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ALLIED INTEGRALS, FUNCTIONS, AND SERIES FOR THE UNIT SPHERE *

O. DZAGNIDZE

ABSTRACT. Among the functions defined on the two-dimensional unit sphere we distinguish functions generalizing the conjugate integral, the conjugate function, and the conjugate series which depend on one variable. We establish the properties of these functions whose structures essentially differ from those of integrals, functions, and series based on the theory of analytic functions of two complex variables.

INTRODUCTION

0.1. Let a function $F(X,Y,Z)$ be defined on the two-dimensional unit sphere $\sigma=\{(X,Y,Z): Z^2+Y^2+Z^2=1\}$ and $F\in L(\sigma)$. Denote by $U_F(z,y,z)$ the Poisson integral for the three-dimensional unit ball $\mathbb{B}=\{(x,y,z): x^2+y^2+z^2<1\}$ with density $F(X,Y,Z)$, $(X,Y,Z)\in\sigma$, $(x,y,z)\in\mathbb{B}$, which has the form

$$U_F(x,y,z)=$$
$$=\frac{1}{4\pi}\int\limits_{\sigma} F(X,Y,Z)\,\frac{1-(x^2+y^2+z^2)}{[(X-x)^2+(Y-y)^2+(Z-z)^2]^{3/2}}\,dS. \qquad (0.1)$$

By introducing the spherical coordinates (r,θ,ϕ), $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$ $(0\le r<1,\ 0\le\theta\le\pi,\ 0\le\phi\le 2\pi)$, the function F is transformed to a function $f(\theta,\phi)$ defined on the rectangle $R=\{(\theta,\phi): 0\le\theta\le\pi,\ 0\le\phi\le 2\pi\}$ so that $f(\theta,\phi)\cdot\sin\theta\in L(R)$. Hence $U_F(x,y,z)$

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* The results of this paper were announced in the author's paper [1], where the terms used differ slightly from those of this paper.

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takes the form (see, e.g., [2], p. 445)

$$\mathcal{U}_f(r,\theta,\phi)=\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} f(\theta',\phi')P_r(\theta,\phi;\theta,'\phi')\sin\theta'\,d\theta' d\phi', \tag{0.2}$$

which is called the spherical Poisson integral of $f(\theta,\phi)$, $(\theta,\phi)\in R$.

The spherical Poisson kernel P_r can be represented as follows (see, e.g., [3], pp. 335 and 143):

$$P_r(\theta,\phi;\theta',\phi')=\frac{1-r^2}{(1-2r[\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi')]+r^2)^{3/2}} \tag{0.3}$$

and

$$P_r(\theta,\phi;\theta',\phi')=1+\sum_{n=1}^{\infty}(2n+1)r^nP_n(\cos\theta)P_n(\cos\theta')+$$
$$+2\sum_{n=1}^{\infty}(2n+1)r^n\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_{nm}(\cos\theta)P_{nm}(\cos\theta')\cos m(\phi-\phi'). \tag{0.4}$$

First and second order derivatives with respect to θ and to ϕ of the spherical Poisson integral $\mathcal{U}_f(r,\theta,\phi)$ have been investigated, together with their boundary values, in [4]–[9].

0.2. The main objects for one-dimensional Fourier analysis are functions defined on or inside the circle. Among them functions which in the circle can be represented by the Poisson integrals, and their conjugate functions (conjugate Poisson integrals) play a special role, since the Poisson integral and the conjugate Poisson integral make up a pair which is an analytic function inside the circle. This analyticity enables one to represent the radial derivative of the Poisson integral in terms of the derivative of the conjugate Poisson integral with respect to a polar angle. If the density of the Poisson integral possesses the summable conjugate function with a finite derivative at some point ϕ_0, then by virtue of Smirnov's theorem ([10], p. 263; [11], p. 583) and Fatou's theorem ([10], p. 100) this derivative is an angular limit at the point $(1,\phi_0)$ for the radial derivative of the Poisson integral.

Analogous questions naturally arise for the spherical Poisson integral $\mathcal{U}_f(r,\theta,\phi)$. The representation of the radial derivative $\frac{\partial}{\partial r}\mathcal{U}_f(r,\theta,\phi)$ becomes a more difficult problem, in particular, because of the fact that for functions defined in the three-dimensional real ball there is no theory that would be analogous to the theory of analytic functions in the disk. The formula for $\frac{\partial}{\partial r}\mathcal{U}_f(r,\theta,\phi)$ considered in the author's paper "The radial derivative with boundary values of the spherical Poisson integral" has turned out to be closely connected with functions that have not been considered previously.

This paper is dedicated to the investigation of such functions having an independent interest for Fourier analysis on the sphere.

Namely, the triple of harmonic functions $\mathcal{U}_f^*(r,\theta,\phi)$, $\widetilde{\mathcal{U}}_f(r,\theta,\phi)$ and $\widetilde{\mathcal{U}}_f^*(r,\theta,\phi)$ in the ball is connected with the spherical Poisson integral $\mathcal{U}_f(r,\theta,\phi)$. These functions are called allied with $\mathcal{U}_f$ harmonic functions with respect to θ, to ϕ, and to (θ,ϕ), respectively (see §1). The triple of harmonic functions $P_r^*(\theta,\phi;\theta',\phi')$, $\widetilde{P}_r(\theta,\phi;\theta',\phi')$ and $\widetilde{P}_r^*(\theta,\phi;\theta',\phi')$ in the ball is connected with the spherical Poisson kernel $P_r(\theta,\phi;\theta',\phi')$. These functions are called allied with P_r kernels with respect to θ, to ϕ, and to (θ,ϕ). These kernels can be regarded as generalizations of the conjugate Poisson kernel $Q_r(t) = \sum_{n=1}^{\infty} r^n \sin nt$ (see §2). Moreover, the triple of allied harmonic functions admits representations in the form of integrals with allied kernels and are called by the author allied integrals with $\mathcal{U}_f$. These integrals are generalizations of the conjugate Poisson integral (see §3). If $f(\theta,\phi)\in L^2$, then in L^2 there exist functions $f^*(\theta,\phi)$, $\widetilde{f}(\theta,\phi)$, and $\widetilde{f}^*(\theta,\phi)$ called allied with $f(\theta,\phi)$ functions with respect to θ, ϕ, and (θ,ϕ). In that case the allied integrals are Poisson integrals for the allied functions (see §4). We have derived an estimate of the L^2-norm of the functions f^*, $\widetilde{f}$, and $\widetilde{f}^*$ through the L^2-norms of the function $f\in L^2$. The obtained inequalities can be regarded as generalizations of M. Riesz's inequality ([10], p. 253) for the ball when $p=2$ (see §5). The application of the above arguments to the Fourier–Laplace series $S[f]$, $f\in L(R)$, enables us to obtain the allied Laplace series $S^*[f]$, $\widetilde{S}[f]$, and $\widetilde{S}^*[f]$ with respect to θ, to ϕ, and to (θ,ϕ), respectively. For $f\in L^2$ these Laplace series become Fourier–Laplace series $S[f^*]$, $S[\widetilde{f}]$, and $S[\widetilde{f}^*]$, respectively. This fact generalizes, for the ball, Smirnov's equality ([10], p. 263; [11], p. 583) and M. Riesz's equality ([10], p. 253) when $p=2$ (see §6).

0.3. In what follows we shall use the representation of the spherical Poisson integral as a series (see, e.g., [2], p. 444)

$$\mathcal{U}_f(r,\theta,\phi) = a_{00} + \sum_{n=1}^{\infty} a_{n0} r^n P_n(\cos\theta) + \\ + \sum_{n=1}^{\infty} r^n \sum_{m=1}^{n} \big(a_{nm}\cos m\phi + b_{nm}\sin m\phi\big) P_{nm}(\cos\theta), \tag{0.5}$$

coinciding with the Abel–Poisson mean values (A-mean values) of the following Fourier–Laplace series $S[f]$ for the function $f(\theta,\phi)\in L(R)$ (see, e.g., [2], p. 444):

$$S[f] = a_{00} + \sum_{n=1}^{\infty} a_{n0} P_n(\cos\theta) +$$

$$+\sum_{n=1}^{\infty}\sum_{m=1}^{n}\big(a_{nm}\cos m\phi+b_{nm}\sin m\phi\big)P_{nm}(\cos\theta), \tag{0.6}$$

where the Fourier–Laplace coefficients a_{n0}, a_{nm}, b_{nm} of the function f are defined by the following equalities:

$$a_{n0}=\frac{2n+1}{4\pi}\int_0^{\pi}\int_0^{2\pi}f(\theta',\phi')P_n(\cos\theta')\sin\theta'\,d\theta'd\phi', \tag{0.7}$$

$$a_{nm}=\frac{2n+1}{2\pi}\cdot\frac{(n-m)!}{(n+m)!}\int_0^{\pi}\int_0^{2\pi}f(\theta',\phi')P_{nm}(\cos\theta')\times$$
$$\times\cos m\phi'\sin\theta'\,d\theta'd\phi', \tag{0.8}$$

$$b_{nm}=\frac{2n+1}{2\pi}\frac{(n-m)!}{(n+m)!}\int_0^{\pi}\int_0^{2\pi}f(\theta',\phi')P_{nm}(\cos\theta')\times$$
$$\times\sin m\phi'\sin\theta'\,d\theta'd\phi'. \tag{0.9}$$

The Legendre polynomials $P_n(x)$ and the associated Legendre functions $P_{nm}(x)$ figuring in these equalities are defined on $[-1,1]$ by the following equalities:

$$P_n(x)=\frac{1}{n!2^n}\cdot\frac{d^n}{dx^n}(x^2-1)^n,\quad n=0,1,2,\dots, \tag{0.10}$$

$$P_{nm}(x)=(1-x^2)^{\frac{1}{2}m}\frac{d^m}{dx^m}P_n(x)= \tag{0.11}$$

$$=\frac{(1-x^2)^{\frac{1}{2}m}}{n!2^n}\frac{d^{n+m}}{dx^{n+m}}(x^2-1)^n, \tag{0.12}$$
$$1\le m\le n,\quad n=1,2,\dots,$$

where $(1-x^2)^{\frac{1}{2}}$ is a non-negative value of the square root. Note that $P_n^0(x)=P_n(x)$, and $P_{nm}(x)=0$ for $m>n$.

Remark 0.1. In defining the associated Legendre functions, some authors use the multiplier $(-1)^m$ (see, e.g., [3], p. 98 or 107; [12], p. 240, formula (7.12.7); [13], p. 67, 191), while others do not (see, e.g., [14], p. 481; [15], p. 246; [16], p. 384).

Remark 0.2. In the case where the associated Legendre functions are defined by equality (0.11) or (0.12), they are sometimes called Ferrers' functions ([15], p. 246) and denoted by the symbol $T_n^m(x)$ ([3], p. 93) or $P_{n,m}(x)$ ([14], p. 481; [16], p. 384[1]).

Remark 0.3. In what follows $P_{nm}(x)$ will be assumed to be defined by equality (0.11) or (0.12). If $P_{nm}(x)$ were defined by the multiplier $(-1)^m$, then the terms of series (0.4) would have $(-1)^m \cdot (-1)^m = 1$ as multipliers, while in series (0.5) and (0.6) $(-1)^m$ would arise the first time from the coefficients a_{nm} and b_{nm} and the second time from the Legendre functions. As a result, the products $a_{nm}P_{nm}(x)$ and $b_{nm}P_{nm}(x)$ would have $(-1)^m \cdot (-1)^m = 1$ as multipliers. Thus, what will be proved for the Fourier series with functions $P_{nm}(x)$ defined by equality (0.11) or (0.12) will also be true for the Fourier series with associated Legendre functions with multipliers $(-1)^m$.

§ 1. Allied Harmonic Functions

1.1. We introduce the following functions connected with representation (0.5) of the harmonic function $\mathcal{U}_f(r,\theta,\phi)$ in the ball $\mathbb{B}$:

$$\mathcal{U}_f^*(r,\theta,\phi) = \sum_{n=1}^{\infty} a_{n0}\lambda_{n0}r^n P_n(\cos\theta) + \\ + \sum_{n=1}^{\infty} r^n \sum_{m=1}^{n} \lambda_{nm}\big(a_{nm}\cos m\phi + b_{nm}\sin m\phi\big)P_{nm}(\cos\theta) \tag{1.1}$$

and

$$\widetilde{\mathcal{U}_f^*}(r,\theta,\phi) = \sum_{n=1}^{\infty} r^n \sum_{m=1}^{n} \lambda_{nm}\big(a_{nm}\sin m\phi - b_{nm}\cos m\phi\big)P_{nm}(\cos\theta), \tag{1.2}$$

where the numbers λ_{nm} are defined by the equalities

$$\lambda_{nm} = \frac{1}{n+m} + \frac{1}{n-m+1} \quad \text{for} \quad 0 \le m \le n, \;\; n = 1,2,\dots . \tag{1.3}$$

Besides the functions $\mathcal{U}_f^*$ and $\widetilde{\mathcal{U}_f^*}$, we introduce one more function again connected with representation (0.5) of the function $\mathcal{U}_f$,

$$\widetilde{\mathcal{U}}_f(r,\theta,\phi) = \sum_{n=1}^{\infty} r^n \sum_{m=1}^{n} \big(a_{nm}\sin m\phi - b_{nm}\cos m\phi\big)P_{nm}(\cos\theta). \tag{1.4}$$

It will be proved below that the functions $\mathcal{U}_f^*$, $\widetilde{\mathcal{U}_f^*} = (\widetilde{\mathcal{U}}_f)^* \equiv \widetilde{\mathcal{U}}_f^*$ and $\widetilde{\mathcal{U}}_f$ are harmonic in the ball $\mathbb{B}$ (see Theorem 1.1).

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[1] On pages 420–423 of [16] one will also find the tables of surface spherical harmonics and solid spherical harmonics for $0 \le n \le 4$, $0 \le m \le 4$.

The functions $\mathcal{U}_f^*$, $\widetilde{\mathcal{U}}_f$, and $\widetilde{\mathcal{U}}_f^*$ will be called allied with $\mathcal{U}_f$ harmonic functions with respect to θ, to ϕ, and to (θ,ϕ), respectively.

Speaking in general, if there is a harmonic function in the ball $\mathbb{B}$

$$\begin{aligned}\mathcal{U}(r,\theta,\phi)=A_{00}+\sum_{n=1}^{\infty}A_{n0}r^nP_n(\cos\theta)+\\ +\sum_{n=1}^{\infty}r^n\sum_{m=1}^{n}\big(A_{nm}\cos m\phi+B_{nm}\sin m\phi\big)P_{nm}(\cos\theta),\end{aligned}\tag{1.5}$$

then the function

$$\widetilde{\mathcal{U}}(r,\theta,\phi)=\sum_{n=1}^{\infty}r^n\sum_{m=1}^{n}\big(A_{nm}\sin m\phi-B_{nm}\cos m\phi\big)P_{nm}(\cos\theta)\tag{1.6}$$

will be called the allied function of $\mathcal{U}(r,\theta,\phi)$ with respect to ϕ.

Similarly, using the numbers λ_{nm}, one can define the functions $\mathcal{U}^*(r,\theta,\phi)$ and $\widetilde{\mathcal{U}}^*$ allied with the function $\mathcal{U}(r,\theta,\phi)$ with respect to θ and to (θ,ϕ).

1.2. Now we shall prove the theorem on the functions $\mathcal{U}_f^*$, $\widetilde{\mathcal{U}}_f^*$, and $\widetilde{\mathcal{U}}_f$ being harmonic for $f\in L(R)$.

Theorem 1.1. *For every function $f(\theta,\phi)$, summable on the rectangle R, the functions $\mathcal{U}_f^*(r,\theta,\phi)$, $\widetilde{\mathcal{U}}_f^*(r,\theta,\phi)$, and $\widetilde{\mathcal{U}}_f(r,\theta,\phi)$ defined by equalities* (1.1)–(1.4) *are hamonic functions in the ball $\mathbb{B}$.*

Proof. If we introduce the functions

$$\begin{aligned}\alpha_n(\theta)&=a_{n0}\lambda_{n0}P_n(\cos\theta),\\ \beta_n(\theta,\phi)&=\sum_{m=1}^{n}\lambda_{nm}\big(a_{nm}\cos m\phi+b_{nm}\sin m\phi\big)P_{nm}(\cos\theta),\end{aligned}\tag{1.7}$$

then series (1.1) will take the form

$$\mathcal{U}_f^*(r,\theta,\phi)=\sum_{n=1}^{\infty}r^n\alpha_n(\theta)+\sum_{n=1}^{\infty}r^n\beta_n(\theta,\phi).\tag{1.8}$$

By virtue of equality (0.8) we shall have

$$\begin{aligned}a_{nm}P_{nm}(\cos\theta)=\frac{2n+1}{2\pi}\int_0^{\pi}\int_0^{2\pi}f(\theta'\phi')\,\frac{(n-m)!}{(n+m)!}\times\\ \times P_{nm}(\cos\theta)P_{nm}(\cos\theta')\cos m\phi'\sin\theta'\,d\theta'd\phi'.\end{aligned}\tag{1.9}$$

But since for $0 \le \theta \le \pi$, $0 \le \theta' \le \pi$, $1 \le m \le n$, $n = 1, 2, \ldots$, the inequality

$$\frac{(n-m)!}{(n+m)!}\left|P_{nm}(\cos\theta)P_{nm}(\cos\theta')\right| < 1 \tag{1.10}$$

is fulfilled (see [3], p. 418, or [17], p. 271), we obtain $|a_{nm}P_{nm}(\cos\theta)| < \frac{1}{2}nI(f)$, where

$$I(f) = \int_0^\pi \int_0^{2\pi} |f(\theta'\phi')| \sin\theta' \, d\theta' d\phi'. \tag{1.11}$$

A similar estimate holds for $b_{nm}P_{nm}(\cos\theta)$ and $a_{n0}P_n(\cos\theta)$. Thus we have the estimates

$$\begin{gathered} |a_{n0}P_n(\cos\theta)| < \frac{1}{4}nI(f), \quad |a_{nm}P_{nm}(\cos\theta)| < \frac{1}{2}nI(f), \\ |b_{nm}P_{nm}(\cos\theta)| < \frac{1}{2}nI(f), \end{gathered} \tag{1.12}$$

$$\left|(a_{nm}\cos m\phi + b_{nm}\sin m\phi)P_{nm}(\cos\theta)\right| < nI(f) \tag{1.13}$$

for the above-mentioned n, m, θ and all ϕ.

Since[2]

$$|\lambda_{nm}| \le \frac{3}{2}\,, \quad 0 \le m \le n, \ \ n = 1, 2, \ldots\,, \tag{1.14}$$

from (1.7) we obtain, in view of (1.12) and (1.13), the equalities

$$|\alpha_n(\theta)| < \frac{1}{2}nI(f), \quad |\beta_n(\theta,\phi)| < 2n^2 I(f) \tag{1.15}$$

for all θ, ϕ and $n = 1, 2, \ldots$.

Therefore the power series $\sum_{n=1}^\infty n^2 r^n$ converging for $r < 1$ is, to within a constant multiplier, a majorant series for series (1.1). This implies that series (1.1) converges uniformly and absolutely in every closed ball $\mathbb{B}_0 = \{(x, y, z) : \ x^2 + y^2 + z^2 \le r_0^2\}$, where $0 < r_0 < 1$. Thus the sum of series (1.1) exists and is continuous in the unit open ball. We denote this sum by $\mathcal{U}_f^*(r, \theta, \phi)$.

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[2]By equality (1.3) we have $\lambda_{n0} = \lambda_{n1} = \frac{1}{n} + \frac{1}{n+1} \le \frac{3}{2}$ for all $n = 1, 2, \ldots$. A simple calculation shows that

$$\lambda_{n,m+1} - \lambda_{n,m} = 2m \cdot \frac{2n+1}{[(n+1)^2 - m^2](n^2 - m^2)} > 0$$

for $1 \le m \le n - 1$ and $n = 1, 2, \ldots$. Therefore

$$\lambda_{n1} < \lambda_{n2} < \cdots < \lambda_{n,n-1} < \lambda_{nn} = \frac{1}{2n} + 1 \le \frac{3}{2}\,, \quad n = 1, 2, \ldots\,.$$

Now we shall show that the function $\mathcal{U}_f^*(r,\theta,\phi)$ is harmonic in $\mathbb{B}$. For this it is sufficient to prove that the function $\mathcal{U}_f^*(r,\theta,\phi)$ in the ball $\mathbb{B}$ satisfies the equation

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\,\frac{\partial^2\psi}{\partial\phi^2}=0. \tag{1.16}$$

Using the function $\beta_n(\theta,\phi)$ from equality (1.7), we introduce the function $v(r,\theta,\phi)=r^n\beta_n(\theta,\phi)$ for which we have the equalities

$$\frac{\partial v}{\partial r}=nr^{n-1}\beta_n,\quad r^2\,\frac{\partial v}{\partial r}=nr^{n+1}\beta_n,\quad \frac{\partial}{\partial r}\left(r^2\,\frac{\partial v}{\partial r}\right)=n(n+1)r^n\beta_n; \tag{1.17}$$

$$\begin{gathered}\sin\theta\,\frac{\partial v}{\partial\theta}=r^n\sin\,\frac{\partial}{\partial\theta}\,\beta_n,\\ \frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\Big(\sin\theta\,\frac{\partial v}{\partial\theta}\Big)=r^2\Big(\frac{\partial^2}{\partial\theta^2}\,\beta_n+\cot\theta\,\frac{\partial}{\partial\theta}\,\beta_n\Big);\end{gathered} \tag{1.18}$$

$$\frac{1}{\sin^2\theta}\,\frac{\partial^2 v}{\partial\phi^2}=r^n\,\frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial\phi^2}\,\beta_n. \tag{1.19}$$

By inequality (1.15) we have the estimate

$$\Big|\frac{\partial}{\partial r}\left(r^2\,\frac{\partial v}{\partial r}\right)\Big|\le 4n^4r^nI(f) \tag{1.20}$$

while (1.18), (1.19) give

$$\Big|\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial v}{\partial\theta}\right)\Big|\le r^n\left(\Big|\frac{\partial^2}{\partial\theta^2}\,\beta_n\Big|+\frac{1}{\sin\theta}\,\Big|\frac{\partial}{\partial\theta}\,\beta_n\Big|\right), \tag{1.21}$$

$$\Big|\frac{1}{\sin^2\theta}\,\frac{\partial^2 v}{\partial\phi^2}\Big|\le\frac{r^n}{\sin^2\theta}\,\Big|\frac{\partial^2}{\partial\phi^2}\,\beta_n\Big|. \tag{1.22}$$

It is necessary to estimate $|\frac{\partial}{\partial\theta}\,\beta_n|$, $|\frac{\partial^2}{\partial\theta^2}\,\beta_n|$ and $|\frac{\partial^2}{\partial\phi^2}\,\beta_n|$. To this end, we use S. Bernstein's well-know inequality ([10], p.118; [11], p. 47): the inequality

$$|T_n'(x)|\le n\cdot\max_{0\le x\le 2\pi}|T_n(x)| \tag{1.23}$$

is fulfilled for a trigonometric polynomial $T_n(x)$ of order not higher than n.

Since $P_{nm}(\cos\theta)$, $0\le m\le n$, is a trigonometric polynomial of order n, by (1.23) we conclude that

$$\Big|a_{nm}\,\frac{d}{d\theta}\,P_{nm}(\cos\theta)\Big|\le n\cdot\max_{0\le\theta\le\pi}|a_{nm}P_{nm}(\cos\theta)|, \tag{1.24}$$

$$\Big|b_{nm}\,\frac{d}{d\theta}\,P_{nm}(\cos\theta)\Big|\le n\cdot\max_{0\le\theta\le\pi}|b_{nm}P_{nm}(\cos\theta)| \tag{1.25}$$

hold for all θ, $0\le\theta\le\pi$.

Now using inequality (1.12) we obtain

$$\left|a_{n0}\,\frac{d}{d\theta}\,P_n(\cos\theta)\right| < \frac{1}{4}\,n^2 I(f), \tag{1.26}$$

$$\left|a_{nm}\,\frac{d}{d\theta}\,P_{nm}(\cos\theta)\right| \le \frac{1}{2}\,n^2 I(f), \quad \left|b_{nm}\,\frac{d}{d\theta}\,P_{nm}(\cos\theta)\right| \le \frac{1}{2}\,n^2 I(f) \tag{1.27}$$

for all θ, $0 \le \theta \le \pi$, and $n = 1, 2, \ldots$.

Inequalities (1.26) and (1.27) imply

$$\max_{0\le\theta\le\pi}\left|\frac{d}{d\theta}\,\big(a_{n0}P_n(\cos\theta)\big)\right| \le \frac{1}{4}\,n^2 I(f), \quad n = 1, 2, \ldots, \tag{1.28}$$

$$\max_{\substack{0\le\theta\le\pi\\0\le\phi\le2\pi}}\left|\frac{\partial}{\partial\theta}\,\big(a_{nm}\cos m\phi + b_{nm}\sin m\phi\big)P_{nm}(\cos\theta)\right| \le n^2 I(f) \tag{1.29}$$

for $1 \le m \le n$, $n = 1, 2, \ldots$.

Using (1.7), (1.28), and (1.29) we obtain

$$\left|\frac{d}{d\theta}\,\alpha_n(\theta)\right| \le \frac{1}{2}\,n^2 I(f), \quad n = 1, 2, \ldots, \tag{1.30}$$

$$\left|\frac{\partial}{\partial\theta}\,\beta_n(\theta,\phi)\right| \le 2\,n^3 I(f), \quad n = 1, 2, \ldots, \tag{1.31}$$

for all (θ,ϕ), $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$. By virtue of inequalities (1.23) and (1.31) we obtain the estimate

$$\left|\frac{\partial^2}{\partial\theta^2}\,\beta_n(\theta,\phi)\right| \le 2\,n^4 I(f), \quad n = 1, 2, \ldots. \tag{1.32}$$

Further, since

$$\frac{\partial^2}{\partial\phi^2}\,\beta_n(\theta,\phi) =$$

$$= -\sum_{m=1}^{n} m^2\lambda_{nm}(a_{nm}\cos m\phi + b_{nm}\sin m\phi)P_{nm}(\cos\theta), \tag{1.33}$$

by (1.13) and (1.14) we have

$$\max_{\substack{0\le\theta\le\pi\\0\le\phi\le2\pi}}\left|\frac{\partial^2}{\partial\phi^2}\,\beta_n(\theta,\phi)\right| \le 2n^4 I(f), \quad n = 1, 2, \ldots. \tag{1.34}$$

On account of estimates (1.31), (1.32), and (1.33) inequalities (1.21) and (1.22) imply that

$$\left|\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial v}{\partial\theta}\right)\right| \le \frac{4}{\sin\theta}\,n^4 r^n I(f), \quad n = 1, 2, \ldots, \tag{1.35}$$

$$\Big|\frac{1}{\sin^2\theta}\,\frac{\partial^2 v}{\partial\phi^2}\Big| \le \frac{2}{\sin^2\theta}\,n^4 r^n I(f), \quad n=1,2,\dots, \tag{1.36}$$

for all (θ,ϕ), $0<\theta<\pi$, $0\le\phi\le 2\pi$.

From (1.20), (1.35), and (1.36) it follows that the power series $\sum_{n=1}^{\infty} n^4 r^n$, which converges for $r<1$, is, to within a constant multiplier, the majorant series for the series[3] $\sum_{n=1}^{\infty}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r})r^n\beta_n(\theta,\phi)$, $\sum_{n=1}^{\infty}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})$ $r^n\beta_n(\theta,\phi)$, $\sum_{n=1}^{\infty}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}r^n\beta_n(\theta,\phi)$. Therefore these series uniformly and absolutely converge in the ball $\mathbb{B}_0$ of radius $r_0<1$. Hence in these series the operation of differentiation can be put before the summation sign, which enables us to conclude that the function $\mathcal{U}_f^*(r,\theta,\phi)$ satisfies equation (1.16) so that it is harmonic in $\mathbb{B}$.

Applying similar arguments, it can be proved that the functions $\widetilde{\mathcal{U}}_f^*(r,\theta,\phi)$ and $\widetilde{\mathcal{U}}_f(r,\theta,\phi)$ are harmonic in the ball $\mathbb{B}$. □

1.3. The proof of Theorem 1.1 enables one to state that the harmonicity preserves after the transformation of the coefficients.

Theorem 1.2. *Let the function $\Phi(\theta,\phi)$ be summable on R. Denote its Fourier–Laplace coefficients by A_{n0}, A_{nm}, B_{nm} and consider the harmonic in the ball $\mathbb{B}$ function which is the spherical Poisson integral for $\Phi(\theta,\phi)$*

$$\mathcal{U}_\Phi(r,\theta,\phi) = A_{00} + \sum_{n=1}^{\infty}\Big[A_{n0}r^n P_n(\cos\theta) + \\ + r^n\sum_{m=1}^{n}(A_{nm}\cos m\phi + B_{nm}\sin m\phi)P_{nm}(\cos\theta)\Big]. \tag{1.37}$$

Assume that a sequence of real numbers μ_{nm} is given such that the relations

$$|\mu_{nm}|\le cn^p \quad for \quad 1\le m\le n, \ \ n=1,2,\dots, \tag{1.38}$$

are fulfilled for some fixed constants $c>0$ and $p>0$.

Then the three series

$$\sum_{n=1}^{\infty}\frac{\partial}{\partial r}\Big(r^2\frac{\partial}{\partial r}\Big)\Big[\mu_{n0}A_{n0}r^n P_n(\cos\theta) + \\ + r^n\sum_{m=1}^{n}\mu_{nm}(A_{nm}\cos m\phi + B_{nm}\sin m\phi)P_{nm}(\cos\theta)\Big], \tag{1.39}$$

$$\sum_{n=1}^{\infty}\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial}{\partial\theta}\Big)\Big[\mu_{n0}A_{n0}r^n +$$

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[3] It is a well-known fact that the values $\theta=0$ and $\theta=\pi$ do not prevent one from making the same conclusions for the whole ball $\mathbb{B}$ ([2], p. 273).

$$+r^n \sum_{m=1}^{n} \mu_{nm}(A_{nm}\cos m\phi + B_{nm}\sin m\phi)P_{nm}(\cos\theta)\Big], \qquad (1.40)$$

$$\sum_{n=1}^{\infty} \frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial\phi^2}\Big[\mu_{n0}A_{n0}r^n +$$

$$+r^n \sum_{m=1}^{n} \mu_{nm}(A_{nm}\cos m\phi + B_{nm}\sin m\phi)P_{nm}(\cos\theta)\Big] \qquad (1.41)$$

converge absolutely and uniformly in each closed ball $\mathbb{B}_0$ *of radius* $r_0 < 1$ *and the sum of the series*

$$\sum_{n=1}^{\infty} \Big[\mu_{n0}A_{n0}r^n P_n(\cos\theta) +$$

$$+r^n \sum_{m=1}^{n} \mu_{nm}(A_{nm}\cos m\phi + B_{nm}\sin m\phi)P_{nm}(\cos\theta)\Big] \qquad (1.42)$$

is a harmonic function in the ball $\mathbb{B}$.

§ 2. Allied Kernels

We introduce the functions

$$\begin{aligned} P_r^*(\theta,\phi;\theta',\phi') &= \sum_{n=1}^{\infty}(2n+1)\lambda_{n0}r^n P_n(\cos\theta)P_n(\cos\theta') + \\ &+ 2\sum_{n=1}^{\infty}(2n+1)r^n \sum_{m=1}^{n} \lambda_{nm}\cdot\frac{(n-m)!}{(n+m)!}\times \\ &\times P_{nm}(\cos\theta)P_{nm}(\cos\theta')\cos m(\phi-\phi'), \qquad (2.1) \\ \widetilde{P_r^*}(\theta,\phi;\theta',\phi') &= 2\sum_{n=1}^{\infty}(2n+1)r^n \sum_{m=1}^{n} \lambda_{nm}\cdot\frac{(n-m)!}{(n+m)!}\times \\ &\times P_{nm}(\cos\theta)P_{nm}(\cos\theta')\sin m(\phi-\phi'), \qquad (2.2) \end{aligned}$$

where λ_{nm} are defined by equality (1.3). Besides, we introduce one more function

$$\begin{aligned} \widetilde{P}_r(\theta,\phi;\theta',\phi') &= 2\sum_{n=1}^{\infty}(2n+1)r^n \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!}\times \\ &\times P_{nm}(\cos\theta)P_{nm}(\cos\theta')\sin m(\phi-\phi'). \qquad (2.3) \end{aligned}$$

The functions P_r^*, $\widetilde{P}_r$, and $\widetilde{P_r^*} = (\widetilde{P}_r)^* \equiv \widetilde{P}_r^*$ will be called allied kernels with the Poisson kernel P_r with respect to θ, to ϕ, and to (θ,ϕ), respectively.

Let us establish some properties of the functions P_r^*, $\widetilde{P}_r^*$, and $\widetilde{P}_r$.

2.1. First, we shall prove that the following equalities are valid:

$$\int_0^{2\pi} \widetilde{P}_r^*(\theta,\phi;\theta',\phi')\,d\phi' = 0, \tag{2.4}$$

$$\int_0^{2\pi} \widetilde{P}_r(\theta,\phi;\theta',\phi')\,d\phi' = 0, \tag{2.5}$$

$$\int_0^{\pi}\int_0^{2\pi} P_r^*(\theta,\phi;\theta',\phi')\sin\theta'\,d\theta' d\phi' = 0, \tag{2.6}$$

$$\int_0^{\pi}\int_0^{2\pi} \widetilde{P}_r^*(\theta,\phi;\theta',\phi')P_k(\cos\theta')\sin\theta'\,d\theta' d\phi' = 0, \quad k=1,2,\dots, \tag{2.7}$$

$$\int_0^{\pi}\int_0^{2\pi} \widetilde{P}_r(\theta,\phi;\theta',\phi')P_k(\cos\theta')\sin\theta'\,d\theta' d\phi' = 0, \quad k=1,2,\dots. \tag{2.8}$$

Equalities (2.4) and (2.5) are immediately obtained from equalities (2.2) and (2.3) by taking into account that the termwise integration is correct on account of inequalities (1.10) and (1.14).

To prove (2.6) note that equality (2.1) implies

$$\int_0^{2\pi} P_r^*(\theta,\phi;\theta',\phi')\,d\phi' = 2\pi\sum_{n=1}^{\infty}(2n+1)\lambda_{n0}r^nP_n(\cos\theta)P_n(\cos\theta'). \tag{2.9}$$

Hence

$$\int_0^{\pi}\int_0^{2\pi} P_r^*\sin\theta'\,d\theta' d\phi' =$$

$$= 2\pi\sum_{n=1}^{\infty}\lambda_{n0}r^nP_n(\cos\theta)\int_0^{\pi}(2n+1)P_n(\cos\theta')\sin\theta'\,d\theta' =$$

$$= 2\pi\sum_{n=1}^{\infty}\lambda_{n0}r^nP_n(\cos\theta)\int_{-1}^{1}(2n+1)P_n(x)\,dx.$$

If we now use the equality (see, for instance, [3], p. 33, equality (34), or [12], p. 228, equality (7.8.2))

$$(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x), \quad P_{-1}(x) = 0 \tag{2.10}$$

and the equalities $P_n(1) = 1$, $P_n(-1) = (-1)^n$, then we shall obtain the relations

$$\int_{-1}^{1} P_n(x)\,dx = 0 = \int_{0}^{\pi} P_n(\cos\theta')\sin\theta'\,d\theta', \quad n = 1, 2, \dots . \tag{2.11}$$

We have thus proved equality (2.6).

Equalities (2.7), (2.8) are obtained from (2.2), (2.3) by virtue of the fact that on the rectangle R each of the systems $(P_{nm}(\cos\theta')\cdot\cos m\phi)$ and $(P_{nm}(\cos\theta')\cdot\sin m\phi)$ is orthogonal to the system $(P_k(\cos\theta'))$ with respect to the measure $\sin\theta'\,d\theta' d\phi'$, i.e.,

$$\int_0^{\pi}\int_0^{2\pi} P_k(\cos\theta')P_{nm}(\cos\theta')\cos m\phi'\sin\theta'\,d\theta' d\phi' = 0, \tag{2.12}$$
$$1 \le m \le n, \quad k = 1, 2, \dots , \quad n = 1, 2, \dots ,$$

$$\int_0^{\pi}\int_0^{2\pi} P_k(\cos\theta')P_{nm}(\cos\theta')\sin m\phi'\sin\theta'\,d\theta' d\phi' = 0, \tag{2.13}$$
$$1 \le m \le n, \quad k = 1, 2, \dots , \quad n = 1, 2, \dots .$$

Equalities (2.7), (2.8) also hold for $k = 0$ but in that case they are weaker than equalities (2.4), (2.5).

2.2. Let us now prove

Theorem 2.1. *The functions P_r^*, $\widetilde{P}_r^*$, and $\widetilde{P}_r$ defined by equalities* (2.1)–(2.3) *are harmonic functions in the unit open ball $\mathbb{B}$.*

Proof. In the first place, inequalities (1.10) and (1.14) imply that the power series $\sum_{n=1}^{\infty} n^2 r^n$, converging for $r < 1$, is the majorant one for series (2.1)–(2.3) so that the functions P_r^*, $\widetilde{P}_r^*$, and $\widetilde{P}_r$ are continuous in $\mathbb{B}$.

To show that these functions are harmonic in $\mathbb{B}$ we shall use the same arguments as in proving Theorem 1.1 with the only difference that there the presence of the function $f \in L(R)$ was used to obtain adequate estimates. By analogy, we introduce the function

$$\gamma_n(\theta,\phi;\theta',\phi') = \sum_{m=1}^{n} 2(2n+1)\lambda_{nm}\frac{(n-m)!}{(n+m)!} \times$$
$$\times P_{nm}(\cos\theta)P_{nm}(\cos\theta')\cos m(\phi-\phi'), \tag{2.14}$$

for which by virtue of eastimates (1.10) and (1.14) we have the estimate

$$\left|\gamma_n(\theta,\phi';\theta',\phi')\right| \le 9n^2, \tag{2.15}$$

which repeats estimate (1.15) to within a constant multiplier.

If we introduce the function $w(r,\theta,\phi;\theta',\phi') = r^n\gamma_n(\theta,\phi';\theta',\phi')$, for w we shall obtain inequalities similar to (1.20)–(1.22).

Further, by inequality (1.23) we obtain the estimate

$$\Big|\frac{d}{d\theta}\,P_{nm}(\cos\theta)\Big| \le n\cdot \max_{0\le\theta\le\pi}|P_{nm}(\cos\theta)|, \tag{2.16}$$

which with (1.10) taken into account give the inequality

$$\Big|\frac{\partial}{\partial\theta}\Big(\frac{(n-m)!}{(n+m)!}\,P_{nm}(\cos\theta)P_{nm}(\cos\theta')\cos m(\phi-\phi')\Big)\Big| \le n. \tag{2.17}$$

Therefore

$$\Big|\frac{\partial}{\partial\theta}\,\gamma_n(\theta,\phi;\theta',\phi')\Big| \le 9n^3, \tag{2.18}$$

which is similar to estimate (1.31).

It is now clear how to complete the proof of the theorem. □

§ 3. Allied Integrals

Theorem 3.1. *For every function $f(\theta,\phi)\in L(R)$ the functions $\mathcal{U}^*_f(r,\theta,\phi)$, $\widetilde{\mathcal{U}}^*_f(r,\theta,\phi)$, and $\widetilde{\mathcal{U}}_f(r,\theta,\phi)$ defined by equalities* (1.1)–(1.4) *admit the following integral representations:*

$$\mathcal{U}^*_f(r,\theta,\phi) = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} f(\theta',\phi')P^*_r(\theta,\phi;\theta',\phi')\sin\theta'\,d\theta' d\phi', \tag{3.1}$$

$$\widetilde{\mathcal{U}}^*_f(r,\theta,\phi) = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} f(\theta',\phi')\widetilde{P}^*_r(\theta,\phi;\theta',\phi')\sin\theta'\,d\theta' d\phi', \tag{3.2}$$

$$\widetilde{\mathcal{U}}_f(r,\theta,\phi) = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} f(\theta',\phi')\widetilde{P}_r(\theta,\phi;\theta',\phi')\sin\theta'\,d\theta' d\phi'. \tag{3.3}$$

Proof. Taking into account equalities (0.7)–(0.8), from equality (2.1) we obtain

$$\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} f(\theta',\phi')P^*_r(\theta,\phi;\theta',\phi')\sin\theta'\,d\theta' d\phi' =$$

$$= \sum_{n=1}^{\infty}\lambda_{n0}r^nP_n(\cos\theta)\cdot\frac{2n+1}{4\pi}\int_0^{\pi}\int_0^{2\pi} f(\theta',\phi')P_n(\cos\theta')\sin\theta'\,d\theta' d\phi' +$$

$$+\sum_{n=1}^{\infty} r^n \sum_{m=1}^{n} \lambda_{nm} P_{nm}(\cos\theta)\Bigg[\cos m\phi \cdot \frac{2n+1}{2\pi}\cdot\frac{(n-m)!}{(n+m)!}\times$$

$$\times\int_0^{\pi}\int_0^{2\pi} f(\theta',\phi')P_{nm}(\cos\theta')\cos m\phi'\sin\theta'\,d\theta' d\phi' +$$

$$+\sin m\phi\cdot\frac{2n+1}{2\pi}\cdot\frac{(n-m)!}{(n+m)!}\times$$

$$\times\int_0^{\pi}\int_0^{2\pi} f(\theta',\phi')P_{nm}(\cos\theta')\sin m\phi'\sin\theta'\,d\theta' d\phi'\Bigg] = \mathcal{U}_f^*(r,\theta,\phi).$$

Here termwise integration is correct because of the uniform convergence of series (1.1) with respect to $(\theta,\phi)\in R$ for $r<1$ (see the proof of Theorem 1.1). $\square$

Equalities (3.2) and (3.3) are proved similarly.

Integrals (3.1), (3.3), and (3.2) will be called allied integrals with the integral $\mathcal{U}_f(r,\theta,\phi)$ with respect to θ, to ϕ, and to (θ,ϕ), respectively.

§ 4. Allied Functions

By Theorem 1.1, to each function $f\in L(R)$ there corresponds with the aid of the Poisson integral $\mathcal{U}_f$ a triple of the harmonic functions $\mathcal{U}_f^*$, $\widetilde{\mathcal{U}}_f^*$, and $\widetilde{\mathcal{U}}_f$ in the ball $\mathbb{B}$, which admits representations (1.1), (1.2), and (1.4) in the form of allied series, and representations (3.1)–(3.3) in the form of allied integrals.

We pose the natural question under what conditions these series will be Fourier–Laplace series and the integrals will be spherical Poisson integrals.

For the integrals (for the series see §6) the answer for the space $L^2(R)$ is stated as

Theorem 4.1. *For every function $f(\theta,\phi)\in L^2(R)$ there exist functions $f^*(\theta,\phi)\in L^2(R)$, $\widetilde{f}^*(\theta,\phi)\in L^2(R)$, and $\widetilde{f}(\theta,\phi)\in L^2(R)$ such that the equalities*

$$\mathcal{U}_f^*(r,\theta,\phi) = \mathcal{U}_{f^*}(r,\theta,\phi), \tag{4.1}$$

$$\widetilde{\mathcal{U}}_f^*(r,\theta,\phi) = \mathcal{U}_{\widetilde{f}^*}(r,\theta,\phi), \tag{4.2}$$

$$\widetilde{\mathcal{U}}_f(r,\theta,\phi) = \mathcal{U}_{\widetilde{f}}(r,\theta,\phi) \tag{4.3}$$

hold in the ball $\mathbb{B}$, *i.e.* (*see equalities* (3.1)–(3.3)),

$$\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} fP_r^* \sin\theta'\, d\theta' d\phi' = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} f^* P_r \sin\theta'\, d\theta' d\phi', \tag{4.4}$$

$$\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} f\widetilde{P}_r^* \sin\theta'\, d\theta' d\phi' = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} \widetilde{f}^* P_r \sin\theta'\, d\theta' d\phi', \tag{4.5}$$

$$\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} f\widetilde{P}_r \sin\theta'\, d\theta' d\phi' = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} \widetilde{f} P_r \sin\theta'\, d\theta' d\phi'. \tag{4.6}$$

Proof. For $f \in L^2(R)$ we have Parseval's equality (see, for instance, [16], p. 349)

$$\int_0^{\pi}\int_0^{2\pi} f^2(\theta',\phi')\sin\theta'\, d\theta' d\phi' = 4\pi a_{00}^2 + 4\pi\sum_{n=1}^{\infty}\frac{1}{2n+1}\, a_{n0}^2 +$$

$$+2\pi\sum_{n=1}^{\infty}\frac{1}{2n+1}\sum_{m=1}^{n}\frac{(n+m)!}{(n-m)!}\,(a_{nm}^2+b_{nm}^2). \tag{4.7}$$

Series (4.7) is therefore convergent. In that case the series obtained from (4.7) will converge if instead of (a_{nm}, b_{nm}) we successively put

$$(\lambda_{nm}a_{nm}, \lambda_{nm}b_{nm}), \quad (-\lambda_{nm}b_{nm}, \lambda_{nm}a_{nm}), \quad (-b_{nm}, a_{nm}). \tag{4.8}$$

Hence by the Riesz–Fischer theorem we obtain the existence of the functions $f^* \in L^2(R)$, $\widetilde{f}^* \in L^2(R)$ and $\widetilde{f} \in L^2(R)$ with the Fourier–Laplace coefficients (4.8). Series (1.1), (1.2), and (1.4) are therefore A-mean values of the Fourier–Laplace series $S[f^*]$, $S[\widetilde{f}^*]$ and $S[\widetilde{f}]$. This in turn means that the harmonic functions $\mathcal{U}_f^*(r,\theta,\phi)$, $\widetilde{\mathcal{U}}_f^*(r,\theta,\phi)$, and $\widetilde{\mathcal{U}}_f(r,\theta,\phi)$ are the spherical Poisson integrals for f^*, $\widetilde{f}^*$, and $\widetilde{f}$, respectively. This is equivalent to equalities (4.1)–(4.3), which completes the proof of the theorem. $\square$

The functions f^*, $\widetilde{f}$, and $\widetilde{f}^*$ from Theorem 4.1 will be called allied with f functions with respect to θ, to ϕ, and to (θ,ϕ).

§ 5. L^2-Inequalities for f^*, $\widetilde{f}^*$, and $\widetilde{f}$

For the operators transforming $f \in L^2(R)$ into $f^* \in L^2(R)$, $\widetilde{f}^* \in L^2(R)$, and $\widetilde{f} \in L^2(R)$ the following statement holds.

Theorem 5.1. *For every function $f \in L^2(R)$ we have the inequalities*

$$\int_0^\pi \int_0^{2\pi} (\widetilde{f^*})^2 \sin\theta'\, d\theta' d\phi' \le \int_0^\pi \int_0^{2\pi} (f^*)^2 \sin\theta'\, d\theta' d\phi', \tag{5.1}$$

$$\int_0^\pi \int_0^{2\pi} (f^*)^2 \sin\theta'\, d\theta' d\phi' \le \frac{9}{4} \int_0^\pi \int_0^{2\pi} f^2 \sin\theta'\, d\theta' d\phi', \tag{5.2}$$

$$\int_0^\pi \int_0^{2\pi} (\widetilde{f})^2 \sin\theta'\, d\theta' d\phi' \le \int_0^\pi \int_0^{2\pi} f^2 \sin\theta'\, d\theta' d\phi' \tag{5.3}$$

and the equalities

$$\int_0^\pi \int_0^{2\pi} (\widetilde{f^*})^2 \sin\theta'\, d\theta' d\phi' =$$

$$= \int_0^\pi \int_0^{2\pi} (f^*)^2 \sin\theta'\, d\theta' d\phi' - 4\pi \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}\, a_{n0}^2, \tag{5.4}$$

$$\int_0^\pi \int_0^{2\pi} (\widetilde{f})^2 \sin\theta'\, d\theta' d\phi' = 2\pi \sum_{n=1}^{\infty} \frac{1}{2n+1} \sum_{m=1}^{n} \frac{(n+m)!}{(n-m)!}\,(a_{nm}^2 + b_{nm}^2), \tag{5.5}$$

$$\int_0^\pi \int_0^{2\pi} (\widetilde{f})^2 \sin\theta'\, d\theta' d\phi' =$$

$$= \int_0^\pi \int_0^{2\pi} f^2 \sin\theta'\, d\theta' d\phi' - 4\pi a_{00}^2 - 4\pi \sum_{n=1}^{\infty} \frac{1}{2n+1}\, a_{n0}^2. \tag{5.6}$$

Proof. Using the same reasoning as for the proof of Theorem 4.1, we obtain the equalities

$$\int_0^\pi \int_0^{2\pi} (\widetilde{f^*})^2 \sin\theta'\, d\theta' d\phi' =$$

$$= 2\pi \sum_{n=1}^{\infty} \frac{1}{2n+1} \sum_{m=1}^{n} \frac{(n+m)!}{(n-m)!}\, \lambda_{nm}^2 (a_{nm}^2 + b_{nm}^2), \tag{5.7}$$

$$\int_0^\pi \int_0^{2\pi} (f^*)^2 \sin\theta'\, d\theta' d\phi' = 4\pi \sum_{n=1}^{\infty} \frac{1}{2n+1}\, \lambda_{n0}^2 a_{n0}^2 +$$

$$+2\pi\sum_{n=1}^{\infty}\frac{1}{2n+1}\sum_{m=1}^{n}\frac{(n+m)!}{(n-m)!}\,\lambda_{nm}^2(a_{nm}^2+b_{nm}^2), \tag{5.8}$$

$$\int_0^{\pi}\int_0^{2\pi}(\widetilde{f}^*)^2\sin\theta'\,d\theta' d\phi' =$$

$$=\int_0^{\pi}\int_0^{2\pi}(f^*)^2\sin\theta'\,d\theta' d\phi' - 4\pi\sum_{n=1}^{\infty}\frac{1}{2n+1}\,\lambda_{n0}^2 a_{n0}^2. \tag{5.9}$$

Equalities (5.7) and (5.8) imply (5.1) and (5.2) with (1.14) and (4.7) taken into account. Equality (5.4) is obtained from (5.9) by virtue of the fact that $\lambda_{n0}=\frac{2n+1}{n(n+1)}$. Furthermore, since $(-b_{nm},a_{nm})$ are the Fourier–Laplace coefficients for the function $\widetilde{f}(\theta,\phi)$, we obtain (5.5) and (5.6) from Parseval's equality. Equality (5.3) follows in turn from (5.6). □

§ 6. Allied Series

The technique used to make series (1.1), (1.2), and (1.4) correspond to series (0.5) enables one to write the allied series S^*, $\widetilde{S}^*$, and $\widetilde{S}$ for any Laplace series S obtained formally from series (1.5) by substituting $r=1$. In particular, the series

$$\begin{aligned}S^*[f] &= \sum_{n=1}^{\infty}\Big(\frac{1}{n}+\frac{1}{n+1}\Big)a_{n0}P_n(\cos\theta)+\\ &+\sum_{n=1}^{\infty}\sum_{m=1}^{n}\Big(\frac{1}{n+m}+\frac{1}{n-m+1}\Big)\times\\ &\times\big(a_{nm}\cos m\phi+b_{nm}\sin m\phi\big)P_{nm}(\cos\theta),\end{aligned} \tag{6.1}$$

$$\begin{aligned}\widetilde{S}^*[f] &= \sum_{n=1}^{\infty}\sum_{m=1}^{n}\Big(\frac{1}{n+m}+\frac{1}{n-m+1}\Big)\times\\ &\times\big(a_{nm}\sin m\phi-b_{nm}\cos m\phi\big)P_{nm}(\cos\theta),\end{aligned} \tag{6.2}$$

$$\widetilde{S}[f]=\sum_{n=1}^{\infty}\sum_{m=1}^{n}\big(a_{nm}\sin m\phi-b_{nm}\cos m\phi\big)P_{nm}(\cos\theta) \tag{6.3}$$

are the allied Laplace series to the Fourier–Laplace series (0.6) with respect to θ, to (θ,ϕ), and to ϕ, respectively.

The proof of Theorem 4.1 gives rise to

Theorem 6.1. *For every function $f\in L^2(R)$ there exist functions $f^*\in L^2(R)$, $\widetilde{f}^*\in L^2(R)$, and $\widetilde{f}\in L^2(R)$ for which the series $S^*[f]$, $\widetilde{S}^*[f]$ and $\widetilde{S}[f]$ are the Fourier–Laplace series, i.e., $S^*[f]=S[f^*]$, $\widetilde{S}^*[f]=S[\widetilde{f}^*]$, $\widetilde{S}[f]=S[\widetilde{f}]$.*

Remark 6.1. Using the numbers λ_{nm} defined by equality (1.3), we introduce the polynomials

$$P_n^*(x) = \Big(\frac{1}{n} + \frac{1}{n+1}\Big) P_n(x) \quad (n = 1, 2, \dots) \tag{6.4}$$

and

$$P_{nm}^* = \Big(\frac{1}{n+m} + \frac{1}{n-m+1}\Big) P_{nm}(x) \tag{6.5}$$
$$(1 \le m \le n, \quad n = 1, 2, \dots),$$

which will be called allied Legendre polynomials and allied Legendre functions, respectively.

Then equalities (1.1), (1.2), (6.1) and (6.2) can be rewritten as

$$\mathcal{U}_f^*(r,\theta,\phi) = \sum_{n=1}^{\infty} a_{n0} r^n P_n^*(\cos\theta) +$$
$$+ \sum_{n=1}^{\infty} r^n \sum_{m=1}^{n} \big(a_{nm}\cos m\phi + b_{nm}\sin m\phi\big) P_{nm}^*(\cos\theta), \tag{6.6}$$

$$\widetilde{\mathcal{U}}_f^*(r,\theta,\phi) = \sum_{n=1}^{\infty} r^n \sum_{m=1}^{n} \big(a_{nm}\sin m\phi - b_{nm}\cos m\phi\big) P_{nm}^*(\cos\theta), \tag{6.7}$$

$$S^*[f] = \sum_{n=1}^{\infty} a_{n0} P_n^*(\cos\theta) +$$
$$+ \sum_{n=1}^{\infty} \sum_{m=1}^{n} \big(a_{nm}\cos m\phi + b_{nm}\sin m\phi\big) P_{nm}^*(\cos\theta), \tag{6.8}$$

$$\widetilde{S}^*[f] = \sum_{n=1}^{\infty} \sum_{m=1}^{n} \big(a_{nm}\sin m\phi - b_{nm}\cos m\phi\big) P_{nm}^*(\cos\theta). \tag{6.9}$$

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Author's address:
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 380093
Georgia