l;0,\tau)) \big\| \leq \big\| v_j; W_p^{s^*+t_j,(s^*+t_j)/2b}(\Omega(l;0,\tau)) \big\| \tag{13}$$

holds, where the constant C_2^j *depends only on* l, m, p.

Proof. As in proving Lemma 3, it is easy to see that

$$\|v_j; W_p^{s^*+t_j,(s^*+t_j)/2b}(\Omega(l;0,\tau))\| = \sum_{|\alpha|+2b\beta\leq s^*+t_j} \langle D^\alpha_{x,x_0}\frac{\partial^\beta}{\partial t^\beta} v_j\rangle_{p,0}^{\Omega(l;0,\tau)} \geq$$

$$\geq \langle D_{x_0}^{s^*+t_j} v_j\rangle_{p,0}^{\Omega(l;0,r)} = \Big(\int\limits_{\Omega(l;0,\tau)} \mu^{(s^*+t_j)p}|w_j|^p dx_0\ dx\ dt\Big)^{1/p} =$$

$$= \mu^{s^*+t_j} e^{1/p}\|w_j; L_p(\omega(l;0,\tau))\|.$$

Hence follows inequality (13) for $C_2^j = \mu^{s^*+t_j} l^{\frac{1}{p}}$. □

Lemma 5. *For the function w, for any $l>0$, $0<\tau\leq T$, and $0<\rho\leq \min(1,d(\omega),\tau^{1/2b})$ the inequality*

$$\sum_{j=1}^N C_3^j(\mu,\rho,l)\rho^{t^*-t_j}\|w_j; L_p(\omega(l;0,\tau))\| \leq$$
$$\leq \sum_{j=1}^N \rho^{t^*-t_j}\|w_j; L_p(\omega(l+M(\omega)\rho;0,\tau))\| \tag{14}$$

holds, where the constant C_3^j depends only on μ, ρ, l, p.

Proof. With (12) and (13) taken into account, inequality (11) implies

$$\sum_{j=1}^N \rho^{s^*+t^*} C_2^j(\mu,l)\|w_j; L_p(\omega(l;0,\tau))\| \leq$$
$$\leq C_0\sum_{j=1}^N \rho^{t^*-t_j} C_1^j(l+M(\omega)\rho)\|w_j; L_p(\omega(l+M(\omega)\rho;0,\tau))\|.$$

Hence follows inequality (14) for $C_3^j = e^{\frac{1}{p}}(l+M(\omega)\rho)^{-1/p}\frac{1}{C_0}(\mu\rho)^{s^*+t_j}$. □

Lemma 6. *Let u be a solution of problem (1), (3), (5). Then for the function w, for any $m_1\geq m_0=\mathrm{const}>0$ and $0<\tau\leq T$ the inequality*

$$\sum_{j=1}^N \|w_j; L_p(\omega(2^{m_1};0,\tau))\| \leq e^{-\frac{k(m_1)}{2}}\sum_{j=1}^N \|w_j; L_p(\omega(2^{m_1+1},2^{m_1};0,\tau))\| \tag{15}$$

holds, where $k(m_1) = \big[\lambda 2^{\frac{2b}{2b-1}m_1}\big]$ and λ is a positive number, while $\omega(l_2,l_1;0,\tau)=\omega(l_2;0,\tau)\backslash\omega(l_1;0,\tau)$ for $l_2>l_1$.

Proof. Assume $l_1 = 2^{m_1}$ and $l_2 = 2l_1$. Let $\rho(m_1) = 2^{m_1}/(M(\omega)k(m_1))$.

Let m_0 be a sufficiently large number such that for $m_1 \geq m_0$ the inequality

$$2^{m_1}/(M(\omega)K(m_1)) \leq \min(1, d(\omega), \tau^{\frac{1}{2b}}) \tag{16}$$

is fulfilled.

Let $l_1(n_1) = l_1 + n_1 M(\omega)\rho$ and $l_1(0) = l_1$. Then inequality (14) implies

$$\sum_{j=1}^{N} C_3^j \rho^{t^*-t_j} \big\| w_j; L_p(\omega(l_1 + n_1 M(\omega)\rho; 0, \tau)) \big\| \leq$$
$$\leq \sum_{j=1}^{N} \rho^{t^*-t_j} \big\| w_j; L_p(\omega(l_1 + (n+1) M(\omega)\rho; 0, \tau)) \big\|.$$

It is easy to see that

$$C_3^j = \Big(\frac{2^{m_1} + n_1 M(\omega)\rho}{2^{m_1} + (n_1+1) M(\omega)\rho}\Big)^{\frac{1}{p}} \frac{1}{C_0} (\mu\rho)^{s^*+t_j} \geq 2^{-\frac{1}{p}} C_0^{-1} (\mu\rho(m_1))^{s^*+t_j}.$$

Let $\mu = \lambda b_0 2^{\frac{m_1}{2b-1}}$. Since $\rho(m_1) \geq 2^{m_1}(M(\omega)\lambda)^{-1} 2^{-\frac{2bm_1}{2b-1}} = (M(\omega)\lambda 2^{\frac{m_1}{2b-1}})^{-1}$, we have $C_3^j \geq 2^{-\frac{1}{p}} C_0^{-1} b_0^{s^*+t_j}$. It is easy to check that if $b_0 = (2^{\frac{p+1}{p}} C_0 e)^{\frac{1}{s^*}}$, then $C_3^j \geq 2e$ for any $j = 1, 2, \dots, N$. In that case we obtain

$$\sum_{j=1}^{N} 2e\rho^{t^*-t_j} \big\| w_j; L_p(\omega(l_1(n_1); 0, \tau)) \big\| \leq \sum_{j=1}^{N} \rho^{t^*-t_j} \big\| w_j; L_p(\omega(l_2(n_1); 0, \tau)) \big\|,$$

where $l_2(n_1) = l_1(n_1 + 1)$. Hence we find

$$\sum_{j=1}^{N} (2e-1) \rho^{t^*-t_j} \big\| w_j; L_p(\omega(l_1(n_1); 0, \tau)) \big\| \leq$$
$$\leq \sum_{j=1}^{N} \rho^{t^*-t_j} \big\| w_j; L_p(\omega(l_2(n_1), l_1(n_1); 0, \tau)) \big\|.$$

But since $2e - 1 > 0$, we have

$$\sum_{j=1}^{N} \rho^{t^*-t_j} \big\| w_j; L_p(\omega(l_1(n_1); 0, \tau)) \big\| \leq$$
$$\leq e^{-1} \sum_{j=1}^{N} \rho^{t^*-t_j} \big\| w_j; L_p(\omega(l_2(n_1), l_1(n_1); 0, \tau)) \big\|. \tag{17}$$

If we now apply inequality (17) corresponding to $n_1 = \nu + 1$ to estimate the right-hand side of this inequality corresponding to $n_1 = \nu$ and assume successively that $\nu = 0, 1, 2, \dots, (k-1)$ and also take into account the fact that the domain $\omega(l_2(k(m_1)-1), l_1(k(m_1)-1); 0, \tau)$ is contained in the domain $\omega(l_2, l_1; 0, \tau)$, then, since $\rho \le 1$, we obtain

$$\sum_{j=1}^{N} \rho^{t^*-t)j} \big\| w_j; L_p(\omega(2^{m_1}; 0, \tau)) \big\| \le e^{-k(m_1)} \sum_{j=1}^{N} \big\| w_j; L_p(\omega(2^{m_1+1}, 2^{m_1}; 0, \tau)) \big\|.$$

Since $t^* - t_j \ge 0$ for any j, we have

$$(\rho(m_1))^{t^*-t_j} \ge \big(2^{m_1}/(M(\omega)k(m_1))\big)^{t^*-t_j} \ge (M(\omega)k(m_1))^{t_j-t^*}.$$

Since $k(m_1) \to \infty$ as $m_1 \to \infty$, the estimate $\big(M(\omega)k(m_1)\big)^{t^*-t_j} \le \exp(k(m_1)/2)$ holds for $m_1 \ge m_0$, where m_0 is a sufficiently large number.

Thus for $m_1 \ge m_0$ the function w satisfies the inequality

$$\sum_{j=1}^{N} \big\| w_j; L_p(\omega(2^{m_1}; 0, \tau)) \big\| \le e^{-\frac{k(m_1)}{2}} \Big\} \sum_{j=1}^{N} \big\| w_j; L_p(\omega(2^{m_1+1}, 2^{m_1}; 0, \tau)) \big\|. \qquad \square$$

Theorem. *Let the domain ω be such that for a solution u of problem* (1), (3), (5) *in the domain ω the equality*

$$\lim_{R\to\infty} \exp\{-\sigma R^{\frac{2b}{2b-1}}\} \cdot \big\| u; L_p(\omega(2R, R; 0, T)) \big\| = 0 \tag{18}$$

holds, where $1 \le \rho < \infty$ and σ is a positive constant. Then $u \equiv 0$ in ω.

Proof. Since $w(x,t) = \exp\{-p_1 \mu^{2b} t\} u(x,t)$, inequality (15) implies

$$\sum_{j=1}^{N} \big\| u_j; L_p(\omega(2^{m_1}; 0, \tau)) \big\| \le \exp\Big\{ -\frac{1}{2} k(m_1) + p_1 \mu^{2b}(m_1)\tau \Big\} \times$$
$$\times \sum_{j=1}^{N} \big\| u_j; L_p(\omega(2^{m_1+1}, 2^{m_1}; 0, \tau)) \big\|,$$

where $k(m_1) = \big[2^{\frac{2bm_1}{2b-1}} \lambda\big]$, $\mu(m_1) = \lambda b_0 2^{\frac{m_1}{2b-1}}$.

Let $\lambda = 4\sigma$ and $\tau_0 = (4p_1 \lambda^{2b-1} b_0^{2b})^{-1}$. Then for any $\tau \le \tau_0$ we have

$$\sum_{j=1}^{N} \big\| u_j; L_p(\omega(2^{m_1}; 0, \tau)) \big\| \le \exp\Big\{ -\sigma 2^{\frac{2b}{2b-1} m_1} \Big\} \times$$
$$\times \sum_{j=1}^{N} \big\| u_j; L_p(\omega(2^{m_2+1}, 2^{m_1}; 0, \tau)) \big\|.$$

Passing in the latter inequality to the limit as $m_1 \to \infty$ and taking into account condition (18), we find that $u \equiv 0$ in the domain $\omega \cap \{x,t : 0 \le t \le \tau_0\}$. Next we find that $u \equiv 0$ in the domains $\omega \cap \{x,t : \tau_0 \le t \le 2\tau_0\}$, $\omega \cap \{x,t : 2\tau_0 \le t \le 3\tau_0\}, \dots$. Therefore $u(x,t) \equiv 0$ in ω. $\square$

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