GEORGIAN MATHEMATICAL JOURNAL: Vol. 5, No. 3, 1998, 263-276

THE TENSOR CATEGORY OF LINEAR MAPS AND LEIBNIZ ALGEBRAS

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ABSTRACT. We equip the category $\mathcal{LM}$ of linear maps of vector spaces with a tensor product which makes it suitable for various constructions related to Leibniz algebras. In particular, a Leibniz algebra becomes a Lie object in $\mathcal{LM}$ and the universal enveloping algebra functor UL from Leibniz algebras to associative algebras factors through the category of cocommutative Hopf algebras in $\mathcal{LM}$. This enables us to prove a Milnor–Moore type theorem for Leibniz algebras.

The relationship between Lie algebras, associative algebras and Hopf algebras can be briefly summarized by the following diagram, in which $\leftrightarrows$ indicates a pair of adjoint functors ($\longrightarrow$ left adjoint to $\longleftarrow$):

$$\begin{tikzcd} & (Hopf) \arrow[dl, shift right, "P"'] \arrow[dr, "forgetful"] & \\ (Lie) \arrow[ur, shift right, "U"'] \arrow[rr, shift right, "U"'] & & (As) \arrow[ll, shift right, "(\ )_L"'] \end{tikzcd}$$

In this diagram (Lie), (As) and $(Hopf)$ stand for the categories of Lie algebras, associative and unital algebras, Hopf co-commutative algebras, respectively.

In [1] (see also [2] and [3]), there is a definition of non-commutative version of Lie algebras called Leibniz algebras. Explicitly, a Leibniz algebra is an algebra whose product denoted by $[-,-]$ satisfies the relation

$$[x,[y,z]]-[[x,y],z]+[[x,z],y]=0.$$

Note that Lie algebras are particular examples of Leibniz algebras. In [2] we constructed and studied the universal enveloping algebra $UL(\mathfrak{g})$ of a Leibniz algebra. Unlike the functor U, the functor UL does not factor

1991 *Mathematics Subject Classification.* 18D10, 17A30, 17B.
Key words and phrases. Tensor categories, Lie algebras, Leibniz algebras.

through (*Hopf*). The purpose of this paper is to show that there exists some factorization provided that one replaces (*Hopf*) by the category of "Hopf objects in the category of linear maps". More precisely, we embed the functor UL into the diagram

$$\begin{tikzcd}
 & & (Hopf\ in\ \mathcal{LM}) \arrow[dl, shift right, "P"'] \arrow[dr, "forgetful"] & & \\
(Leib) \arrow[r, shift right] \arrow[rrrr, bend right, "\mathrm{UL}"'] & (Lie\ in\ \mathcal{LM}) \arrow[l, shift right] \arrow[ur, shift right, "U"'] \arrow[rr, shift right, "U"'] & & (As\ in\ \mathcal{LM}) \arrow[ll, shift right, "(\ )_L"'] \arrow[r] & (As)
\end{tikzcd}$$

The motto is to replace the tensor category of vector spaces by the tensor category $\mathcal{LM}$ of linear maps $f : V \to W$. The key point is the definition of the tensor product of two linear maps (called the *infinitesimal tensor product*) :

$$\begin{pmatrix} V \\ \downarrow \\ W \end{pmatrix} \otimes \begin{pmatrix} V' \\ \downarrow \\ W' \end{pmatrix} := \begin{pmatrix} V \otimes W' \oplus W \otimes V' \\ \downarrow \\ W \otimes W' \end{pmatrix}.$$

In this tensor category one easily defines the notions of associative algebras, Lie algebras, Hopf algebras and the related functors. One can also prove all classical theorems like the Poincaré–Birkhoff–Witt theorem and the Milnor–Moore theorem.

Any Leibniz algebra $\mathfrak{g}$ gives rise to a Lie object in $\mathcal{LM}$, namelly $\mathfrak{g} \twoheadrightarrow \mathfrak{g}_{Lie}$, where $\mathfrak{g}_{Lie}$ is simply $\mathfrak{g}$ quotiented by the ideal generated by the elements $[x, x]$ for $x \in \mathfrak{g}$. This functor $(Leib) \to (Lie\ in\ \mathcal{LM})$ permits us to construct the factorization of UL indicated in the above diagram. The field $\mathbb{K}$ is fixed throughout the paper. The category of vector spaces over $\mathbb{K}$ is denoted by *Vect*. It is a tensor category for the tensor product $\otimes_{\mathbb{K}}$ of vector spaces abbreviated into $\otimes$.

1. The infinitesimal tensor category of linear maps

1.1. The category $\mathcal{LM}$. The objects of the category of linear maps $\mathcal{LM}$ are the $\mathbb{K}$-linear maps $f : V \to W$, where V and W are $\mathbb{K}$-vector spaces. We sometimes denote it by $\begin{pmatrix} V \\ f\downarrow \\ W \end{pmatrix}$, or $\begin{pmatrix} V \\ \downarrow \\ W \end{pmatrix}$, or (V, W), if no confusion can arise. The image of $v \in V$ under f is denoted by $f(v)$ or $\bar{v}$. A morphism

$(\alpha, \bar{\alpha})$ in $\mathcal{LM}$ from (V, W) to (V', W') is a commutative diagram in *Vect*

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & V' \\ f\downarrow & & \downarrow f' \\ W & \xrightarrow{\bar{\alpha}} & W'. \end{array}$$

The composition in $\mathcal{LM}$ is obvious.

We define the *infinitesimal tensor product* of two objects (V, W) and (V', W') of $\mathcal{LM}$ by

$$\begin{pmatrix} & V \\ f & \downarrow \\ & W \end{pmatrix} \otimes \begin{pmatrix} & V' \\ f' & \downarrow \\ & W' \end{pmatrix} \quad := \quad \begin{pmatrix} V \otimes W' \oplus W \otimes V' \\ {\scriptstyle f\otimes 1_{W'}} \downarrow {\scriptstyle +1_W \otimes f'} \\ W \otimes W' \end{pmatrix}. \qquad (1.1.1)$$

Note that there is another tensor product given by $(V \otimes V', W \otimes W')$, but we do not use this tensor product in the present paper.

We will sometimes write $(VW' + WV', WW')$ for the infinitesimal tensor product. In the sequel we refer to f as the "vertical map". "Upstairs" and "downstairs" refer to V and W, respectively.

For two linear maps (V, W) and (V', W') the *interchange map*

$$\tau = \tau_{(V,W),(V',W')} : (V, W) \otimes (V', W') \to (V', W') \otimes (V, W) \quad (1.1.2)$$

is given by $v \otimes w' + w \otimes v' \mapsto v' \otimes w + w' \otimes v$ upstairs and $w \otimes w' \mapsto w' \otimes w$ downstairs. It is clear that $\tau_{(V',W'),(V,W)} = \tau^{-1}_{(V,W),(V',W')}$.

1.2. In the sequel a *tensor category* $\mathcal{X}$ is a $\mathbb{K}$-linear category which is strict monoidal. It means the following : for any two objects X and X', $Hom(X, X')$ is a $\mathbb{K}$-vector space for which the composition is bilinear. Moreover, there is a functor $\mathcal{X} \times \mathcal{X} \to \mathcal{X}, (X, Y) \mapsto X \otimes Y$ (the monoid law, cf. [4]), which is strictly associative, compatible with the composition $((f \circ g) \otimes (f' \circ g') = (f \otimes f') \circ (g \otimes g')$ and $id_x \otimes id_y = id_{x \otimes y})$ and unital $(\exists\ 1 \in \mathcal{O}b\,\mathcal{X}$ such that $X \otimes 1 = 1 \otimes X = X)$.

A tensor category is *symmetric* if for any objects X and Y an isomorphism $\tau = \tau_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$, is given, which is functorial in X and Y and such that $\tau_{Y,X} = \tau^{-1}_{X,Y}$.

The paradigm is the category of vector spaces over $\mathbb{K}$.

Obviously, the infinitesimal tensor category of linear maps, as defined in 1.1, is a symmetric tensor category. Its unit element 1 is $(0, \mathbb{K})$. Since the two functors

$$Vect \longrightarrow \mathcal{M} \longrightarrow Vect,$$
$$W \longmapsto (0, W), \quad (V, W) \longmapsto W,$$

preserve $\otimes$ and since the composite is the identity, it is clear that the downstairs object of an associative algebra (resp. Lie algebra, etc ...) in $\mathcal{LM}$ is an associative algebra (resp. Lie algebra, etc ...) in the classical sense (cf. Section 2 below).

1.3. From definition (1.1) it immediately follows that the *iterated tensor product* of n copies of (V, W) is

$$(V, W)^{\otimes n} = \Big(\bigoplus_{\substack{i+j=n-1 \\ i \geq 0, j \geq 0}} W^{\otimes i} \otimes V \otimes W^{\otimes j}, W^{\otimes n} \Big).$$

The *symmetric product* $S^2(V, W)$ is defined as the linear map which is universal for the morphisms ϕ originating from $(V, W)^{\otimes 2}$ which satisfy $\phi = \tau \circ \phi$. Downstairs we get $S^2 W$ and upstairs we get $(VW + WV)/\approx$, where the equivalence relation $\approx$ is given by $v \otimes w \approx w \otimes v$ so that $S^2(V, W) \cong (V \otimes W, S^2 W)$. More generally, the n-th symmetric product is

$$S^n(V, W) \cong (V \otimes S^{n-1} W, S^n W).$$

The *exterior product* $\Lambda^2(V, W)$ is universal for morphisms from $(V, W)^{\otimes 2}$ satisfying $\phi = -\tau \circ \phi$. Downstairs we get $\Lambda^2 W$ and upstairs we get $(VW + WV)/\approx$, where the equivalence relation is now given by $v \otimes w \approx -w \otimes v$. So we get $\Lambda^2(V, W) \cong (V \otimes W, \Lambda^2 W)$. The n-th exterior product is therefore

$$\Lambda^n(V, W) \cong (V \otimes \Lambda^{n-1} W, \Lambda^n W).$$

2. Algebra in $\mathcal{LM}$ and a bimodule over algebra

2.1. Definition. An *associative algebra* in the tensor category $\mathcal{X}$ is an object X equipped with a morphism $\mu : X \otimes X \to X$ such that $\mu(\mu \otimes 1) = \mu(1 \otimes \mu)$. It is *unital* if there a morphism $\varepsilon : 1 \to X$ is given such that both composites $X \cong X \otimes 1 \overset{1 \otimes \varepsilon}{\to} X \otimes X \overset{\mu}{\to} X$ and $X \cong 1 \otimes X \overset{\varepsilon \otimes 1}{\to} X \otimes X \overset{\mu}{\to} X$ are the identity. It is *commutative* if $\mu \circ \tau = \mu$.

2.2. Definition. A bimodule over the algebra (M, R) is a $\mathbb{K}$-linear map $f : M \to R$, where R is an associative algebra, M is an R-bimodule and f is an R-bimodule map.

A bimodule over the algebra (M, R) is said to be *commutative* if R is commutative and the bimodule M is symmetric. It is *unital* if R is unital and M is a unitary bimodule.

2.3. Proposition *An associative algebra in the infinitesimal tensor category $\mathcal{L}\mathcal{M}$ is equivalent to a bimodule over the algebra.*
Proof. By definition (M, R) is an algebra in $\mathcal{L}\mathcal{M}$ if a morphism

$$\mu : (MR + RM, RR) = (M, R) \otimes (M, R) \longrightarrow (M, R)$$

in $\mathcal{L}\mathcal{M}$ is given which is associative. Downstairs it simply means that R is an algebra. Upstairs it provides us with two linear maps $M \otimes R \to M$ and $R \otimes M \to M$. The associativity condition implies that the two possible maps $MRR + RMR + RRM \longrightarrow M$ are the same. This ensures that M is an R-bimodule.

The commutativity of the diagram associated to the defining morphism implies that f is a bimodule map.

Note that the infinitesimal tensor product of two bimodules over algebras is still a bimodule over algebra.

We leave the reader the task of translating Definitions 2.1, 2.2 and Proposition 2.3 into the coalgebra framework.

2.4. Free bimodule over algebra. To any linear map (V, W) one associates the bimodule over the algebra $(T(W) \otimes V \otimes T(W),\ T(W))$, where $T(W)$ is the tensor algebra over W. The $T(W)$-bimodule structure of $T(W) \otimes V \otimes T(W)$ is clear. The vertical map sends $\omega \otimes v \otimes \omega'$ to the product $\omega\bar{v}\omega'$. This is a free object among bimodules over algebras in the following sense : the functor $(V, W) \mapsto (T(W) \otimes V \otimes T(W),\ T(W))$ is left adjoint to the forgetful functor which assigns to a bimodule over algebra its underlying linear map.

2.5. Commutative bimodule over algebra. A commutative algebra in $\mathcal{L}\mathcal{M}$ is an algebra (M, R) in $\mathcal{L}\mathcal{M}$ for which the product map $\mu : (M, R) \otimes (M, R) \to (M, R)$ satisfies $\mu \circ \tau = \mu$. It is immediate to check that this is equivalent to : R is commutative and M is symmetric.

The free commutative bimodule over algebra associated to the linear map (V, W) is $(S(W) \otimes V,\ S(W))$, where $S(W)$ is the classical symmetric algebra over W (e.g. polynomial algebra if W is finite dimensional), and $S(W) \otimes V$ is a symmetric $S(W)$-bimodule.

2.6. Module over a bimodule over algebra. For an algebra object (M, R) in $\mathcal{L}\mathcal{M}$ a left (M, R)-module is an object (V, W) of $\mathcal{L}\mathcal{M}$ equipped with a morphism

$$(M, R) \otimes (V, W) \to (V, W)$$

satisfying the obvious associativity axiom.

Explicitly, this is equivalent to a left R-module map $V \to W$, together with an R-module map

$$\alpha : M \otimes_R W \to V$$

satisfying some obvious compatibility with the vertical maps.

2.7. Algebra associated to a bimodule over algebra. In this subsection we construct a functor

$$(Bimod/Alg) \longrightarrow (As)$$

from the bimodules over algebra to the associative algebras.

2.8. Proposition. *Let (M, R) be a bimodule over algebra. The formula*

$$(m + r)(m' + r') = f(m)m' + mr' + rm' + rr'$$

for $r, r' \in R$ and $m, m' \in M$, endows the direct sum $M \oplus R$ with an algebra structure. This algebra is denoted by $M \oplus_f R$. If (M, R) is commutative, then $M \oplus_f R$ is a commutative algebra.

Proof. The proof is straightforward and follows from the bimodule properties of M and of f.

Note that if $f = 0$, then $M \oplus_f R$ is the dual number extension (also called the infinitesimal extension) of R by M. It means that M is a square-zero ideal.

There is a dual construction based on the product $mm' = mf(m')$.

If (V, W) is a left (M, R)-module (cf. 2.6), then the direct sum $V \oplus W$ becomes a left $M \oplus_f R$-module by

$$(m + r)(v + w) = \alpha(m \otimes \bar{v} + m \otimes w) + rv + rw.$$

3. Lie algebra in $\mathcal{LM}$ and Leibniz algebras

3.1. Definition. A Lie algebra in a tensor category $\mathcal{X}$ is an object X equipped with a morphism

$$\mu : X \otimes X \to X$$

satisfying

(i) $\mu \circ \tau = -\mu$,

(ii) $\mu(1 \otimes \mu) - \mu(\mu \otimes 1) + \mu(\mu \otimes 1)(1 \otimes \tau) = 0$.

3.2. Proposition. *A Lie object in $\mathcal{LM}$ is equivalent to a linear map $f : M \to \mathfrak{g}$, where $\mathfrak{g}$ is a Lie algebra, M is a (right) $\mathfrak{g}$-module and f is $\mathfrak{g}$-equivariant.*

We denote by $[m, g]$ the action of $g \in \mathfrak{g}$ on $m \in M$.

Proof. Let $\mu : (M, \mathfrak{g}) \otimes (M, \mathfrak{g}) \to (M, \mathfrak{g})$ be the defining morphism. Downstairs it is equivalent to a Lie algebra structure on $\mathfrak{g}$. Upstairs it provides a linear map $M \otimes \mathfrak{g} + \mathfrak{g} \otimes M \to M$. By the symmetry property (i) the map $\mathfrak{g} \otimes M \to M$ can be deduced from the map $M \otimes \mathfrak{g} \to \mathfrak{g}$. By property (ii) this latter map equips M with a right $\mathfrak{g}$-module structure. Commutation of the diagram associated with μ ensures that $M \to \mathfrak{g}$ is $\mathfrak{g}$-equivariant.

3.3. Free Lie object in $\mathcal{LM}$. Let (V,W) be a linear map and let $\mathcal{L}(W)$ be the free Lie algebra on W. In the following sequence of maps

$$V \otimes T(W) \otimes \mathcal{L}(W) \hookrightarrow V \otimes T(W) \otimes T(W) \to V \otimes T(W)$$

the first map is induced by the inclusion $\mathcal{L}(W) \hookrightarrow T(W)$ and the second map is induced by the product in the tensor algebra $T(W)$. The composite defines a right $\mathcal{L}(W)$-module structure on $V \otimes T(W)$.

3.4. Proposition. *The linear map $\gamma : V \otimes T(W) \to \mathcal{L}(W)$ induced by $\gamma(v \otimes w_1 w_2 ... w_n) = [...[[\bar{v}, w_1], w_2], ..., w_n]$ and $\gamma(v \otimes 1) = \bar{v}$ is the free Lie algebra in $\mathcal{LM}$ on (V,W).*

Proof. The $\mathcal{L}(W)$-module structure of $V \otimes T(W)$ was defined above. Let us show that γ is $\mathcal{L}(W)$-equivariant. It suffices to treat the case $V = W$. Then $W \otimes T(W)$ is identified with $\overline{T}(W) = W \oplus \cdots \oplus W^{\otimes n} \oplus \cdots$, by multiplication. The $\mathcal{L}(W)$-equivariance follows from the formula $\gamma(xy) = [\gamma(x), y]$ in $T(W)$ valid for any $x \in \overline{T}(W)$ and any $y \in \mathcal{L}(W) \subset \overline{T}(W)$.

So we have proved that $(V \otimes T(W),\ \mathcal{L}(W))$ is a Lie object in $\mathcal{LM}$. Let us prove the universal freeness property.

Let $(V,W) \xrightarrow{\phi} (M, \mathfrak{g})$ be a morphism in $\mathcal{LM}$ where $(M, \mathfrak{g})$ is a Lie object. This data defines a unique Lie algebra homomorphism $\mathcal{L}(W) \to \mathfrak{g}$. We complete it into a morphism of Lie objects by defining

$$\tilde{\phi} : V \otimes T(W) \to M \ , \ v \otimes w_1, ... w_n \mapsto [...[\phi(v), \bar{\phi}(w_1)], ..., \bar{\phi}(w_n)].$$

Since the maps from V to M and from W to $\mathfrak{g}$ are prescribed and since we need $\tilde{\phi}$ to be coherent with the Lie actions, there is no other possible choice. This proves the existence and uniqueness of $(V \otimes T(W), \mathcal{L}(W)) \to (M, \mathfrak{g})$.

For instance, the free Lie object on (W,W) is $(\overline{T}(W), \mathcal{L}(W))$.

3.5. Definition [1-3]. A right Leibniz algebra $\mathfrak{h}$ is a vector space equipped with a bilinear map

$$[-,-] : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$$

which satisfies the Leibniz identity

$$[x,[y,z]] - [[x,y],z] + [[x,z],y] = 0$$

for all $x, y, z \in \mathfrak{h}$.

The notion of a morphism is obvious. The category of Leibniz algebras is denoted by $(Leib)$.

Note that a Lie algebra is a particular case of a Leibniz algebra.

3.6. Lemma. *For any Lie object* $(M, \mathfrak{g})$ *in* $\mathcal{LM}$ *the vector space* M *equipped with the Lie bracket*

$$[m, m'] := [m, \overline{m'}]$$

is a Leibniz algebra. Moreover, $M \to \mathfrak{g}$ *is a Leibniz homomorphism.*

Proof. The $\mathfrak{g}$-action on $m \in M$ satisfies

$$[m, [g, g']] = [[m, g], g'] - [[m, g'], g]$$

for any $g, g' \in \mathfrak{g}$. Applied to $g = \overline{m'}$ and $g' = \overline{m''}$, this relation gives precisely the Leibniz relation. The rest is obvious from the definition.

3.7. (Leib) and (Lie in $\mathcal{LM}$). The quotient of the Leibniz algebra $\mathfrak{h}$ by the 2-sided ideal generated by the elements $[x, x]$ for $x \in \mathfrak{h}$ is a Lie algebra denoted by $\mathfrak{h}_{Lie}$. The surjective map $\mathfrak{h} \to \mathfrak{h}_{Lie}$ is a Lie object in $\mathcal{LM}$.

So we have constructed two functors

$$(Leib) \leftrightarrow (Lie\ in\ \ \mathcal{LM})$$

which are obviously adjoint to each other :

$$Hom_{Lie\,\mathcal{LM}}(\mathfrak{h} \to \mathfrak{h}_{Lie}, M \to \mathfrak{g}) = Hom_{Leib}(\mathfrak{h}, M).$$

Moreover, starting from $(Leib)$ the composite is the identity.

Note that the Leibniz algebra associated to the free Lie object $(\overline{T}(W), \mathcal{L}(W))$ is the free Leibniz object on W as constructed in [2].

3.8. Example. In [2] we showed that for any associative algebra A the quotient $A \otimes A/\,Im\,b$ can be equipped with a structure of Leibniz algebra (b is the Hochschild boundary, cf. [1]). This Leibniz algebra comes in fact from the Lie object $A \otimes A/\,Im\,b \xrightarrow{b} A$, which is the basic object of study of [5].

3.9. Exercise. Describe a Leibniz object in $\mathcal{LM}$.

4. Universal enveloping algebra in $\mathcal{LM}$

In this section we construct the functor U in the $\mathcal{LM}$ case.

4.1. From (As in $\mathcal{LM}$) to (Lie in $\mathcal{LM}$). The classical functor $(As) \to (Lie)$ associates to the associative algebra A the Lie algebra A_L with bracket given by $[a, b] := ab - ba$. It admits a generalization

$$(-)_L : (As\ in\ \ \mathcal{LM}) \to (Lie\ in\ \ \mathcal{LM})$$

given by $(M, R) \mapsto (M, R_L)$, where the Lie-action of R_L on M is simply $[m, r] := mr - rm$. It is immediate to verify that (M, R_L) is a Lie algebra in $\mathcal{LM}$. Before constructing the adjoint functor we need to prove the following.

4.2. Lemma. *Let $(M,\mathfrak{g})$ be a Lie algebra object in $\mathcal{LM}$ and denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. There is a unique right $U(\mathfrak{g})$-action on $U(\mathfrak{g})\otimes M$ satisfying*

$$(x\otimes m).g = xg\otimes m + x\otimes[m,g],$$

for any $x\in U(\mathfrak{g})$, $m\in M$ and $g\in\mathfrak{g}$. As a result, $U(\mathfrak{g})\otimes M$ becomes a $U(\mathfrak{g})$-bimodule.

Proof. Let us first show the following formula:

$$(1\otimes m).[g,h] = ((1\otimes m).g).h - ((1\otimes m).h).g.$$

On one hand, one has

$$\begin{aligned}(1\otimes m).[g,h] &= [g,h]\otimes m + 1\otimes[m,[g,h]]\\ &= [g,h]\otimes m + 1\otimes[[m,g],h] - 1\otimes[[m,h],g].\end{aligned}$$

On the other hand, one has

$$\begin{aligned}((1\otimes m).g).h &= (g\otimes m + 1\otimes[m,g]).h\\ &= gh\otimes m + g\otimes[m,h] + h\otimes[m,g] + 1\otimes[[m,g],h]\end{aligned}$$

so that

$$\begin{aligned}((1\otimes m).g).h) - ((1\otimes m).h).g &= (gh-hg)\otimes m + 1\otimes[[m,g],h]\\ &\quad - 1\otimes[[m,h],g].\end{aligned}$$

This shows that the right action of $U(\mathfrak{g})$ on $U(\mathfrak{g})\otimes M$ is well-defined.

Since the left action is given by $\omega.(\omega'\otimes m) = (\omega\omega'\otimes m)$, it is clear that $U(\mathfrak{g})\otimes M$ is a $U(\mathfrak{g})$-bimodule.

4.3. Definition. For any Lie algebra object $(M,\mathfrak{g})$ in $\mathcal{LM}$ the *universal enveloping algebra* in $\mathcal{LM}$ denoted by $U(M,\mathfrak{g})$ is the associative algebra in $\mathcal{LM}$

$$(U(\mathfrak{g})\otimes M, U(\mathfrak{g})),$$

where:

– $U(\mathfrak{g})$ is the classical enveloping algebra of $\mathfrak{g}$,
– $U(\mathfrak{g})\otimes M$ is the $U(\mathfrak{g})$-bimodule of Lemma 4.2,
– the vertical map is induced by $1\otimes m\mapsto\bar{m}\in\mathfrak{g}\subset U(\mathfrak{g})$.

The vertical map is a $U(\mathfrak{g})$-bimodule map, in particular, a right $U(\mathfrak{g})$-module map because $\bar{m}g = g\bar{m} + [\bar{m},g]$ in $U(\mathfrak{g})$.

Like in the classical case we verify that

$$U((M,\mathfrak{g})\oplus(M',\mathfrak{g}')) \cong U(M,\mathfrak{g})\otimes U(M',\mathfrak{g}'),$$

where the tensor product is the infinitesimal tensor product in $\mathcal{LM}$.

4.4. Proposition. *The functor U : (Lie in $\mathcal{LM}$) $\to$ (As in $\mathcal{LM}$) is left adjoint to the functor $(-)_L$.*

Proof. The proof is straightforward and left to the reader.

4.5. Example. Let $(V \otimes T(W),\ \mathcal{L}(W))$ be the free Lie object over the linear map (V, W). From the definition of U it comes

$$\begin{aligned} U(V \otimes T(W), \mathcal{L}(W)) &= (U(\mathcal{L}(W)) \otimes (V \otimes T(W)), U(\mathcal{L}(W)) \\ &\cong (T(W) \otimes V \otimes T(W), T(W)), \end{aligned}$$

which is the free associative algebra on (V, W) as expected.

4.6. Universal enveloping algebra of a Leibniz algebra. For any Leibniz algebra $\mathfrak{h}$ the universal enveloping algebra $UL(\mathfrak{h})$ was defined in [2] as follows. Let $\mathfrak{h}^\ell$ and $\mathfrak{h}^r$ be two copies of $\mathfrak{h}$ with elements ℓ_x and r_x corresponding to $x \in \mathfrak{h}$. Then $UL(\mathfrak{h})$ is the quotient of the associative algebra $T(\mathfrak{h}^\ell \oplus \mathfrak{h}^r)$ by the 2-sided ideal generated by the elements

$$\begin{cases} \text{(i)} & r_{[x,y]} - (r_x r_y - r_y r_x), \\ \text{(ii)} & \ell_{[x,y]} - (\ell_x r_y - r_y \ell_x), \\ \text{(iii)} & (r_y + \ell_y)\ell_x, \end{cases}$$

for all $x, y \in \mathfrak{h}$.

4.7. Theorem. *The functor UL is precisely the composite*

$$\begin{array}{ccccccc} (\mathit{Leib}) & \to & (\mathit{Lie\ in}\ \mathcal{LM}) & \overset{U}{\to} & (\mathit{As\ in}\ \mathcal{LM}) & \to & (\mathit{As}) \\ \mathfrak{h} & \mapsto & (\mathfrak{h}, \mathfrak{h}_{Lie}) & & (M, R) & \mapsto & M \oplus_f R \end{array} .$$

Proof. Up to a change of sign this is Proposition 2.4 of [2]. Let $\mathfrak{h}$ be a Leibniz algebra. Its image under the above composite is $M \oplus R$, where $R := U(\mathfrak{h}_{Lie})$ as an algebra, $M := U(\mathfrak{h}_{Lie}) \otimes \mathfrak{h}$ as an R-bimodule (cf. Lemma 4.2), and the product of the two elements $1 \otimes g$ and $1 \otimes h$ of M is

$$f(1 \otimes g)(1 \otimes h) = \bar{g}(1 \otimes h) = \bar{g} \otimes h.$$

Define an algebra map $UL(\mathfrak{h}) \to M \oplus R$ by $r_x \mapsto \bar{x} \in R$, $\ell_y \mapsto -(1 \otimes y) \in M$. Relation (i) is fullfilled, since it is the defining relation of $U(\mathfrak{h}_{Lie})$. Relation (ii) follows from the definition of the right action of R on M. Relation (iii) is a consequence of the formula $mm' = f(m)m'$; indeed,

$$\begin{aligned} (\bar{y} - (1 \otimes y))(-1 \otimes x) &= -\bar{y}(1 \otimes x) + (1 \otimes y)(1 \otimes x) \\ &= -\bar{y}(1 \otimes x) + \bar{y}(1 \otimes x) = 0. \end{aligned}$$

It is easy to check that this well-defined map is an isomorphism of vector spaces.

4.8. Homology. In [1, §10.6] we defined the homology groups $HL_n(\mathfrak{g})$ of a Leibniz algebra $\mathfrak{g}$. In [2] we proved that

$$HL_n(\mathfrak{g}) \cong Tor_n^{UL(\mathfrak{g})}(\mathbb{K}, U(\mathfrak{g}_{Lie})).$$

It is important to notice that this Tor-interpretation of $HL(\mathfrak{g})$ does not depend simply on $UL(\mathfrak{g})$ but rather on the associative algebra in $\mathcal{LM}$ $U(\mathfrak{g} \to \mathfrak{g}_{Lie})$. So it is natural to extend the HL-theory to Lie objects in $\mathcal{LM}$ by

$$HL_n(M \to \mathfrak{g}) := Tor_n^{U(M \to \mathfrak{g})}(\mathbb{K}, U(\mathfrak{g})).$$

5. Hopf algebra in $\mathcal{LM}$

5.1. Bialgebra and Hopf algebra in $\mathcal{LM}$. Let (M, H) be a *bialgebra in $\mathcal{LM}$*. Then H is a bialgebra in the classical sense, M is an H-bimodule and an H-bi-comodule. Moreover, these bimodule and bi-comodule structures are compatible in the following sense : the maps

$$\Delta_1 : M \to M \otimes H \textit{ and } \Delta_2 : M \to H \otimes M$$

defining the two co-module structures are H-bimodule maps. When this bialgebra has an antipode S, then it is called a *Hopf algebra* in $\mathcal{LM}$. By definition, a Hopf algebra in $\mathcal{LM}$ is *irreducible* if it contains a unique simple subcoalgebra. In fact, a Hopf algebra in $\mathcal{LM}$ is *irreducible* iff the downstairs Hopf algebra is irreducible in the classical sense (cf. for instance [6]).

5.2. Theorem. *The universal enveloping functor*

$$U : (Lie\ in\ \mathcal{LM}) \longrightarrow (As\ in\ \mathcal{LM})$$

factors through the category (Hopf in $\mathcal{LM}$) of cocommutative Hopf algebras in the category of linear maps.

Proof. Let $(M, \mathfrak{g})$ be a Lie object in $\mathcal{LM}$. We define a coproduct

$$\Delta : (U\,\mathfrak{g} \otimes M, U\,\mathfrak{g}) \to (U\,\mathfrak{g} \otimes M, U\,\mathfrak{g}) \otimes (U\,\mathfrak{g} \otimes M, U\,\mathfrak{g})$$

as follows. Downstairs Δ is the classical coproduct of the universal enveloping algebra of $\mathfrak{g}$. Recall that it is induced by $\Delta(g) = g \otimes 1 + 1 \otimes g$ for $g \in \mathfrak{g}$.

Upstairs the map

$$\Delta : U\,\mathfrak{g} \otimes M \to (U\,\mathfrak{g} \otimes M) \otimes U\,\mathfrak{g} + U\,\mathfrak{g} \otimes (U\,\mathfrak{g} \otimes M)$$

is induced by $\Delta(1 \otimes m) = (1 \otimes m) \otimes 1 + 1 \otimes (1 \otimes m)$. It is extended to $U\,\mathfrak{g} \otimes M$ thanks to the left $U\,\mathfrak{g}$-module structure. Let us show that Δ is also a right $U\,\mathfrak{g}$-module map :

$$\begin{aligned}\Delta((1\otimes m).g) = \Delta(g\otimes m + 1\otimes[m,g]) = \Delta(g)\Delta(m) + \Delta(1\otimes[m,g]) =\\ = (g\otimes 1 + 1\otimes g)((1\otimes m)\otimes 1 + 1\otimes(1\otimes m)) +\\ +((1\otimes[m,g])\otimes 1 + 1\otimes(1\otimes[m,g])) =\\ = (g\otimes m)\otimes 1 + (1\otimes m)\otimes g + g\otimes(1\otimes m) + 1\otimes(g\otimes m) +\\ +((1\otimes[m,g])\otimes 1 + 1\otimes(1\otimes[m,g]) =\\ = (1\otimes m)\otimes g + ((g\otimes m)\otimes 1 + (1\otimes[m,g])\otimes 1) +\\ +(1\otimes(g\otimes m) + 1\otimes(1\otimes[m,g])) + g\otimes(1\otimes m) =\\ = ((1\otimes m)\otimes 1 + 1\otimes(1\otimes m))(g\otimes 1 + 1\otimes g) = \Delta(1\otimes m)\Delta(g).\end{aligned}$$

Checking that Δ is co-associative is straightforward.

It suffices, now, to show the existence of an antipode $S : U(M,\mathfrak{g})\to U(M,\mathfrak{g})$. As we know, it is induced by $S(g) = -g$ on $U\,\mathfrak{g}$. On $U\,\mathfrak{g}\otimes M$ we induce it from $S(1\otimes m) = -1\otimes m$.

5.3. The functor P **:** $(Bialg\ in\ \mathcal{LM})\to(Lie\ in\ \mathcal{LM})$**.** Let (M,H) be a bialgebra in $\mathcal{LM}$. Let us define

$$P(M) := \{m\in M \mid \Delta(m) = m\otimes 1 + 1\otimes m \in M\otimes H + H\otimes M\}$$
$$P(H) := \{x\in H \mid \Delta(x) = x\otimes 1 + 1\otimes x \in H\otimes H\}.$$

The vertical map sends any element in $P(M)$ to an element in $P(H)$, therefore $P(M,H) := (P(M),P(H))$ is a well-defined linear map.

5.4. Lemma. *For any bialgebra (M,H), the linear map $P(M,H)$ is a Lie object in $\mathcal{LM}$.*

Proof. That $P(H)$ is a Lie algebra is folklore. The action of $P(H)$ on $P(M)$ is given, as expected, by the formula

$$[m,x] := mx - xm\ ,\ m\in P(M)\ ,\ x\in P(H).$$

Let us prove that $[m,x]\in P(M)$.

One has

$$\begin{aligned}\Delta([m,x]) = \Delta(mx) - \Delta(xm) = \Delta(m)\Delta(x) - \Delta(x)\Delta(m) =\\ = (m\otimes 1 + 1\otimes m)(x\otimes 1 + 1\otimes x) - (x\otimes 1 + 1\otimes x)(m\otimes 1 + 1\otimes m) =\\ = mx\otimes 1 + 1\otimes mx - xm\otimes 1 - 1\otimes xm = [m,x]\otimes 1 + 1\otimes[m,x].\end{aligned}$$

As above, the rest of the proof follows the classical pattern.

Note that by composition (5.3 and 3.5) any bialgebra (M,H) in $\mathcal{LM}$ gives rise to a Leibniz algebra $P(M)$.

5.5. Poincaré-Birkhoff-Witt in $\mathcal{LM}$. From now on we suppose that the ground field $\mathbb{K}$ contains $\mathbb{Q}$. In any symmetric tensor category one can perform the following construction. Let $\mathcal{X}$ be a Lie object and $U(\mathcal{X})$ its universal enveloping algebra. The canonical map $\mathcal{X} \to U(\mathcal{X})$ extends to an algebra map $p : T(\mathcal{X}) \to U(\mathcal{X})$. For a fixed integer n and any permutation $\sigma \in S_n$ there exists a morphism $\sigma : \mathcal{X}^{\otimes n} \to \mathcal{X}^{\otimes n}$. Restricting p to $\mathcal{X}^{\otimes n}$ one can form $\frac{1}{n!}\sum_{\sigma \in S_n} p \circ \sigma$, which obviously factors through the symmetric power $S^n\mathcal{X}$. Whence a morphism $e : S(\mathcal{X}) \to U(\mathcal{X})$.

Applied to the Lie object $(M, \mathfrak{g})$ of $\mathcal{LM}$ this construction gives a morphism

$$\underline{e} : (S(\mathfrak{g}) \otimes M, S(\mathfrak{g})) \to (U(\mathfrak{g}) \otimes M, U(\mathfrak{g})).$$

Downstairs this map is the classical symmetrization map : $e(x_1 ... x_n) = \frac{1}{n!}\sum_\sigma x_{\sigma(1)} ... x_{\sigma(n)}$. Upstairs this map is slightly more complicated (and is *not* $e \otimes 1_M$). For instance, $\underline{e}(g \otimes m) = g \otimes m + 1/2(1 \otimes [m, g])$.

As in the classical case, $\underline{e}$ is an isomorphism of coalgebra objects in $\mathcal{LM}$ (cf., for instance, [7] p. 281). This assertion is, essentially, a rephrasing of the PBW-theorem for Leibniz algebras obtained in [2].

5.6. Milnor–Moore theorem in $\mathcal{LM}$. *The functor* $U : (M, \mathfrak{g}) \mapsto U(M, \mathfrak{g}) = (U(\mathfrak{g}) \otimes M, U(\mathfrak{g}))$ *is an equivalence between the category of Lie algebras in* $\mathcal{LM}$ *and the category of irreducible cocommutative Hopf algebras in* $\mathcal{LM}$*, the quasi-inverse functor being* P *.*

Proof. The isomorphism $PU(M, \mathfrak{g}) \cong (M, \mathfrak{g})$ is a consequence of 5.5. Conversely, if $(M, \mathfrak{h})$ is an irreducible cocommutative Hopf algebra in $\mathcal{LM}$, then $\mathfrak{h} \cong UP(\mathfrak{h})$ thanks to the classical Milnor–Moore theorem. It follows from the cocommutative conditions that

$$P(M) = \{x \in M, \Delta_1(x) = x \otimes 1\}.$$

Since M is a right Hopf module, the natural map $UP(M, H) \to (M, H)$ is an isomorphism by Theorem 4.1.1 in [6].

5.7. The internal hom-functor of $\mathcal{LM}$. In a reasonable tensor category the tensor product is left adjoint to a hom-functor :

$$hom(A, hom(B, C)) \cong hom(A \otimes B, C).$$

In order for such an isomorphism to exist it is necessary for $hom(B, C)$ to be an object in the tensor category. Such a functor is called an *internal* hom-functor and then the category is a "closed category" in the sense of MacLane [4].

There exists an internal hom-functor in $\mathcal{LM}$ which is described as follows. Let $f : V \to W$ and $f' : V' \to W'$ be two linear maps. Then hom ((V,W),(V',W')) is the linear map $\varphi : X \to Y$ where
$Y = Hom_{\mathcal{LM}}((V, W), (V', W')) = \{\alpha : V \to V', \beta : W \to W' \mid f' \circ \alpha = \beta \circ f\}$,

$X = \{(\alpha, \beta; \tilde{\beta}) \mid (\alpha, \beta)\}$ as above and $\tilde{\beta} : W \to V'$ such that $\beta = f' \circ \tilde{\beta}\}$, φ consists in forgetting $\tilde{\beta}$.

Acknowledgement

Thanks to Mamuka Jibladze for helpful comments.

The second author is very grateful to the Louis Pasteur University at Strasbourg for providing PAST position at the time of writing this paper. He was also partially supported by INTAS grant No. 93 - 2618 - EXT.

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(Received 29.04.1996)

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