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# ON THE BOUNDARY VALUE PROBLEM IN A DIHEDRAL ANGLE FOR NORMALLY HYPERBOLIC SYSTEMS OF FIRST ORDER

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ABSTRACT. Some structural properties as well as a general three-dimensional boundary value problem for normally hyperbolic systems of partial differential equations of first order are studied. A condition is given which enables one to reduce the system under consideration to a first-order system with the spliced principal part. It is shown that the initial problem is correct in a certain class of functions if some conditions are fulfilled.

## § 1. Some Structural Properties of Normally Hyperbolic Systems of First Order

In the Euclidean space $\mathbb{R}^{n+1}$, $n \geq 2$, of independent variables $(x,t)$, $x=(x_1,\dots,x_n)$, we consider the system of partial differential equations of first order

$$A_0 u_t + \sum_{i=1}^{n} A_i u_{x_i} + Bu = F, \tag{1.1}$$

where $A_i$, $i=0,1,\dots,n$, $B$ are the given real $m \times m$ matrix-functions, $m \geq 2$, $F$ is the given and $u$ is the unknown $m$-dimensional real vector-function. It is assumed that $\det A_0 \neq 0$.

Denote by $p(x,t;\lambda,\xi)$ the characteristic determinant of system (1.1), i.e., $p(x,t;\lambda,\xi) \equiv \det Q(x,t;\lambda,\xi)$, where

$$Q(x,t;\lambda,\xi) \equiv A_0\lambda + \sum_{i=1}^{n} A_i\xi_i, \quad \lambda \in \mathbb{R}, \quad \xi=(\xi_1,\dots,\xi_n) \in \mathbb{R}^n.$$

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Since $\det A_0 \neq 0$, we have the representation

$$p(x,t;\lambda,\xi) = \det A_0 \prod_{i=1}^{l} \big(\lambda - \lambda_i(x,t;\xi)\big)^{k_i}, \quad \sum_{i=1}^{l} k_i = m,$$
$$l = l(x,t;\xi), \quad k_i = k_i(x,t;\xi), \quad i = 1,\dots,l.$$

System (1.1) is said to be hyperbolic at the point $(x,t)$ if all roots $\lambda_1(x,t;\xi),\dots,\lambda_l(x,t;\xi)$ of the polynomial $p(x,t;\lambda,\xi)$ are real numbers.

One can easily verify that

$$k_i(x,t;\xi) \geq m - \operatorname{rank} Q\big(x,t;\lambda_i(x,t;\xi),\xi\big), \quad i = 1,\dots,l.$$

The hyperbolic system (1.1) is said be normally hyperbolic at the point $(x,t)$, if the equalities

$$k_i(x,t;\xi) = m - \operatorname{rank} Q\big(x,t;\lambda_i(x,t;\xi),\xi\big), \quad i = 1,\dots,l,$$

are fulfilled (see, e.g., [1], [2]).

Note that strictly hyperbolic systems, i.e., when $l = m$, $k_i = 1$, $i = 1,\dots,m$, form a subclass of normally hyperbolic systems.

Since $\det A_0 \neq 0$, it can be assumed without loss of generality that $A_0 = E$, where $E$ is the $m\times m$ unit matrix. For simplicity, we shall always assume that (i) $n = 2$, $x_1 = x$, $x_2 = y$; (ii) the matrices $A_1$ and $A_2$ are constant; (iii) system (1.1) is normally hyperbolic.

In our assumptions, in the space of independent variables $x$, $y$ and $t$, system (1.1) is rewritten as

$$u_t + A_1 u_x + A_2 u_y + Bu = F. \tag{1.2}$$

It is easy to show that since system (1.2) is normally hyperbolic, each of the matrices $A_i$, $i = 1,2$, has only real characteristic roots so that the corresponding eigenvectors of the operator $A_i$, $1 \leq i \leq 2$, form a complete system, i.e., a basis in the space $\mathbb{R}^m$. Therefore the matrices $A_i$, $i = 1,2$, are diagonalizable, i.e., there exist real nondegenerate matrices $C_i$, $i = 1,2$, such that the matrices $C_i^{-1}A_iC_i$, $i = 1,2$, are diagonal.

The normally hyperbolic system (1.2) will be said to be diagonalizable if there exists a real nondegenerate matrix $C$ such that the matrices $C^{-1}A_iC$, $i = 1,2$, are diagonal. We have

**Lemma 1.1.** *The normally hyperbolic system* (1.2) *is diagonalizable if and only if the matrices $A_1$ and $A_2$ are commutative, i.e., $A_1A_2 = A_2A_1$.*

*Proof.* The necessity readily follows from the fact that the diagonal matrices $C^{-1}A_1C$ and $C^{-1}A_2C$ are commutative, $C^{-1}A_1CC^{-1}A_2C = C^{-1}A_2CC^{-1}A_1C$, i.e., $C^{-1}A_1A_2C = C^{-1}A_2A_1C$. This immediately implies $A_1A_2 = A_2A_1$.

To prove sufficiency note that since system (1.2) is normally hyperbolic, we have $\dim \mathbb{R}_{\lambda_i} = k_i$, where $\mathbb{R}_{\lambda_i} \equiv \mathrm{Ker}\ (A_1 - \lambda_i E)$, $1 \le i \le l$. Clearly, $\hat{A}_1(\mathbb{R}_{\lambda_i}) \subset \mathbb{R}_{\lambda_i}$, $1 \le i \le l$, where $\hat{A}_1$ stands for the linear transform corresponding to the matrix $A_1$.

Let $\{\nu_{ij}\}_{j=1}^{k_i}$ be an arbitrary basis of the space $\mathbb{R}_{\lambda_i}$, $1 \le i \le l$. By the definition of the space $\mathbb{R}_{\lambda_i}$, the vectors $\nu_{i1}, \dots, \nu_{ik_i}$ are the eigenvectors for the transform $\hat{A}_1$ and correspond to the eigenvalue $\lambda_i$, $1 \le i \le l$. Therefore, the matrix of the transform $\hat{A}_1$ in the basis $\{\nu_{ij}\}_{j=1}^{k_i}$ of the space $\mathbb{R}_{\lambda_i}$ will be diagonal of order $(k_i \times k_i)$ and written as diag $[\underbrace{\lambda_i, \dots, \lambda_i}_{k_i-\text{times}}]$, $1 \le i \le l$. Hence, recalling that the decomposition of the space $\mathbb{R}^m$ as the direct sum of subspaces $\mathbb{R}_{\lambda_i}$, $i = 1, \dots, l$, i.e., $\mathbb{R}^m = \mathbb{R}_{\lambda_1} \oplus \cdots \oplus \mathbb{R}_{\lambda_l}$ is unique, we can write the matrix $D_1$ of the transform $\hat{A}_1$ in the basis $\{\nu_{ij};\ i = 1, \dots, l;\ j = 1, \dots, k_i\}$ as $D_1 = \mathrm{diag}\,[\underbrace{\lambda_1, \dots, \lambda_1}_{k_1-\text{times}}, \dots, \underbrace{\lambda_l, \dots, \lambda_l}_{k_l-\text{times}}]$.

Let $\widetilde{A_{2i}}$ be the matrix corresponding to the linear transform $\hat{A}_2$ of the subspace $\mathbb{R}_{\lambda_i}$, $1 \le i \le l$ in the basis $\{\nu_{ij}\}_{j=1}^{k_i}$. Since the matrices $A_1$ and $A_2$ are commutative, the subspace $\mathbb{R}_{\lambda_i}$ is invariant with respect to the linear transform $\hat{A}_2$, i.e., $\hat{A}_2(\mathbb{R}_{\lambda_i}) \subset \mathbb{R}_{\lambda_i}$, $1 \le i \le l$ (see, e.g., [3]). Therefore, in the basis $\{\nu_{ij};\ i = 1, \dots, l;\ j = 1, \dots, k_i\}$ of the space $\mathbb{R}^m$, the matrix $\widetilde{A_2}$ corresponding to $\hat{A}_2$ will be block-diagonal and have, on its principal diagonal, matrices $\widetilde{A_{2i}}$, $i = 1, \dots, l$. It is well known that matrices giving the same linear transform in different bases are similar. At the same time, similar matrices have the same characteristic equation. Therefore we have

$$\det(A_2 - \lambda E) = \det(\widetilde{A_2} - \lambda E) = \det(\widetilde{A_{21}} - \lambda E_{k1}) \times \cdots \times \det(\widetilde{A_{2l}} - \lambda E_{kl}).$$

Since system (1.2) is normally hyperbolic, the matrix $A_2$ has only real characteristic roots. Thus for the linear transform $\hat{A}_2 : \mathbb{R}_{\lambda_i} \to \mathbb{R}_{\lambda_i}$ there exists a basis $\{\mu_{ij}\}_{j=1}^{k_i}$ which consists of the real vectors of the subspace $\mathbb{R}_{\lambda_i}$, and where the matrix $A_{2i}^*$ of the above-mentioned transform is of Jordan form, $1 \le i \le l$. Therefore, in the basis $\{\mu_{ij};\ i = 1, \dots, l;\ j = 1, \dots, k_i\}$ of the space $\mathbb{R}^m$, the matrix $A_2^*$ of the transform $\hat{A}_2$ will also be of Jordan form. But, since system (1.2) is normally hyperbolic, in the space $\mathbb{R}^m$ there exists a basis $\{\sigma_i\}_{i=1}^m$ in which the matrix of the transform $\hat{A}_2$ is diagonal. Further, as is well known, a Jordan matrix similar to the diagonal one is diagonal too. Therefore in the basis $\{\mu_{ij};\ i = 1, \dots, l;\ j = 1, \dots, k_i\}$ of the space $\mathbb{R}^m$ the matrix $A_2^*$ is diagonal, but the matrix of the transform $\hat{A}_1$ is diagonal in any basis of the transform $\mathbb{R}_{\lambda_i}$, in particular, in the basis $\{\mu_{ij}\}_{j=1}^{k_i}$, $1 \le i \le l$. Therefore the matrices of the transforms $\hat{A}_1$ and $\hat{A}_2$ will be diagonal in the basis $\{\mu_{ij};\ i = 1, \dots, l;\ j = 1, \dots, k_i\}$ of the space $\mathbb{R}^m$. $\square$

## § 2. Statement of the Problem and Some Notations

In the discussion below the matrices $A_1$ and $A_2$ will always be assumed to be commutative, i.e., the equality

$$A_1A_2 = A_2A_1 \tag{2.1}$$

is valid.

After introducing a new unknown function $v$ by the formula $u = Cv$ with the nondegenerate matrix $C$ whose existence was proved by condition (2.1) in §1, system (1.2) takes the form

$$v_t + D_1v_x + D_2v_y + B_0v = F_0, \tag{2.2}$$

where by virtue of Lemma 1.1 the matrices $D_i = C^{-1}A_iC$, $i = 1,2$, are diagonal, i.e., $D_1 = \operatorname{diag}[\nu_1,\dots,\nu_m]$, $D_2 = \operatorname{diag}[\mu_1,\dots,\mu_m]$, $B_0 = C^{-1}BC$, $F_0 = C^{-1}F$.

It is obvious that the directions defined by the vectors $l_i = (\nu_i,\mu_i,1)$, $i = 1,\dots,m$, are bicharacteristic.

Let $\hat{A}_j$ be the linear transform corresponding to he matrix $A_j$, $1 \le j \le 2$. Denote by $\Lambda_i$ an $m$-dimensional vector which is the eigenvalue of the transform $\hat{A}_1$, corresponding to the eigenvalue $\nu_i$, $1 \le i \le m$. By virtue of (2.1) the vector $\Lambda_i$ is also the eigenvector of the transform $\hat{A}_2$ corresponding to the eigenvalue $\mu_i$, $1 \le i \le m$. By Lemma 1.1, the vectors $\Lambda_i$, $i = 1,\dots,m$, can be chosen such that the $(m \times m)$ matrix $C = [\Lambda_1,\dots,\Lambda_m]$, whose columns consist of these vectors, will reduce the matrices $A_1$ and $A_2$ to the diagonal form, namely, to $D_i = C^{-1}A_iC$, $i = 1,2$.

Obviously, the vectors $l_i = (\nu_i,\mu_i,1)$ and $l_j = (\nu_j,\mu_j,1)$ define the same bicharacteristic direction if the equalities $\nu_i = \nu_j$, $\mu_i = \mu_j$, $1 \le i \ne j \le m$, are fulfilled. In this context, we divide the set of vectors $\{l_1,\dots,l_m\}$ into nonintersecting classes $\{l_{11},\dots,l_{1s_1}\},\dots,\{l_{m_01},\dots,l_{m_0s_{m_0}}\}$ whose representatives with respective "multiplicities" $s_1,\dots,s_{m_0}$, will be denoted by $\widetilde{l}_1,\dots,\widetilde{l}_{m_0}$, $m_0 \le m$. Now the matrix $C = [\Lambda_1,\dots,\Lambda_m]$ can be represented as

$$C = [\Lambda_{11},\dots,\Lambda_{1s_1};\dots;\Lambda_{m_01},\dots,\Lambda_{m_0s_{m_0}}],$$

or as $C = (\widetilde{C_1},\widetilde{C_2})$, where

$$\begin{aligned}
\widetilde{C_1} &= [\Lambda_{11},\dots,\Lambda_{1s_1};\dots;\Lambda_{q1},\dots,\Lambda_{qs_q}],\\
\widetilde{C_2} &= [\Lambda_{q+11},\dots,\Lambda_{q+1s_{q+1}};\dots;\Lambda_{m_01},\dots,\Lambda_{m_0s_{m_0}}]
\end{aligned}$$

and $q$ will be defined below.

Denote by $D^*$ the dihedral angle

$$D^* \equiv \{(x,y,t) \in \mathbb{R}^3,\ t-y>0,\ t+y>0\}.$$

For bicharacteristic directions of system (1.2) we make the following assumption: bicharacteristics passing through any point of the edge $\Gamma^* \equiv \{(x,y,t) \in \mathbb{R}^3 : y = t = 0, x \in \mathbb{R}\}$ of the angle $D^*$ have no common points with the set $\overline{D^*} \backslash \Gamma^*$. This is equivalent to the fulfillment of the inequalities

$$|\mu_i| > 1, \quad i = 1, \dots, m. \tag{2.3}$$

Let $P_0 = P_0(x_0, y_0, t_0)$ be an arbitrary fixed point of the set $\overline{D^*} \backslash \Gamma^*$, and let $S_1 \supset \Gamma^*$ and $S_2 \supset \Gamma^*$ be the two-dimensional edges of $D^*$, i.e., $\partial D^* = S_1 \cup S_2$, $S_1 \equiv \{(x,y,t) \in \mathbb{R}^3 : x \in \mathbb{R},\ y = t,\ t \in \overline{\mathbb{R}}_+\}$, $S_2 \equiv \{(x,y,t) \in \mathbb{R}^3 : x \in \mathbb{R},\ y = -t,\ t \in \overline{\mathbb{R}}_+\}$, $\mathbb{R}_+ \equiv (0, \infty)$. From the point $P_0$ we draw the bicharacteristic beam $\widetilde{L_i}(P_0)$ of system (2.2) which corresponds to the vector $\widetilde{l}_i$, is directed towards the decreasing values of the $t$-coordinate of a moving point $\widetilde{L_i}(P_0)$, and intersects one of the edges $S_1$ or $S_2$ at a point $\widetilde{P_0^i}$, $1 \le i \le m_0$. It can be assumed without loss of generality that bicharacteristic beams defined by the vectors $\widetilde{l}_1, \dots, \widetilde{l}_q$ and passing through the point $P_0$ intersect the edge $S_1$, while those defined by $\widetilde{l}_{q+1}, \dots, \widetilde{l}_{m_0}$ intersect the edge $S_2$.

Below, it will be assumed for simplicity that $q = 2$ and $m_0 = 3$, $\widetilde{l}_i = (\widetilde{\nu}_i, \widetilde{\mu}_i, 1)$, $i = 1, 2, 3$, and also rank $(\widetilde{l}_1, \widetilde{l}_2, \widetilde{l}_3) = 3$.

Through the point $P_0$ draw a plane $P_{\widetilde{l}_i, \widetilde{l}_j}$, parallel to the vectors $\widetilde{l}_i$ and $\widetilde{l}_j$, $1 \le i < j \le 3$. We introduce the following notation:

$P_{\widetilde{l}_2, \widetilde{l}_3}$ and $P_{\widetilde{l}_1, \widetilde{l}_3}$ are respectively the intersection points of the planes $Q_1$ and $Q_2$ with the edge $\Gamma^*$;

$D$ is a the domain forming a pentahedron with the vertices at the points $P_0, \widetilde{P_0^2}, \widetilde{P_0^1}, Q_2, \widetilde{P_0^3}, Q_1$;

$\Delta_1$ and $\Delta_2$ are respectively a triangle and rectangle with the vertices at the points $Q_2, \widetilde{P_0^3}, Q_1$ and $Q_1, \widetilde{P_0^2}, \widetilde{P_0^1}, Q_2$, respectively.

For system (1.2) we consider the boundary value problem formulated as follows: Find, in the domain $D$, a regular solution $u(x,y,t)$ of system (1.2) satisfying the boundary conditions

$$B_i u \big|_{\overline{\Delta_i}} = f_i, \tag{2.4}$$

where $B_i$ are the given $(m_i \times m)$ matrix-functions and $f_i$ are the given $m_i$-dimensional vector-functions, $i = 1, 2, m_1 = s_1 + \dots + s_q, m_2 = s_{q+1} + \dots + s_{m_0}$. It is obvious that $m_1 + m_2 = m$ though we do not exclude the cases with $m_1 = 0$ or $m_2 = 0$, which correspond to the Cauchy problem. Below it will always be assumed that $0 < m_i < m$, $i = 1, 2$.

A function $u(x,y,t)$ which, together with its partial derivatives $u_x, u_y, u_t$, is continuous in $D$ and satisfies system (1.2) is called a regular solution of system (1.2).

Some analogs of the Goursat problem for hyperbolic systems of first order with two independent variables have been studied in [4]–[8]. A lot of papers are devoted to general boundary value problems of the Darboux type for normally hyperbolic systems of second order on a plane (see, e.g., [1], [2]). Some multidimensional problems of the Goursat and Darboux type are considered in several papers (see, e.g., [9]–[11]) both for a hyperbolic equation and for a system of equations in a dihedral angle. For hyperbolic equations of third order, a boundary value problem in a dihedral angle is investigated in [12].

Denote by $\Delta_i^*$ the orthogonal projection of the polygons $\underline{\Delta_i^*}$, $i=1,2$, onto the plane $x,t$. The restrictions of $B_i$ and $f_i$ on the sets $\overline{\Delta_i^*}$, $i=1,2$, will be denoted as before.

In the domains $D$ and $\Delta_i^*$, we introduce the following functional spaces

$$\overset{0}{C}_\alpha(\overline{D}) \equiv \Big\{ w \in C(\overline{D}) : w|_\Gamma = 0, \sup_{(x,y,t)\in\overline{D}\setminus\Gamma^*} \rho^{-\alpha}||w(x,y,t)||_{\mathbb{R}^m} < \infty \Big\},$$

$$\overset{0}{C}_\alpha(\overline{\Delta_i^*}) \equiv \Big\{ \psi \in C(\overline{\Delta_i^*}) : \psi|_{\Gamma_1} = 0, \sup_{(x,t)\in\overline{\Delta_i^*}\setminus\Gamma_1^*} t^{-\alpha}||\psi(x,t)||_{\mathbb{R}^m} < \infty \Big\},$$

where $\Gamma \equiv \overline{D}\cap\Gamma^*, \Gamma_1 \equiv \overline{\Delta_i^*}\cap\Gamma_1^*, i=1,2, \Gamma_1^* \equiv \{(x,t)\in\mathbb{R}^2 : x\in\mathbb{R}, t=0\}$, $\rho$ is the distance from the point $(x,y,t)\in\overline{D}\setminus\Gamma^*$ to the edge $\Gamma^*$ of the domain $D^*$, the real parameter $\alpha = const \geq 0$. For $a=(a_1,\dots,a_m)\in\mathbb{R}^m$, $m\geq 2$, denote $\|a\|_{\mathbb{R}_m} = |a_1|+\cdots+|a_m|$.

Obviously, the spaces $\overset{0}{C}_\alpha(\overline{D})$ and $\overset{0}{C}_\alpha(\overline{\Delta_i^*})$, $i=1,2$, are Banach ones with the norms

$$||w||_{\overset{0}{C}_\alpha(\overline{D})} = \sup_{(x,y,t)\in\overline{D}\setminus\Gamma^*} \rho^{-\alpha}||w(x,y,t)||_{\mathbb{R}^m},$$

$$||\psi||_{\overset{0}{C}_\alpha(\overline{\Delta_i^*})} = \sup_{(x,t)\in\overline{\Delta_i^*}\setminus\Gamma_1^*} t^{-\alpha}||\psi(x,t)||_{\mathbb{R}^m}.$$

*Remark* 2.1. Since the estimate $1\leq \rho/t \leq \sqrt{2}$, $(x,y,t)\in D^*$, is uniform, the value $\rho$ in the definition of the space $\overset{0}{C}_\alpha(\overline{D})$ below will be replaced by the variable $t$.

It is easy to verify that the fact that $w\in\overset{0}{C}(\overline{D})$ and $\psi\in\overset{0}{C}(\overline{\Delta_i^*})$ belong to the spaces $\overset{0}{C}_\alpha(\overline{D})$ and $\overset{0}{C}_\alpha(\overline{\Delta_i^*})$, respectively, is equivalent to the fulfillment of the inequalities

$$\begin{aligned} &||w(x,y,t)||_{\mathbb{R}^m} \leq ct^\alpha, \ \ (x,y,t)\in\overline{D}, \\ &||\psi(x,t)||_{\mathbb{R}^m} \leq ct^\alpha, (x,t)\in\overline{\Delta_i^*}, \ \ i=1,2. \end{aligned} \tag{2.5}$$

We shall investigate the boundary value problem (1.2), (2.4) in the Banach space

$$\overset{0}{C}{}^{1,1,1}_{\alpha}(\overline{D}) \equiv \Big\{u : \frac{\partial^{|i|}u}{\partial x^{i_1}\partial y^{i_2}\partial t^{i_3}} \in \overset{0}{C}_{\alpha}(\overline{D}),\ |i| \leq 1,\ |i| = \sum_{j=1}^{3} i_j\Big\},$$

with respect to the norm

$$\|u\|_{\overset{0}{C}{}^{1,1,1}_{\alpha}(\overline{D})} = \sum_{|i|\leq 1} \Big\| \frac{\partial^{|i|}u}{\partial x^{i_1}\partial y^{i_2}\partial t^{i_3}} \Big\|_{\overset{0}{C}_{\alpha}(\overline{D})}$$

assuming that the matrix-functions $B \in C(\overline{D})$, $B_i \in C(\overline{\Delta_i^*})$ and the vector-functions $F \in \overset{0}{C}_{\alpha}(\overline{D})$, $f_i \in \overset{0}{C}_{\alpha}(\overline{\Delta_i^*})$, $i = 1, 2$.

## § 3. Equivalent Reduction of Problem (1.2), (2.4) to a System of Integro-Differential Equations

From an arbitrary point $P(x, y, t) \in \overline{D}\backslash\Gamma$ we draw the bicharacteristic beam $\widetilde{L_i}(P)$ of system (2.2) which corresponds to the vector $\widetilde{l_i}$ and is directed towards the decreasing values of the $t$-coordinate of a moving point of $\widetilde{L_i}(P)$, $1 \leq i \leq 3$. The points of intersection of beams $\widetilde{L_i}(P)$, $i = 1, 2, 3$, with the faces $S_1$ and $S_2$ are $\widetilde{P^i} \in S_1$, $i = 1, 2$, and $\widetilde{P^3} \in S_2$. Denote by $\big(\omega_i^1(x, y, t), \omega_i^2(x, y, t)\big)$ the coordinates of orthogonal projection of the point $\widetilde{P^i}$ onto the plane $(x, t)$, $1 \leq i \leq 3$. A simple calculation yields

$$\begin{aligned} &\left.\begin{aligned}\omega_i^1(x, y, t) &= x + \widetilde{\nu}_i(1 - \widetilde{\mu}_i)^{-1}(y - t),\\ \omega_i^2(x, y, t) &= t + (1 - \widetilde{\mu}_i)^{-1}(y - t),\end{aligned}\right. \qquad i = 1, 2,\\ &\omega_3^1(x, y, t) = x - \widetilde{\nu}_3(1 + \widetilde{\mu}_3)^{-1}(y + t),\ \ \omega_3^2(x, y, t) = t - (1 + \widetilde{\mu}_3)^{-1}(y + t). \end{aligned}$$

Let $\xi = x_i(x, y, t; \tau)$, $\eta = y_i(x, y, t; \tau)$, $\zeta = \tau$ be the parametrization of a segment $\widetilde{L_i}(P) \cap \overline{D}$, where $\omega_i^2(x, y, t) \leq \tau \leq t$, $1 \leq i \leq 3$.

After integrating the $(q_i + j)$-th equation of system (2.2), where $q_1 = 0$, $q_i = s_1 + \cdots + s_{i-1}$, $i \geq 2$, $j = 1, \dots, s_i$, along the $i$-th bicharacteristic $\widetilde{L_i}(P)$ drawn from an arbitrary point $P(x, y, t) \in \overline{D}\backslash\Gamma$ and lying between the point $P(x, y, t)$ and the point of intersection of $\widetilde{L_i}(P)$ with the face $S_1$ or $S_2$ (depending on the index $i$ of $\widetilde{L_i}(P)$), we obtain

$$\begin{aligned} v_{q_i+j}(x, y, t) &= v_{q_i+j}\big(\omega_i^1(x, y, t), \omega_i^2(x, y, t)\big) + \\ &+ \int\limits_{\omega_i^2(x,y,t)}^{t} \Big(\sum_{p'=1}^{m} b_{ijp'}v_{p'}\Big)\big(x_i(x, y, t; \tau), y_i(x, y, t; \tau), \tau\big)d\tau + \\ &+ F_{ij}(x, y, t),\ \ 1 \leq i \leq 3,\ \ 1 \leq j \leq s_i, \end{aligned} \tag{3.1}$$

where $v_{q_i+j}$ are the components of the vector $v$, $b_{ijp'}$ and $F_{ij}$ are the well-defined functions depending only on the coefficients and the right-hand side of system (2.2).

We set

$$\begin{aligned}&\varphi^1_{q_i+j}(x,t)\equiv v_{q_i+j}\big|_{\overline{\Delta_1}}\equiv v_{q_i+j}(x,t,t),\ (x,t)\in\overline{\Delta_1^*}, i=1,2; j=1,\dots,s_i,\\&\varphi^2_{q_i+j}(x,t)\equiv v_{q_i+j}\big|_{\overline{\Delta_2}}\equiv v_{q_i+j}(x,-t,t),\ (x,t)\in\overline{\Delta_2^*}, i=3; j=1,\dots,s_i.\end{aligned}\tag{3.2}$$

It is obvious that the number of components of the vectors

$$\begin{aligned}&\varphi^1(x,t)\equiv\big(\varphi^1_{q_i+j}(x,t)\big),\ \ (x,t)\in\overline{\Delta_1^*},\ i=1,2;\ j=1,\dots,s_i,\\&\varphi^2(x,t)\equiv\big(\varphi^2_{q_i+j}(x,t)\big),\ \ (x,t)\in\overline{\Delta_2^*},\ i=3;\ j=1,\dots,s_i,\end{aligned}$$

is equal to the numbers $m_1$ and $m_2$, respectively.

By substituting the expressions of $v$ from equality (3.1) into the boundary conditions (2.4) and taking into account (3.2) we have

$$\begin{aligned}&Q^1_0(x,t)\varphi^1(x,t)+Q^1_3(x,t)\varphi^2\big(\sigma_3(x,t)\big)+(T_1v)(x,t)=f^1(x,t),\ (x,t)\in\overline{\Delta_1^*},\\&Q^2_0(x,t)\varphi^2(x,t)+\sum_{i=1}^{2}Q^2_i(x,t)\varphi^1\big(\sigma_i(x,t)\big)+(T_2v)(x,t)=\\&\qquad=f^2(x,t),\ \ (x,t)\in\overline{\Delta_2^*},\end{aligned}\tag{3.3}$$

where

$$\begin{aligned}&\big(T_1v\big)(x,t)\equiv\int\limits_{\omega^2_3(x,t,t)}^{t}\big(\widetilde{A_3}v\big)\big(x_3(x,t,t;\tau),y_3(x,t,t;\tau),\tau\big)\,d\tau,\\&\big(T_2v\big)(x,t)\equiv\sum_{i=1}^{2}\int\limits_{\omega^2_i(x,-t,t)}^{t}\big(\widetilde{A_i}v\big)\big(x_i(x,-t,t;\tau),y_i(x,-t,t;\tau),\tau\big)\,d\tau,\end{aligned}\tag{3.4}$$

and $Q^1_3$, $\widetilde{A_3}$, $Q^2_i$, $\widetilde{A_i}$, $f^i$, $i=1,2$, are respectively the well-defined matrices and vectors.

It is obvious that $Q^i_0$ from (3.3) are matrices of order $(m_i\times m_i)$ which can be represented as the product

$$Q^i_0=B_i\times\widetilde{C_i},\quad i=1,2,\tag{3.5}$$

and the functions $\sigma_i$ are defined by the equalities

$$\begin{aligned}&\sigma_i:(x,t)\to\big(\omega^1_i(x,-t,t),\omega^2_i(x,-t,t)\big),\ i=1,2,\\&\sigma_i:(x,t)\to\big(\omega^1_i(x,t,t),\omega^2_i(x,t,t)\big),\ i=3.\end{aligned}$$

Assuming that

$$\det Q_0^i\big|_{\overline{\Delta_i^*}} \neq 0, \quad i=1,2, \tag{3.6}$$

where the matrices $Q_0^i$ are given by (3.5), we rewrite system (3.3) as

$$\begin{aligned}
&\varphi^1(x,t)-\sum_{i=1}^{2} G_i^1(x,t)\varphi^1\big(J_i^1(x,t)\big)+\big(T_3v\big)(x,t)=f^3(x,t),\ (x,t)\in\overline{\Delta_1^*},\\
&\varphi^2(x,t)-\sum_{i=1}^{2} G_i^2(x,t)\varphi^2\big(J_i^2(x,t)\big)+\big(T_4v\big)(x,t)=f^4(x,t),\ (x,t)\in\overline{\Delta_2^*},
\end{aligned} \tag{3.7}$$

where $G_i^p$ are the known matrix-functions of order $(m_p\times m_p)$, $p=1,2$, $f^{i+2}$ are the known vector-functions,

$$\begin{aligned}
&J_i^1(x,t)\equiv\sigma_i\big(\sigma_3(x,t)\big), \quad (x,t)\in\overline{\Delta_1^*},\\
&J_i^2(x,t)\equiv\sigma_3\big(\sigma_i(x,t)\big), \quad (x,t)\in\overline{\Delta_2^*}, \ \ i=1,2.
\end{aligned}$$

It is easy to verify that by virtue of equalities (3.4) the linear integral operators $T_3$ and $T_4$ can be represented as

$$\begin{aligned}
\big(T_3v\big)(x,t)=\sum_{i=1}^{2}\int\limits_{\widetilde{\omega}_i^2(x,t)}^{\omega_3^2(x,t,t)}&\big(\widetilde{B}_iv\big)\big(x_i\big(\omega_3^1(x,t,t),-\omega_3^2(x,t,t),\omega_3^2(x,t,t);\tau\big),\\
&y_i\big(\omega_3^1(x,t,t),-\omega_3^2(x,t,t),\omega_3^2(x,t,t);\tau\big),\tau\big)\,d\tau+\\
&+\int\limits_{\omega_3^2(x,t,t)}^{t}\big(\widetilde{B}_3v\big)\big(x_3(x,t,t;\tau),y_3(x,t,t;\tau),\tau\big)\,d\tau,\\
\big(T_4v\big)(x,t)=\sum_{i=1}^{2}\int\limits_{\widetilde{\omega}_3^2(x,t)}^{\omega_i^2(x,-t,t)}&\big(E_iv\big)\big(x_3\big(\omega_i^1(x,-t,t),\omega_i^2(x,-t,t),\omega_i^2(x,-t,t);\tau\big),\\
&y_3\big(\omega_i^1(x,-t,t),\omega_i^2(x,-t,t),\omega_i^2(x,-t,t);\tau\big),\tau\big)\,d\tau+\\
&+\sum_{i=1}^{2}\int\limits_{\omega_i^2(x,-t,t)}^{t}\big(H_iv\big)\big(x_i(x,-t,t;\tau),y_i(x,-t,t;\tau),\tau\big)\,d\tau,
\end{aligned}$$

where $\widetilde{\omega}_i^2(x,t)\equiv\omega_i^2\big(\omega_3^1(x,t,t),\ -\omega_3^2(x,t,t),\ \omega_3^2(x,t,t)\big)$, $\widetilde{\omega}_3^2(x,t)\equiv\omega_3^2\big(\omega_i^1(x,-t,t),\ \omega_i^2(x,-t,t),\ \omega_i^2(x,-t,t)\big)$, and $E_i$, $H_i$, $i=1,2$, $\widetilde{B_j}$, $j=1,2,3$, are the well-defined matrices.

For the functions $J_i^k : \overline{\Delta_k^*} \to \overline{\Delta_k^*}$ we have the formulas

$$J_i^k : (x,t) \to (x+\delta_i^k t, \tau_i t), \quad (x,t) \in \overline{\Delta_k^*},$$

where $\delta_i^k, \tau_i$, $i,k = 1,2$, are the well-defined constants written in terms of $\widetilde{\nu}_i$, $\widetilde{\mu}_i$, $i = 1,2,3$.

*Remark* 3.1. Note that by virtue of condition (2.3) it is easy to establish that $0 < \tau_i < 1$, $i = 1,2$.

*Remark* 3.2. It is obvious that when conditions (3.6) are fulfilled, problem (1.2), (2.4) in the class $\overset{0}{C}{}_\alpha^{1,1,1}(\overline{D})$ is equivalently reduced to system (3.7) for the unknown vector-function $\varphi^i$ of the class $\overset{0}{C}_\alpha(\overline{\Delta_i^*})$, $i = 1,2$. Furthermore, if $u \in \overset{0}{C}{}_\alpha^{1,1,1}(\overline{D})$, then $\varphi^i \in \overset{0}{C}_\alpha(\overline{\Delta_i^*})$, $i = 1,2$. Vice versa, if $\varphi^i \in \overset{0}{C}_\alpha(\overline{\Delta_i^*})$, $i = 1,2$, then with regard to inequality (2.5) equalities (3.1), (3.2) and $u = Cv$ readily imply that $u \in \overset{0}{C}{}_\alpha^{1,1,1}(\overline{D})$.

## § 4. Investigation of the System of Integro-Functional Equations (3.1), (3.7) and the Proof of the Main Result

Let us consider the system of functional equations

$$(K_p\varphi^p)(x,t) \equiv \varphi^p(x,t) - \sum_{i=1}^{2} G_i^p(x,t)\varphi^p\big(J_i^p(x,t)\big) = g_p(x,t), \ (x,t) \in \overline{\Delta_p^*}, \tag{4.1}$$

and introduce the notation

$$\begin{gathered} h_p(\rho) \equiv \sum_{i=1}^{2} \eta_{ip}\tau_i^\rho, \ \ \eta_p \equiv \max_{1\le i\le 2} \sup_{(x,t)\in\overline{\Delta_p^*}} \|G_i^p(x,t)\|, \\ \eta_{ip} \equiv \sup_{x\in[Q_1,Q_2]} \|G_i^p(x,0)\|, \ \ i,p = 1,2, \ \ \rho \in \mathbb{R}. \end{gathered} \tag{4.2}$$

Here and in what follows by $\|.\|$ we understand the norm of a matrix operator acting from one Euclidean space into another.

If all values $\eta_{ip} = 0$, then it is assumed that $\rho_p = -\infty$, $i = 1,2$, $1 \le p \le 2$. Let now for some value of the index $i$ the value $\eta_{ip}$, $1 \le p \le 2$, be different from zero. In that case, by Remark 3.1, the function $h_p : \mathbb{R} \to \overline{\mathbb{R}}_+$ is continuous and strictly decreasing on $\mathbb{R}$; also, $\lim_{\rho\to-\infty} h_p(\rho) = +\infty$ and $\lim_{\rho\to+\infty} h_p(\rho) = 0$, $1 \le p \le 2$. Therefore there exists a unique real number $\rho_p$ such that $h_p(\rho_p) = 1$, $1 \le p \le 2$. It is assumed that $\rho_0 \equiv \max\,(\rho_1, \rho_2)$.

**Lemma 4.1.** *If $\alpha > \rho_0$, then equation* (4.1) *is uniquely solvable in the space* $\overset{0}{C}\alpha(\overline{\Delta_p^*})$ *and for the solution* $\varphi^p = K_p^{-1} g^p$ *the estimate*

$$\|\varphi^p(x,t)\|_{\mathbb{R}^{m_p}} = \|(K_p^{-1} g_p)(x,t)\|_{\mathbb{R}^{m_p}} \leq$$
$$\leq C_{2+p} t^\alpha \|g_p\|_{\overset{0}{C}\alpha(\overline{\Delta_p^*} \cap \{t_1 \leq t\})}, \quad (x,t) \in \overline{\Delta_p^*}, \tag{4.3}$$

*holds, where $C_{2+p}$ is a positive constant not depending on the function $g_p$,* $1 \leq p \leq 2$.

*Proof.* We shall consider the case $p = 1$, since the case $p = 2$ is considered analogously. The condition $\alpha > \rho_0$ and the definition of the function $h_1$ from (4.2) imply

$$h_1(\alpha) = \sum_{i=1}^{2} \eta_{i1} \tau_i^\alpha < 1. \tag{4.4}$$

By inequality (4.4) and the continuity of the functions $G_i^1$, $i = 1, 2$, there exist positive numbers $\varepsilon_1$ ($\varepsilon_1 < t^0$) and $\delta_1$ such that the inequalities

$$\|G_i^1(x,t)\| \leq \eta_{i1} + \delta_1, \quad i = 1, 2, \tag{4.5}$$

$$\sum_{i=1}^{2} (\eta_{i1} + \delta_1) \tau_i^\alpha \equiv \gamma_1 < 1 \tag{4.6}$$

hold for $(x,t) \in \overline{\Delta_1^*} \cap \{0 \leq t \leq \varepsilon_1\}$.

By Remark 3.1 there is a natural number $r_0$ such that for $r \geq r_0$

$$\tau_{i_r} \tau_{i_{r-1}} \cdots \tau_{i_1} t \leq \varepsilon_1, \quad 0 \leq t \leq t^0, \tag{4.7}$$

where $1 \leq i_s \leq 2$, $s = 1, \ldots, r$.

We introduce the operators $\Lambda_1$ and $K_1^{-1}$ acting by the formulas

$$(\Lambda_1 \varphi^1)(x,t) = \sum_{i=1}^{2} G_i^1(x,t) \varphi^1\Big(J_i^1(x,t)\Big), \ \ (x,t) \in \overline{\Delta_1^*}, \ \ K_1^{-1} = I + \sum_{r=1}^{\infty} \Lambda_1^r,$$

where $I$ is the identical operator. Obviously, the operator $K_1^{-1}$ is the formally inverse operator to the operator $K_1$ defined by equality (4.1). Hence it is sufficient for us to show that $K_1^{-1}$ is continuous in the space $\overset{0}{C}_\alpha(\overline{\Delta_1^*})$.

As is easily seen, the expression $\Lambda_1^r g_1$ is the sum consisting of terms of the form

$$I_{i_1 \cdots i_r}(x,t) = G_{i_1}^1(x,t) G_{i_2}^1(J_{i_1}^1(x,t)) G_{i_3}^1(J_{i_2}^1(J_{i_1}^1(x,t))) \cdots$$
$$\cdots G_{i_r}^1(J_{i_{r-1}}^1(J_{i_{r-2}}^1(\cdots(J_{i_1}^1(x,t))\cdots))) g_1(J_{i_r}^1(J_{i_{r-1}}^1(\cdots(J_{i_1}^1(x,t))\cdots))),$$

where $1 \leq i_s \leq 2$, $s = 1, \ldots, r$.

Hence, using (4.2), (4.5), (3.7) and Remark 3.1, we obtain: for $r > r_0$, $g_1 \in \overset{0}{C}_\alpha(\overline{\Delta_1^*})$

$$
\begin{aligned}
&\|I_{i_1\cdots i_r}(x,t)\|_{\mathbb{R}^{m_1}} \le \\
&\le \|G^1_{i_1}(x,t)\| \cdots \|G^1_{i_{r_0}}(J^1_{i_{r_0-1}}(J^1_{i_{r_0-2}}(\cdots(J^1_{i_1}(x,t))\cdots)))\| \times \\
&\times \|G^1_{i_{r_0+1}}(J^1_{i_{r_0}}(J^1_{i_{r_0-1}}(\cdots(J^1_{i_1}(x,t))\cdots)))\| \cdots \\
&\cdots \|G^1_{i_r}(J^1_{i_{r-1}}(J^1_{i_{r-2}}(\cdots(J^1_{i_1}(x,t))\cdots)))\| \times \\
&\times \|g_1(J^1_{i_r}(J^1_{i_{r-1}}(\cdots(J^1_{i_1}(x,t))\cdots)))\|_{\mathbb{R}^{m_1}} \le \\
&\le \eta_1^{r_0}(\eta_{i_{r_0+1}1}+\delta_1)\cdots(\eta_{i_r1}+\delta_1)(\tau_{i_r}\tau_{i_{r-1}}\cdots\tau_{i_1}t)^\alpha \|g_1\|_{\overset{0}{C}_\alpha(\overline{\Delta_1^*}\cap\{t_1\le t\})} \le \\
&\le \eta_1^{r_0}\Big(\prod_{s=r_0+1}^{r}(\eta_{i_s1}+\delta_1)\Big)\Big(\prod_{s=r_0+1}^{r}\tau_{i_s}^\alpha\Big)t^\alpha \|g_1\|_{\overset{0}{C}\alpha(\overline{\Delta_1^*}\cap\{t_1\le t\})} = \\
&= \eta_1^{r_0}\Big(\prod_{s=r_0+1}^{r}(\eta_{i_s1}+\delta_1)\tau_{i_s}^\alpha\Big)t^\alpha \|g_1\|_{\overset{0}{C}\alpha(\overline{\Delta_1^*}\cap\{t_1\le t\})},
\end{aligned}
\tag{4.8}
$$

and for $1 \le r \le r_0$

$$
\begin{aligned}
\|I_{i_1\cdots i_r}(x,t)\|_{\mathbb{R}^{m_1}} &\le \eta_1^r(\tau_{i_r}\tau_{i_{r-1}}\cdots\tau_{i_1}t)^\alpha \|g_1\|_{\overset{0}{C}\alpha(\overline{\Delta_1^*}\cap\{t_1\le t\})} \le \\
&\le \eta_1^r t^\alpha \|g_1\|_{\overset{0}{C}\alpha(\overline{\Delta_1^*}\cap\{t_1\le t\})}.
\end{aligned}
\tag{4.9}
$$

By (4.8), (4.9), and (4.6) we have: for $r > r_0$

$$
\begin{aligned}
&\|(\Lambda_1^r g_1)(x,t)\|_{\mathbb{R}^{m_1}} = \Big\|\sum_{i_1,\dots,i_r} I_{i_1\cdots i_r}(x,t)\Big\|_{\mathbb{R}^{m_1}} \le \\
&\le \Big(\sum_{i_1,\dots,i_{r_0}} 1\Big)^{r_0}\eta_1^{r_0}\Big[\sum_{i=1}^{2}(\eta_{i1}+\delta_1)\tau_i^\alpha\Big]^{r-r_0} t^\alpha \|g_1\|_{\overset{0}{C}_\alpha(\overline{\Delta_1^*}\cap\{t_1\le t\})} \le \\
&\le C_5\gamma_1^r t^\alpha \|g_1\|_{\overset{0}{C}_\alpha(\overline{\Delta_1^*}\cap\{t_1\le t\})},
\end{aligned}
\tag{4.10}
$$

and for $1 \le r \le r_0$

$$
\|(\Lambda_1^r g_1)(x,t)\|_{\mathbb{R}^{m_1}} \le C_6 t^\alpha \|g_1\|_{\overset{0}{C}_\alpha(\overline{\Delta_1^*}\cap\{t_1\le t\})},
\tag{4.11}
$$

where $C_5 \equiv \eta_1^{r_0}\gamma_1^{-r_0}\Big(\sum_{i_1,\dots,i_{r_0}} 1\Big)^{r_0}$, $C_6 \equiv \eta_1^r\Big(\sum_{i_1,\dots,i_r} 1\Big)$.

Inequalities (4.10) and (4.11) finally imply

$$
\begin{aligned}
&\|\varphi^1(x,t)\|_{\mathbb{R}^{m_1}} = \|(K_1^{-1}g_1)(x,t)\|_{\mathbb{R}^{m_1}} \le \\
&\le \|g_1(x,t)\|_{\mathbb{R}^{m_1}} + \sum_{r=1}^{r_0}\|(\Lambda_1^r g_1)(x,t)\|_{\mathbb{R}^{m_1}} + \sum_{r=r_0+1}^{\infty}\|(\Lambda_1^r g_1)(x,t)\|_{\mathbb{R}^{m_1}} \le
\end{aligned}
$$

$$\leq \big(1 + C_6 r_0 + C_5 \gamma_1^{r_0+1}(1-\gamma_1)^{-1}\big) t^\alpha \|g_1\|_{\overset{0}{C}_\alpha(\overline{\Delta_1^*}\cap\{t_1\leq t\})} =$$
$$= C_3 t^\alpha \|g_1\|_{\overset{0}{C}_\alpha(\overline{\Delta_1^*}\cap\{t_1\leq t\})},$$

where $C_3 \equiv 1 + C_6 r_0 + C_5\gamma_1^{r_0+1}(1-\gamma_1)^{-1}$. Hence we conclude that the operator $K_1^{-1}$ is continuous in the space $\overset{0}{C}_\alpha(\overline{\Delta_1^*})$ and therefore Lemma 4.1 is true. □

On the basis of this lemma we have

**Theorem 4.1.** *Let conditions* (3.6) *be fulfilled. If* $\alpha > \rho_0$, *then problem* (1.2), (2.4) *is uniquely solvable in the space* $\overset{0}{C}{}^{1,1,1}_\alpha(\overline{D})$.

*Proof.* First we solve the system of equations (3.1), (3.7) with respect to the unknown functions $v \in \overset{0}{C}{}^{1,1,1}_\alpha(\overline{D})$ and $\varphi^p \in \overset{0}{C}_\alpha(\overline{\Delta_p^*})$, $p = 1,2$, using the method of successive approximations.

Let

$$v_0(x,y,t) \equiv 0, \ \ (x,y,t) \in \overline{D}; \ \ \varphi_0^p(x,t) \equiv 0, \ \ (x,t) \in \overline{\Delta_p^*}, \ \ p = 1,2;$$
$$v_{q_i+j,k}(x,y,t) = \varphi_{q_i+j,k}\big(\omega_i^1(x,y,t),\omega_i^2(x,y,t)\big) +$$
$$+ \int\limits_{\omega_i^2(x,y,t)}^{t} \Big(\sum_{p'=1}^m b_{ijp'} v_{p',k-1}\Big)\big(x_i(x,y,t;\tau), y_i(x,y,t;\tau),\tau\big)d\tau + \tag{4.12}$$
$$+\widetilde{F_{ij}}(x,y,t), \quad 1\leq i\leq 3, \quad 1\leq j\leq s_i,$$

where

$$\varphi_{q_i+j,k}(\omega_i^1(x,y,t),\omega_i^2(x,y,t)) =$$
$$= \begin{cases} \varphi^1_{q_i+j,k}\big(\omega_i^1(x,y,t),\omega_i^2(x,y,t)\big), & 1\leq i\leq 2, \ 1\leq j\leq s_i, \\ \varphi^2_{q_i+j,k}\big(\omega_i^1(x,y,t),\omega_i^2(x,y,t)\big), & i=3, \ 1\leq j\leq s_i, \end{cases} \quad (x,y,t)\in\overline{D}.$$

The values $\varphi_k^p(x,t)$ are defined by the equations

$$(K_p\varphi_k^p)(x,t) + (T_{2+p}v_{k-1})(x,t) = f^{2+p}(x,t), \tag{4.13}$$
$$(x,t)\in\overline{\Delta_p^*}, \ \ p=1,2, \ \ k\geq 1,$$

where the operators $K_p$, $p=1,2$, act by (4.1).

For convenience, system (4.12) is rewritten as

$$v_k(x,y,t) = \varphi_k(x,y,t) +$$

$$+\sum_{i=1}^{3}\int\limits_{\omega_i^2(x,y,t)}^{t}\Big(\Omega_i v_{k-1}\Big)\Big(x_i(x,y,t;\tau),y_i(x,y,t;\tau),\tau\Big)d\tau+$$

$$+\widetilde{F}(x,y,t),\quad (x,y,t)\in\overline{D},\tag{4.14}$$

where the $(q_i+j)$-th component of the vector $\varphi_k(x,y,t)$ is equal to $\varphi_{q_i+j,k}\big(\omega_i^1(x,y,t),\omega_i^2(x,y,t)\big)$, $1\le i\le 3$, $1\le j\le s_i$, $k\ge 1$; $\Omega_i$, $i=1,2,3$, and $\widetilde{F}$ are respectively the well-defined matrices and vector-functions.

We shall now show that the following estimates are true:

$$\|v_{k+1}(x,y,t)-v_k(x,y,t)\|_{\mathbb{R}^m}\le M^*\frac{M_*^k}{k!}t^{k+\alpha},\quad (x,y,t)\in\overline{D},\tag{4.15}$$

$$\|\varphi_{k+1}^p(x,t)-\varphi_k^p(x,t)\|_{\mathbb{R}^{m_p}}\le M^*\frac{M_*^k}{k!}t^{k+\alpha},\quad (x,t)\in\overline{\Delta_p^*},\tag{4.16}$$

where $M_*$ and $M^*$ are well-defined sufficiently large numbers not depending on $k$, $k\ge 1$, $p=1,2$.

Indeed, by the assumptions for $f_p$ and $F$ we have $f^{2+p}\in\overset{0}{C}_{\alpha}(\overline{\Delta_p^*})$, $\widetilde{F}\in\overset{0}{C}_{\alpha}(\overline{D})$, $p=1,2$. Hence, on account of inequalities (2.5) from §2, we conclude that the estimates

$$\|\widetilde{F}(x,y,t)\|_{\mathbb{R}^m}\le\Theta_1 t^{\alpha},\quad (x,y,t)\in\overline{D},\tag{4.17}$$

$$\|f^{2+p}(x,t)\|_{\mathbb{R}^{m_p}}\le\Theta_{1+p}t^{\alpha},\quad (x,t)\in\overline{\Delta_p^*},\tag{4.18}$$

$$p=1,2,\quad \Theta_i=\mathrm{const}\ge 0,\quad i=1,2,3,$$

are fulfilled.

By $v_0\equiv 0$, $\varphi_0^p\equiv 0$, $p=1,2$ and the conditions of Theorem 4.1 estimate (4.3) is true so that (4.13), (4.18) imply

$$\begin{gathered}\|\varphi_1^p(x,t)-\varphi_0^p(x,t)\|_{\mathbb{R}^{m_p}}=\|\varphi_1^p(x,t)\|_{\mathbb{R}^{m_p}}\le C_5\Theta_4 t^{\alpha},\ \ p=1,2,\\ C_7=\max\,(C_3,C_4),\quad \Theta_4=\max\,(\Theta_2,\Theta_3)\end{gathered}\tag{4.19}$$

which in turn gives rise to

$$\begin{aligned}&\|\varphi_1(x,y,t)-\varphi_0(x,y,t)\|_{\mathbb{R}^m}=\|\varphi_1(x,y,t)\|_{\mathbb{R}^m}=\\&=\sum_{1\le i\le 3}\sum_{1\le j\le s_i}\big|\varphi_{q_i+j,1}\big(\omega_i^1(x,y,t),\omega_i^2(x,y,t)\big)\big|\le\\&\le\sum_{1\le i\le 3}\sum_{1\le j\le s_i}C_7\Theta_4\big(\omega_i^2(x,y,t)\big)^{\alpha}\le mC_7\Theta_4 t^{\alpha},\end{aligned}\tag{4.20}$$

since $\sum\limits_{1\le i\le 3}\sum\limits_{1\le j\le s_i}1=m$ and, as shown in §3, $0\le\omega_i^2(x,y,t)\le t$, $i=1,2,3$.

By virtue of (4.17) and (4.20), from (4.14) we have

$$\|v_1(x,y,t)-v_0(x,y,t\|_{\mathbb{R}^m}=\|v_1(x,y,t)\|_{\mathbb{R}^m}\le\|\varphi_1(x,y,t)\|_{\mathbb{R}^m}+$$
$$+\|\widetilde{F}(x,y,t)\|_{\mathbb{R}^m}\le mC_7\Theta_4t^\alpha+\Theta_1t^\alpha=(mC_7\Theta_4+\Theta_1)t^\alpha. \qquad (4.21)$$

Now, assuming that estimates (4.15), (4.16) are fulfilled for $k$, $k>0$, we shall show that they hold for $k+1$ when $M_*$ and $M^*$ are sufficiently large.

Using (4.13), for $p=1$ we have

$$\{K_1(\varphi^1_{k+2}-\varphi^1_{k+1})\}(x,t)=-\{T_3(v_{k+1}-v_k)\}(x,t),\quad (x,t)\in\overline{\Delta^*_1}. \quad (4.22)$$

It is obvious that for the right-hand side of equation (4.22) we have the estimate

$$\big\|\{T_3(v_{k+1}-v_k)\}(x,t)\big\|_{\mathbb{R}^{m_1}}\le$$
$$\le\sum_{i=1}^{2}\int\limits_{\widetilde{\omega}_i^2(x,t)}^{\omega_3^2(x,t,t)}\|\widetilde{B_i}\|\|v_{k+1}-v_k\|_{\mathbb{R}^m}\big(x_i\big(\omega^1_3(x,t,t),-\omega^2_3(x,t,t),$$
$$\omega^2_3(x,t,t);\tau\big),y_i\big(\omega^1_3(x,t,t),-\omega^2_3(x,t,t),\omega^2_3(x,t,t);\tau\big),\tau\big)d\tau+$$
$$+\int\limits_{\omega_3^2(x,t,t)}^{t}\|\widetilde{B_3}\|\|v_{k+1}-v_k\|_{\mathbb{R}^m}\big(x_3(x,t,t;\tau),y_3(x,t,t;\tau),\tau\big)d\tau. \quad (4.23)$$

Denote by $\xi_1$ the largest of the numbers $\max\limits_{x,t,\tau}\|\widetilde{B_i}(x,t,\tau)\|$, $i=1,2,3$. Since $0\le\widetilde{\omega}_i^2(x,t)\le\omega_3^2(x,t,t)\le t$, by (4.15) we find from (4.23) that

$$\big\|\{T_3(v_{k+1}-v_k)\}(x,t)\big\|_{\mathbb{R}^{m_1}}\le$$
$$\le\xi_1M^*\frac{M_*^k}{k!}\Big(\sum_{i=1}^{2}\int\limits_{\widetilde{\omega}_i^2(x,t)}^{\omega_3^2(x,t,t)}\tau^{k+\alpha}d\tau+\int\limits_{\omega_3^2(x,t,t)}^{t}\tau^{k+\alpha}d\tau\Big)\le$$
$$\le\xi_1M^*\frac{M_*^k}{k!}\Big(\sum_{i=1}^{2}1+1\Big)\int\limits_0^t\tau^{k+\alpha}d\tau\le$$
$$\le3\xi_1M^*\frac{M_*^k}{k!}\frac{1}{k+\alpha+1}t^{k+\alpha+1}\le3\xi_1M^*\frac{M_*^k}{(k+1)!}t^{k+1+\alpha}. \qquad (4.24)$$

Now (4.22), (4.24), and (4.3) (for $p=1$) imply

$$\big\|\varphi^1_{k+2}(x,t)-\varphi^1_{k+1}(x,t)\big\|_{\mathbb{R}^{m_1}}\le3C_3\xi_1M^*\frac{M_*^k}{(k+1)!}t^{k+1+\alpha}. \qquad (4.25)$$

Similarly, (4.13) (for $p=2$), (4.15), and (4.3) (for $p=2$) give

$$\left\|\varphi_{k+2}^{2}(x,t)-\varphi_{k+1}^{2}(x,t)\right\|_{\mathbb{R}^{m_2}}\leq 4C_4\xi_2 M^*\frac{M_*^k}{(k+1)!}t^{k+1+\alpha}, \qquad (4.26)$$

where $\xi_2$ denotes the largest of the numbers $\max\limits_{x,t,\tau}\|E_i(x,t,\tau)\|$, $\max\limits_{x,t,\tau}\|H_i(x,t,\tau)\|$, $i=1,2$.

Using the same arguments as in deriving estimate (4.20), from (4.25) and (4.26) we obtain

$$\|\varphi_{k+2}(x,y,t)-\varphi_{k+1}(x,y,t)\|_{\mathbb{R}^m}\leq \xi_4 M^*\frac{M_*^k}{(k+1)!}t^{k+1+\alpha}, \qquad (4.27)$$

where $\xi_4\equiv 4mC_7\xi_3$, $\xi_3\equiv\max\,(\xi_1,\xi_2)$.

We denote by $\eta$ the largest of the numbers $\max\limits_{\overline{D}}\|\Omega_i\|$, where the matrices $\Omega_i$, $i=1,2,3$, are defined by (4.14). By (4.27) and (4.15), from system (4.14) we have

$$\begin{aligned}
&\|v_{k+2}(x,y,t)-v_{k+1}(x,y,t)\|_{\mathbb{R}^m}\leq\|\varphi_{k+2}(x,y,t)-\varphi_{k+1}(x,y,t)\|_{\mathbb{R}^m}+\\
&+\sum_{i=1}^{3}\int\limits_{\omega_i^2(x,y,t)}^{t}\|\Omega_i\|\|v_{k+1}-v_k\|_{\mathbb{R}^m}\big(x_i(x,y,t;\tau),y_i(x,y,t;\tau),\tau\big)\,d\tau\leq\\
&\qquad\leq\xi_4M^*\frac{M_*^k}{(k+1)!}t^{k+1+\alpha}+3\eta\int\limits_0^t M^*\frac{M_*^k}{k!}\tau^{k+\alpha}d\tau\leq\\
&\qquad\leq(\xi_4+3\eta)M^*\frac{M_*^k}{(k+1)!}t^{k+1+\alpha},\quad (x,y,t)\in\overline{D}, \qquad (4.28)
\end{aligned}$$

since $0\leq\omega_i^2(x,y,t)\leq t$, $i=1,2,3$.

If we set

$$M^*=mC_7\Theta_4+\Theta_1,\quad M_*=\max\big(3C_3\xi_1,\,4C_4\xi_2,\,\xi_4+3\eta\big),$$

then by (4.19), (4.21), (4.25), (4.26), (4.28) immediately imply that estimates (4.15), (4.16) hold for any integer $k\geq 0$.

It follows from (4.15), (4.16) that the series

$$v(x,y,t)=\lim_{k\to\infty}v_k(x,y,t)=\sum_{k=1}^{\infty}\big(v_k(x,y,t)-v_{k-1}(x,y,t)\big),\ (x,y,t)\in\overline{D},$$

$$\varphi^p(x,t)=\lim_{k\to\infty}\varphi_k^p(x,t)=\sum_{k=1}^{\infty}\big(\varphi_k^p(x,t)-\varphi_{k-1}^p(x,t)\big),\ (x,t)\in\overline{\Delta_p^*},\ p=1,2,$$

converge in the spaces $\overset{0}{C}{}^{1,1,1}_{\alpha}(\overline{D})$, $\overset{0}{C}{}_{\alpha}(\overline{\Delta^*_p})$, $p=1,2$, and by (4.13), (4.14) the limit functions $v$, $\varphi^p$, $p=1,2$, satisfy system (3.1), (3.7). Finally, since problem (1.2), (2.4) is equivalent to system (3.1), (3.7) and the equality $u=Cv$ holds, we conclude that the obtained function $u(x,y,t)$ is really a solution of problem (1.2), (2.4) in the class $\overset{0}{C}{}^{1,1,1}_{\alpha}(\overline{D})$, $\alpha>\rho_0$.

Now we shall show that under the conditions of Theorem 4.1 problem (1.2), (2.4) has no other solutions in the class $\overset{0}{C}{}^{1,1,1}_{\alpha}(\overline{D})$. Indeed, if $u\in\overset{0}{C}{}^{1,1,1}_{\alpha}(\overline{D})$ is the solution of the homogeneous problem corresponding to (1.2), (2.4), then the corresponding functions $v$, $\varphi^p$, $p=1,2$, satisfy the homogeneous system of equations

$$\begin{gathered}
v_{q_i+j}(x,y,t)=\varphi_{q_i+j}\big(\omega^1_i(x,y,t),\omega^2_i(x,y,t)\big)+\\
+\int\limits_{\omega^2_i(x,y,t)}^{t}\Big(\sum_{p'=1}^{m}b_{ijp'}v_{p'}\Big)\big(x_i(x,y,t;\tau),y_i(x,y,t;\tau),\tau\big)\,d\tau,\ (x,y,t)\in\overline{D},\\
1\le i\le 3,\quad 1\le j\le s_i,\\
\big(K_p\varphi^p\big)(x,t)+\big(T_{2+p}v\big)(x,t)=0,\ (x,t)\in\overline{\Delta^*_p},\ \ p=1,2.
\end{gathered}\tag{4.29}$$

We apply the method of successive approximations to system (4.29), assuming that $v$, $\varphi^1$, $\varphi^2$, are zero approximations. Since these values satisfy system (4.29), each next approximation will coincide with it so that we shall have $v_k(x,y,t)\equiv v(x,y,t)$, $(x,y,t)\in\overline{D}$, $\varphi^p_k(x,t)\equiv\varphi^p(x,t)$, $(x,t)\in\overline{\Delta^*_p}$ for $k\ge 1$ and $p=1,2$. Recalling that these values satisfy estimates of form (4.17), (4.18) and arguing as in the case of deriving estimates (4.15), (4.16), we obtain

$$\|v(x,y,t)\|_{\mathbb{R}^m}=\|v_{k+1}(x,y,t)\|_{\mathbb{R}^m}\le\widetilde{M}^*\frac{\widetilde{M}^k_*}{k!}t^{k+\alpha},\ \ (x,y,t)\in\overline{D},$$

$$\|\varphi^p(x,t)\|_{\mathbb{R}^{m_p}}=\|\varphi^p_{k+1}(x,t)\|_{\mathbb{R}^{m_p}}\le\widetilde{M}^*\frac{\widetilde{M}^k_*}{k!}t^{k+\alpha},\ (x,t)\in\overline{\Delta^*_p},\ k\ge 1,\ p=1,2,$$

whence, as $k\to\infty$, we find in the limit that

$$v(x,y,t)\equiv 0,\ \ (x,y,t)\in\overline{D},\ \ \varphi^p(x,t)\equiv 0,\ \ (x,t)\in\overline{\Delta^*_p},\ \ p=1,2.\qquad\square$$

Next, using inequality (4.15) and recalling that the value $M^*$ is defined by $\Theta_i$, $i=1,2,3$, which are given by the right-hand sides $F$ and $f_i$, $i=1,2$, of problem (1.2), (2.4), we can readily show that for a regular solution of the considered problem the estimate

$$\|u\|_{\overset{0}{C}{}^{1,1,1}_{\alpha}(\overline{D})}\le c\Big(\sum_{i=1}^{2}\|f_i\|_{\overset{0}{C}{}_{\alpha}(\overline{\Delta_i})}+\|F\|_{\overset{0}{C}{}_{\alpha}(\overline{D})}\Big)\tag{4.30}$$

holds, where the positive constant $c$ does not depend on $f_i$, $i = 1, 2$, and $F$. Estimate (4.30) implies that a regular solution of problem (1.2), (2.4) is stable in the space $\overset{0}{C}{}^{1,1,1}_{\alpha}(\overline{D}), \alpha > \rho_0$.

## References


1. A. V. Bitsadze, Some classes of partial equations. (Russian) *Nauka, Moscow*, 1981.
2. S. S. Kharibegashvili, On one boundary value problem for normally hyperbolic systems of second order with variable coefficients. (Russian) *Differentsial'nye Uravneniya* **21**(1985), No. 1, 149–155.
3. I. M. Gelfand, Lectures on linear algebra, 4th edition. (Russian) *Nauka, Moscow*, 1971.
4. S. L. Sobolev, On analytic solutions of systems of partial equations with two independent variables. (Russian) *Matem. Sb.* **38**(1931), Nos. 1, 2, 107–147.
5. V. P. Mikhailov, On analytic solution of the Goursat problem for a system of differential equations. (Russian) *Dokl. Akad. Nauk SSSR* **115**(1957), No. 3, 450–453.
6. V. P. Mikhailov, On non-analytic solutions of the Goursat problem for a system of differential equations with two independent variables. (Russian) *Dokl. Akad. Nauk SSSR* **117**(1957), No. 5, 759–762.
7. L. A. Mel'tser, On non-correct formulation of the Goursat problem. (Russian) *Mat. Sb.* **18(60)**(1946), No. 1, 59–104.
8. Z. O. Melnik, One nonclassical boundary value problem for hyperbolic systems of first order of two independent variables. (Russian) *Differentsial'nie Uravneniya* **17**(1981), No. 6, 1096–1104.
9. J. Hadamard, Lectures on Cauchy's problem in linear partial differential equations. *Yale Univ. Press, New Haven; Oxford Univ. Press, London*, 1923.
10. J. Tolen, Probléme de Cauchy sur la deux hypersurfaces caracteristique sécantes. *C. R. Acad. Sci. Paris, Ser. A–B* **291**(1980), No. 1, A 49–A 52.
11. S. Kharibegashvili, On the solvability of a spatial problem of Darboux type for the wave equation. *Georgian Math. J.* **2**(1995), No. 4, 385–394.
12. O. Jokhadze, Spatial problem of Darboux type for one model equation of third order. *Georgian Math. J.* **3**(1996), No. 6, 547–564.


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