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ASYMPTOTIC SOLUTIONS FOR MIXED-TYPE EQUATIONS WITH A SMALL DEVIATION

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ABSTRACT. An asymptotic matrix solution is formulated for a class of mixed-type linear vector equations with a single variable deviation which is small at infinity. This matrix solution describes the asymptotic behavior of all exponentially bounded solutions. A sufficient condition is obtained for there to be no other solutions.

§ 1. INTRODUCTION

We consider the equation

$$\frac{d}{dt}x(t) = Ax(t - r(t)), \tag{1.1}$$

where A is an $N \times N$ constant matrix, r is measurable on $[\tau, \infty)$ with $-\sigma \leq r(t) \leq \rho$ for some constants $\sigma > 0$ and $\rho > 0$, and r satisfies a suitable condition of smallness as $t \to \infty$. Under the conditions that $0 \leq r(t) \leq \rho$, $\lim_{t\to\infty} r(t) = 0$ and either $r \in L^p(\tau, \infty)$ for some $p \in (1, 2)$ with dr/dt bounded or $r \in L(\tau, \infty)$, Cooke [1] showed for the scalar case of (1.1) that every solution has the asymptotic form

$$x(t) = \exp\left\{At - A^2 \int_0^t r(s)ds\right\}\{c + o(1)\} \tag{1.2}$$

as $t \to \infty$. Kato [2] proved that (1.2) holds also for $N > 1$, still with $0 \leq r(t) \leq \rho$, if either $r \in L(\tau, \infty)$ or $r \in L^2(\tau, \infty)$ provided, in the latter case, r is either Lipschitzian or monotone for large t. This leaves open the case of $r \in L^p(\tau, \infty)$ with $p > 2$. Also, it would be desirable to have a clearer indication of the size of the $o(1)$ term in (1.2), particularly when $c = 0$. If, further, we remove the restriction $r \geq 0$, then another issue arises: in the retarded case all solutions are $O(e^{|A|t})$ as $t \to \infty$ (see [3, §6.1]), but this

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need not be so for the mixed-type equation. These matters are all dealt with in this present paper.

Our results for (1.1) generalize to equations of the form

$$\frac{d}{dt}x(t) = A(t)x(t) + B(t)x(t - r(t)), \tag{1.3}$$

where $A(t) = \sum_{i=1}^{n} a_i(t)M_i$ and $B(t) = \sum_{i=1}^{n} b_i(t)M_i$, a_i and b_i are locally integrable scalar functions and the constant matrices M_i all commute.

The contents of this paper are arranged as follows. We present two main theorems in §2, but leave some of the proofs to §3. In §4 sufficient conditions are given to exclude the possibility of solutions that grow faster than $e^{\gamma t}$ for every $\gamma > 0$. Equation (1.3) is dealt with in §5.

§ 2. Main Results

As we shall see later, (1.1) can be transformed, by a generalization of a substitution of Kato [2], to an equation of the form

$$\frac{d}{dt}x(t) = F(t)x(t) + G(t)\{x(t - r(t)) - x(t)\}, \tag{2.1}$$

where $F \in L(\tau, \infty)$ and $G \in L^\infty(\tau, \infty)$. We shall give a result for (2.1) before tackling (1.1).

It will be assumed that

$$\boldsymbol{\mathcal{M}}\{t \geq \tau : |r(t)| > \alpha\} < \infty \quad \text{for every} \quad \alpha > 0, \tag{2.2}$$

where bM denotes Lebesgue measure. This condition certainly holds if $r \in L^p(\tau, \infty)$ for some $p \in [1, \infty)$ or if $\lim_{t\to\infty} r(t) = 0$. For $s \geq T \geq \tau$, let

$$\mu_T(s) = \boldsymbol{\mathcal{M}}\left\{t \geq T : \min\{t, t - r(t)\} \leq s \leq \max\{t, t - r(t)\}\right\}. \tag{2.3}$$

Since (2.2) implies that, for $\alpha > 0$,

$$\lim_{T\to\infty} \boldsymbol{\mathcal{M}}\{t \geq T : |r(t)| > \alpha\} = 0,$$

for any $\delta > 0$ there exists $T_\delta \geq \tau$ such that $\boldsymbol{\mathcal{M}}\{t \geq T_\delta : |r(t)| \geq \delta/3\} < \delta/3$. Then, for $s \geq T \geq T_\delta$, $\mu_T(s) < \delta$. Thus if

$$\varepsilon_T = \sup_{s \geq T} \mu_T(s), \tag{2.4}$$

then ε_T is finite and tends to zero as $T \to \infty$.

A continuous vector function $x : [T - \rho, \infty) \to \mathbb{C}^N$ is said to be a solution of (2.1) on $[T - \rho, \infty)$ if, on $[T, \infty)$, it is locally absolutely continuous and satisfies (2.1) almost everywhere. A solution is said to be exponentially bounded if it is $O(e^{\gamma t})$ as $t \to \infty$ for some $\gamma > 0$. By a large solution we mean one that is not exponentially bounded. For any $p \in [1, \infty]$ and any

$f \in L^p(T, \infty)$, we denote the norm of f by $\|f\|_p$ or, to mark the relevance of $[T, \infty)$, by $\|f\|_p(T, \infty)$.

Theorem 1. *Assume that* $F \in L(\tau, \infty)$, $G \in L^\infty(\tau, \infty)$, *and* (2.2) *holds. Then, for* $T \geq \tau$ *satisfying*

$$\|F\|_1(T, \infty) + \varepsilon_T \|G\|_\infty(T, \infty) < 1, \tag{2.5}$$

(2.1) *has solutions* $x_n : [T - \rho, \infty) \to \mathbb{C}^N$ $(n = 1, 2, \ldots, N)$ *such that*

$$x_n(t) = e_n + o(1) \tag{2.6}$$

as $t \to \infty$, *where* e_n *is the nth coordinate vector, and* x_n *is constant on* $[T - \rho, T]$. *Moreover, every exponentially bounded solution of* (2.1) *has the asymptotic representation*

$$x(t) = X(t)c + o\big(e^{-\beta t}\big) \tag{2.7}$$

for arbitrary $\beta > 0$, *where* $X = (x_1, x_2, \ldots, x_N)$ *and* $c \in \mathbb{C}^N$ *is a constant dependent on* x.

The proof of Theorem 1 is left to §3. The following two remarks also apply to Theorem 2 given below.

Remarks. (i) Equation (2.1) may have large solutions which certainly cannot be represented by (2.7). For instance, the equation

$$\frac{d}{dt}x(t) = 2te^{-2t^{1/2} - t^{-1}} x(t + t^{-1/2})$$

meets all the requirements of Theorem 1 and it has a large solution $x(t) = e^{t^2}$.

(ii) From representation (2.7) we see that, if every solution of (2.1) is exponentially bounded, then the space of solutions on $[T - \rho, \infty)$ can be decomposed as $\mathbf{S}_1 \oplus \mathbf{S}_2$, where $\mathbf{S}_1$ is N dimensional with a basis X, and $\mathbf{S}_2$ consists of solutions tending to zero faster than any exponential.

Extend r to all of $\mathbb{R} \equiv (-\infty, \infty)$ by putting $r(t) = 0$ for $t < \tau$ and, for $k = 0, 1, 2, \ldots$ and $t \in \mathbb{R}$, let

$$r_{k+1}(t) = \int_t^{t - r(t)} |r_k(s)| ds, \tag{2.8}$$

$$A_{k+1}(t) = A \exp\left\{ \int_t^{t - r(t)} A_k(s) ds \right\} \tag{2.9}$$

with $r_0(t) = r(t)$ and $A_0(t) = A$.

Theorem 2. *Assume that* (2.2) *holds and* $r_M \in L(\tau, \infty)$ *for some integer* $M \geq 0$. *Then, for sufficiently large* $T \geq \tau$, (1.1) *has solutions* x_n *on* $[T-\rho, \infty)$ *satisfying*

$$x_n(t) = \exp\left\{\int_T^t A_M(s)ds\right\}\{e_n + o(1)\} \quad (n = 1, 2, \ldots, N) \tag{2.10}$$

as $t \to \infty$. *Moreover, every exponentially bounded solution of* (1.1) *has the asymptotic representation* (2.7) *with* X *composed of* N *solutions* (2.10).

Proof. Since each A_k in (2.9) can be expanded as a power series in A with scalar functions as coefficients, they all commute with A and with each other. Let

$$c_0 = |A| \quad \text{and} \quad c_{k+1} = |A| \exp\{c_k \max\{\rho, \sigma\}\} \quad (k \geq 0). \tag{2.11}$$

Then, from (2.9) and (2.11),

$$\begin{gathered} \|A_0\|_\infty = c_0, \\ \|A_{k+1}\|_\infty \leq |A| \exp\{\|A_k\|_\infty \max\{\rho, \sigma\}\} \leq c_{k+1} \quad (k \geq 0). \end{gathered} \tag{2.12}$$

Since (2.11) implies $c_k \leq c_{k+1}$ $(k \geq 0)$ and, for any a and b in $\mathbb{C}$,

$$|e^a - e^b| \leq e^{\max\{|a|,|b|\}}|a - b|,$$

we have

$$|A_{k+1}(t) - A_k(t)| \leq c_{k+1}\left|\int_t^{t-r(t)} (A_k(s) - A_{k-1}(s))ds\right| \quad (k \geq 0),$$

where $A_{-1} = 0$. Inductively, from this and (2.8), we obtain

$$|A_{k+1}(t) - A_k(t)| \leq c_0 c_1 \cdots c_{k+1} |r_k(t)| \quad (k \geq 0). \tag{2.13}$$

By the transformation

$$x(t) = \exp\left\{\int_T^t A_M(s)ds\right\} y(t) \tag{2.14}$$

(cf. Kato [2, p163]), equation (1.1) can be rewritten as

$$\frac{d}{dt}y(t) = (A_{M+1}(t) - A_M(t))\, y(t) + A_{M+1}(t)\{y(t - r(t)) - y(t)\}. \tag{2.15}$$

Then, since A_{M+1} is bounded and (2.13), along with the condition $r_M \in L(\tau, \infty)$, implies $A_{M+1} - A_M \in L(\tau, \infty)$, the conclusion follows from Theorem 1 and (2.14). $\square$

Remark. $T \geq \tau$ required in Theorem 2 can be chosen in advance. From (2.5), (2.15), (2.13), and (2.12), we see that it is sufficient for T to satisfy

$$c_0 c_1 \cdots c_{M+1} \int_T^\infty |r_M(s)| ds + \varepsilon_T c_{M+1} < 1. \tag{2.16}$$

Such a choice is possible since $r_M \in L(\tau, \infty)$ and $\varepsilon_T \to 0$ as $T \to \infty$.

Corollary. *Assume that* $r \in L^p[\tau, \infty)$ *for some* $p \in (1, \infty)$. *Then* $r_M \in L(\tau, \infty)$ *and the conclusion of Theorem 2 holds for* $M \geq p$.

The proof of Corollary is left to §3.

Remarks. (i) If $r \in L(\tau, \infty)$, then certainly we can take $M = 0$. If $r \in L^p(\tau, \infty)$ with $p > 1$ and r is either monotone or Lipschitzian (as in Kato [2]), then $r_k \in L^{p/(k+1)}(\tau, \infty)$ for each k and hence $r_M \in L(\tau, \infty)$ for $M \geq p - 1$. This is a consequence of the following inequalities which can easily be shown from (2.8) by induction:

(a) $|r_k(t)| \leq |r(t)|^{k+1}$ if $r \leq 0$ and r is increasing,
(b) $|r_k(t + k\rho)| \leq |r(t)|^{k+1}$ if $r \geq 0$ and r is decreasing,
(c) $|r_k(t)| \leq (\mu + 1)^{k(k+1)/2} |r(t)|^{k+1}$ if $|r(t) - r(s)| \leq \mu |t - s|$.

(ii) For any particular value of $p \geq 1$, we can further simplify the expression (2.10). If $p = 1$, then $M = 0$ and $x_n(t) = e^{A(t-T)}\{e_n + o(1)\}$. If $p \in (1, 2]$ and $M = 1$, we have $A_1(t) = Ae^{-Ar(t)} = A - A^2 r(t) + \boldsymbol{\mathcal{L}}$, where $\boldsymbol{\mathcal{L}}$ stands for a function in $L(\tau, \infty)$, and we can replace (2.10) by

$$\hat{x}_n(t) = \exp\left\{At - A^2 \int_T^t r(s) ds\right\}\{e_n + o(1)\} \quad (n = 1, 2, \ldots, N) \tag{2.17}$$

as $(x_1, x_2, \ldots, x_N) = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_N)E$ for an invertible constant matrix E. Hence, in view of Remark (i), our result includes that of Kato referred to in §1. If $p \in (1, 2]$ and $M = 2$, then

$$A_2(t) = A \exp\left\{\int_t^{t-r(t)} Ae^{-Ar(s)} ds\right\} = A - A^2 r(t) - A^3 \int_t^{t-r(t)} r(s) ds + \boldsymbol{\mathcal{L}}$$

and we can replace (2.10) by

$$\hat{x}_n(t) = \exp\left\{At - A^2 \int_T^t \left(r(s) + A \int_s^{s-r(s)} r(u) du\right) ds\right\}\{e_n + o(1)\}. \tag{2.18}$$

Similarly, we can simplify (2.10) for $p > 2$.

§ 3. Proofs of Theorem 1 and Corollary to Theorem 2

For $t \geq \tau$ and each $k \in \{1, 2, \ldots, N\}$, we consider the integral equation

$$x(t) = e_k - \int_t^\infty \{F(s)x(s) + G(s)\,(x(s - r(s)) - x(s))\}\, ds. \qquad (3.1)$$

Lemma 1. *Assume that the conditions of Theorem* 1 *are met and that* $T \geq \tau$ *satisfies* (2.5). *Then, for each* $k \in \{1, 2, \ldots, N\}$, *(3.1) has a solution on* $[T - \rho, \infty)$ *satisfying* $\lim_{t\to\infty} x_k(t) = e_k$ *and* $x_k(t) = x_k(T)$ *for* $t \in [T - \rho, T]$.

Proof. Let $\mathbf{S}$ be the set of x in $C([T - \rho, \infty), \mathbb{C}^N)$ satisfying $x(t) = x(T)$ for $t \in [T - \rho, T]$, $x \in L^\infty(T, \infty)$ and $x(\cdot - r(\cdot)) - x \in L(T, \infty)$. Then $\mathbf{S}$ is a Banach space under the norm $\|x\|_{\mathbf{S}} = \max\{\|x\|_\infty, \beta\|x\|_0\}$, where

$$\|x\|_0 = \int_T^\infty |x(t - r(t)) - x(t)|dt$$

and $\beta > 0$ is chosen so that $\beta\varepsilon_T < 1$ and $\|F\|_1(T, \infty) + \frac{1}{\beta}\|G\|_\infty(T, \infty) < 1$. For any $x \in \mathbf{S}$ let

$$\boldsymbol{\mathcal{A}}x(t) = -\int_t^\infty \{F(s)x(s) + G(s)\,(x(s - r(s)) - x(s))\}\, ds \quad (t \geq T)$$

and $\boldsymbol{\mathcal{A}}x(t) = \boldsymbol{\mathcal{A}}x(T)$ for $t \in [T - \rho, T]$. Then $\boldsymbol{\mathcal{A}}x$ is continuous and bounded with

$$\|\boldsymbol{\mathcal{A}}x\|_\infty \leq \|F\|_1\|x\|_\infty + \|G\|_\infty\|x\|_0 \leq (\|F\|_1 + \frac{1}{\beta}\|G\|_\infty)\|x\|_{\mathbf{S}}.$$

Further,

$$|\boldsymbol{\mathcal{A}}x(t - r(t)) - \boldsymbol{\mathcal{A}}x(t)| \leq \int\limits_{\max\{\beta(t), T\}}^{\gamma(t)} |F(s)x(s) + G(s)\,(x(s - r(s)) - x(s))|\, ds,$$

where $\gamma(t) = \max\{t, t - r(t)\}$ and $\beta(t) = \min\{t, t - r(t)\}$, and so, with μ_T defined by (2.3),

$$\begin{aligned} &\int_T^\infty |\boldsymbol{\mathcal{A}}x(t - r(t)) - \boldsymbol{\mathcal{A}}x(t)|dt \leq \\ &\leq \int_T^\infty |F(s)x(s) + G(s)\,(x(s - r(s)) - x(s))|\,\mu_T(s)ds \leq \\ &\leq \varepsilon_T\left(\|F\|_1\|x\|_\infty + \|G\|_\infty\|x\|_0\right). \end{aligned}$$

Thus

$$\beta\|\boldsymbol{\mathcal{A}}x\|_0 \leq \|F\|_1\|x\|_\infty + \|G\|_\infty\|x\|_0.$$

Hence $\boldsymbol{\mathcal{A}}$ is a contraction on $\mathbf{S}$ and the conclusion follows. □

Lemma 2. *Assume that* $f : [R - \sigma_1, \infty) \to \mathbb{C}^N$ *satisfies*

$$|f(t)| \leq \omega(t) \|f\|_{[t-\sigma_1, t+\sigma_2]} \quad (t \geq R), \tag{3.2}$$

where $\omega : [R, \infty) \to (0,1)$ *is decreasing,* σ_1 *and* σ_2 *are positive constants, and* $\|f\|_{[a,b]} = \sup_{a \leq s \leq b} |f(s)|$. *If*

$$f(t) = o\left(\exp\left\{ -\frac{1}{\sigma_2} \int_R^{t-2\sigma_2} \ln \omega(s) ds \right\} \right) \tag{3.3}$$

as $t \to \infty$, *then*

$$|f(t)| \leq \exp\left\{ \frac{1}{\sigma_1} \int_R^t \ln \omega(s) ds \right\} \|f\|_{[R-\sigma_1, R]} \quad (t \geq R). \tag{3.4}$$

Proof. From (3.2),

$$\|f\|_{[t,t+\sigma_2]} \leq \omega(t) \|f\|_{[t-\sigma_1, t+2\sigma_2]}$$

and hence, since $\omega(t) < 1$,

$$\|f\|_{[t,t+\sigma_2]} \leq \omega(t) \max\left\{ \|f\|_{[t-\sigma_1,t]}, \|f\|_{[t+\sigma_2, t+2\sigma_2]} \right\}. \tag{3.5}$$

We shall show by induction that, for all $t \geq R$ and non-negative integer k,

$$\|f\|_{[t,t+\sigma_2]} \leq \\ \leq \max\left\{ \omega(t) \|f\|_{[t-\sigma_1,t]}, \left(\prod_{j=0}^{k} \omega(t+j\sigma_2) \right) \|f\|_{[t+(k+1)\sigma_2, t+(k+2)\sigma_2]} \right\} \tag{3.6}$$

and

$$\|f\|_{[t+k\sigma_2, t+(k+1)\sigma_2]} \leq \\ \leq \omega(t+k\sigma_2) \max\left\{ \|f\|_{[t-\sigma_1,t]}, \|f\|_{[t+(k+1)\sigma_2, t+(k+2)\sigma_2]} \right\}. \tag{3.7}$$

For $k = 0$ these both reduce to (3.5). Suppose they hold for k. Then, from (3.7) with t replaced by $t + \sigma_2$,

$$\|f\|_{[t+(k+1)\sigma_2, t+(k+2)\sigma_2]} \leq \omega(t + (k+1)\sigma_2) \times \\ \times \max\left\{ \|f\|_{[t+\sigma_2-\sigma_1, t+\sigma_2]}, \|f\|_{[t+(k+2)\sigma_2, t+(k+3)\sigma_2]} \right\}. \tag{3.8}$$

Further,

$$\|f\|_{[t+\sigma_2-\sigma_1, t+\sigma_2]} \leq \max\left\{ \|f\|_{[t-\sigma_1,t]}, \|f\|_{[t,t+\sigma_2]} \right\} \leq \\ \leq \max\left\{ \|f\|_{[t-\sigma_1,t]}, \|f\|_{[t+(k+1)\sigma_2, t+(k+2)\sigma_2]} \right\}$$

by (3.6), and this together with (3.8) implies (3.7) with k replaced by $k+1$. By substitution we also obtain (3.6) with k replaced by $k+1$.

Now, from (3.3), for any $\varepsilon > 0$,

$$|f(t)| \leq \varepsilon \exp\left\{ -\frac{1}{\sigma_2} \int_R^{t-2\sigma_2} \ln \omega(s) ds \right\}$$

for all sufficiently large t. Hence, for any $t \geq R$ and large enough k,

$$\omega(t+\sigma_2)\omega(t+2\sigma_2)\cdots\omega(t+k\sigma_2)\|f\|_{[t+(k+1)\sigma_2, t+(k+2)\sigma_2]} \leq$$
$$\leq \varepsilon \exp\left\{ \sum_{j=1}^{k} \ln \omega(t+j\sigma_2) - \frac{1}{\sigma_2} \int_R^{t+k\sigma_2} \ln \omega(s) ds \right\} \leq$$
$$\leq \varepsilon \exp\left\{ -\frac{1}{\sigma_2} \int_R^{t} \ln \omega(s) ds \right\},$$

and so, by letting $k \to \infty$ in (3.6), we have

$$\|f\|_{[t,t+\sigma_2]} \leq \omega(t)\|f\|_{[t-\sigma_1,t]}.$$

From this it follows that $\|f\|_{[t,t+\sigma_1]} \leq \omega(t)\|f\|_{[t-\sigma_1,t]}$ for all $t \geq R$ and hence that, for $R + m\sigma_1 \leq t < R + (m+1)\sigma_1$, where m is an integer,

$$|f(t)| \leq \prod_{j=0}^{m} \omega(t-j\sigma_1)\|f\|_{[R-\sigma_1,R]} \leq \exp\left\{ \frac{1}{\sigma_1} \int_R^{t} \ln \omega(s) ds \right\} \|f\|_{[R-\sigma_1,R]}. \quad \square$$

Remark. There are functions that satisfy (3.2) but not (3.3). For instance,

$$f(t) = \exp\left\{ -\frac{1}{\sigma_2} \int_R^{t+\delta} \ln \omega(s) ds \right\} \quad (t \geq R)$$

with $f(t) = f(R)$ for $t \in [R - \sigma_1, R]$, is such a function for any $\delta \geq 0$.

With the help of Lemmas 1 and 2 and an adaptation of the method used in Driver [4], we now complete the proof of Lemma 2.

Proof of Theorem 1. Since each solution of (3.1) is also a solution of (2.1), by Lemma 1 (2.1) has solutions x_n $(n = 1, 2, \ldots, N)$ with asymptotic behavior (2.6) and $X = (x_1, x_2, \ldots, x_N)$ is a matrix solution of (2.1). With r_1 given by (2.8), since (2.2) implies $\lim_{t\to\infty} r_1(t) = 0$, there is $T_1 \geq T + 2\rho$ such that X^{-1} is bounded and $\xi < 1$ on $[T_1 - 3\rho, \infty)$, where

$$\xi(t) = \left(\|X\|_\infty \|X^{-1}\|_\infty \|G\|_\infty\right)^2 \sup_{s \geq t} |r_1(s)|. \tag{3.9}$$

The substitution of $x = Xy$ into (2.1) leads to

$$\frac{d}{dt} y(t) = X^{-1}(t)G(t)X(t - r(t))\{y(t - r(t)) - y(t)\} \quad (t \geq T_1 - 2\rho). \tag{3.10}$$

Integration of (3.10) from t to $t-r(t)$ produces

$$z(t)=\int_t^{t-r(t)} X^{-1}(s)G(s)X(s-r(s))z(s)ds \quad (t\ge T_1-\rho),$$

where $z(t)=y(t-r(t))-y(t)$. Thus $z(t)$ satisfies

$$\begin{aligned} z(t)=&\int_t^{t-r(t)} X^{-1}(s)G(s)X(s-r(s))\times \\ &\times\int_s^{s-r(s)} X^{-1}(u)G(u)X(u-r(u))z(u)du\,ds \end{aligned} \tag{3.11}$$

for $t\ge T_1$. By (3.9) and (3.11),

$$|z(t)|\le \xi(t)\|z\|_{[t-2\rho,t+2\sigma]} \quad (t\ge T_1). \tag{3.12}$$

Now suppose that $x(t)=O(e^{\gamma t})$ as $t\to\infty$ for some $\gamma>0$. Then certainly

$$x(t)=o\Bigg(\exp\Bigg\{-\frac{1}{2\sigma}\int_{T_1}^{t-5\sigma}\ln\xi(s)ds\Bigg\}\Bigg) \tag{3.13}$$

as $t\to\infty$, and, by the boundedness of X^{-1}, (3.13) also holds if x is replaced by y. Hence, from the definition of z,

$$z(t)=o\Bigg(\exp\Bigg\{-\frac{1}{2\sigma}\int_{T_1}^{t-4\sigma}\ln\xi(s)ds\Bigg\}\Bigg).$$

By Lemma 2, z satisfies

$$|z(t)|\le\exp\Bigg\{\frac{1}{2\rho}\int_{T_1}^{t}\ln\xi(s)ds\Bigg\}\|z\|_{[T_1-2\rho,T_1]} \quad (t\ge T_1). \tag{3.14}$$

For any $\delta>0$ and $t\ge T_1$, (3.10) and (3.14) imply that

$$|y(t+\delta)-y(t)|\le\gamma_0\int_t^{t+\delta}\exp\Bigg\{\frac{1}{2\rho}\int_{T_1}^{s}\ln\xi(\theta)d\theta\Bigg\}ds$$

for some constant γ_0. Since ξ decreases to zero as $t\to\infty$ and $\xi(t)<1$, we have

$$\begin{aligned} |y(t+\delta)-y(t)|\le&\gamma_0\Bigg(\exp\Bigg\{\frac{1}{2\rho}\int_{T_1}^{t}\ln\xi(\theta)d\theta\Bigg\}\Bigg)\times \\ &\times\int_t^{t+\delta}\exp\big\{(s-t)(2\rho)^{-1}\ln\xi(T_1)\big\}ds\le \\ &\le\gamma_1\exp\Bigg\{\frac{1}{2\rho}\int_{T_1}^{t}\ln\xi(\theta)d\theta\Bigg\} \quad (t\ge T_1), \end{aligned} \tag{3.15}$$

where γ_1 is a constant. Further, for any $\beta > 0$,

$$\exp\left\{\frac{1}{2\rho}\int_{T_1}^{t} \ln\xi(\theta)d\theta\right\} = o\big(e^{-\beta t}\big) \tag{3.16}$$

as $t \to \infty$. Then, by (3.15), (3.16) and the Cauchy convergence principle, there is $c \in \mathbb{C}^N$ such that $\lim_{t\to\infty} y(t) = c$. Letting $\delta \to \infty$ in (3.15), we obtain $|y(t) - c| = o\big(e^{-\beta t}\big)$ as $t \to \infty$. Thus

$$x(t) = X(t)(y(t) - c) + X(t)c = X(t)c + o\big(e^{-\beta t}\big)\,,$$

i.e., every exponentially bounded solution of (2.1) has form (2.7). □

Remark. Any solution of (2.1) that is not of form (2.7) must fail to satisfy (3.13). This, by (3.9), relates the separation of large solutions of (2.1) from those in (2.7) to the rate at which r_1 tends to zero.

Lemma 3. *If $r \in L^p(\tau,\infty)$ for some $p \in [1,\infty)$, then $r_n \in L^{\beta_n}(\tau,\infty)$, where*

$$\frac{1}{\beta_n} = \frac{n+1}{p} - \frac{n}{p^2} + \frac{n-1}{p^3} - \cdots + (-1)^n\frac{1}{p^{n+1}}.$$

Proof. We show by induction that

$$|r_n(t)| \le |r(t)|^{\eta_n} q_n(t), \tag{3.17}$$

where

$$\eta_n = 1 - \frac{1}{p} + \frac{1}{p^2} - \cdots + (-1)^n\frac{1}{p^n},$$

for some function q_n satisfying

$$\|q_n\|_{[t-\rho,t+\sigma]} \in L^{\beta_{n-1}}(\tau,\infty). \tag{3.18}$$

We take $q_0(t) = 1$ for $t \ge \tau$ and $\beta_{-1} = \infty$, and take $q_n(t) = 0$ for $t < \tau$ and all n. The conclusion of the lemma follows from (3.17) with (3.18) since

$$\frac{\eta_n}{p} + \frac{1}{\beta_{n-1}} = \frac{1}{\beta_n}. \tag{3.19}$$

Suppose then that (3.17) with (3.18) holds for a particular value of n. Then

$$\begin{aligned}|r_{n+1}(t)| &\le \|q_n\|_{[t-\rho,t+\sigma]}\left|\int_t^{t-r(t)} |r(s)|^{\eta_n} ds\right| \le \\ &\le \|q_n\|_{[t-\rho,t+\sigma]}|r(t)|^{1-\eta_n/p}\left|\int_t^{t-r(t)} |r(s)|^p ds\right|^{\eta_n/p}\end{aligned}$$

by Hölder's inequality. Thus we have

$$|r_{n+1}(t)| \le |r(t)|^{\eta_{n+1}} q_{n+1}(t)$$

with
$$q_{n+1}(t) = \|q_n\|_{[t-\rho,t+\sigma]} \left| \int_t^{t-r(t)} |r(s)|^p ds \right|^{\eta_n/p} \quad (t \geq \tau).$$
Then
$$\|q_{n+1}\|_{[t-\rho,t+\sigma]} \leq \|q_n\|_{[t-2\rho,t+2\sigma]} \left| \int_{t-2\rho}^{t+2\sigma} |r(s)|^p ds \right|^{\eta_n/p}$$
and there is in $L^{\beta_n}(\tau,\infty)$ by (3.19). $\square$

Proof of Corollary to Theorem 2. By Lemma 3, $r_n \in L^{\beta_n}(\tau,\infty)$. We have
$$\frac{1}{\beta_n} = \frac{(n+1)p^{n+2} + (n+2)p^{n+1} + (-1)^n}{p^{n+1}(p+1)^2},$$
so $\beta_n \leq 1$ if $n \geq p$. Thus $r_M \in L(\tau,\infty)$ for $M \geq p$. $\square$

§ 4. Elimination of Large Solutions

In this section we give a condition under which (1.1) or (2.1) cannot have any large solutions.

Lemma 4. *Let $\{t_n\}$ be a real sequence satisfying $t_n \to \infty$ as $n \to \infty$ and let $\omega : [R,\infty) \to (0,1)$ be decreasing. If $f : [R-\sigma_1,\infty) \to \mathbb{C}^N$ satisfies*
$$|f(t)| \leq \omega(t)\|f\|_{[t-\sigma_1,t_n]} \tag{4.1}$$
for each $t_n \geq R$ and all $t \in [R,t_n]$, where $\sigma_1 > 0$ is a constant, then inequality (3.4) *holds.*

Proof. For each $t_n \geq R$ and any $t \in [R,t_n]$, the conditions imply that
$$|f(t)| \leq \|f\|_{[t,t_n]} \leq \omega(t)\|f\|_{[t-\sigma_1,t_n]}.$$
Hence $\|f\|_{[t-\sigma_1,t_n]} = \|f\|_{[t-\sigma_1,t]}$ and
$$|f(t)| \leq \omega(t)\|f\|_{[t-\sigma_1,t]}. \tag{4.2}$$
Since $\lim_{n\to\infty} t_n = \infty$, (4.2) actually holds for all $t \geq R$. The conclusion follows as in the last part of the proof of Lemma 2. $\square$

Definition. A point $t_0 \geq \tau$ is said to be a zero-advanced point (ZAP) of (1.1), (1.3), or (2.1) if $t - r(t) \leq t_0$ for all $t \in [\tau, t_0]$.

Note that in the retarded case, $r(t) \geq 0$, every point $t_0 \geq \tau$ is a ZAP. As a simple illustration of the case when r changes sign, consider (1.1) with $r(t) = t^{-\alpha}\sin\beta t$, where $\alpha,\beta > 0$. Clearly, the points $t = (2n+1)\pi/\beta$ are ZAPs for all sufficiently large integer n, and the corollary of Theorem 4, below, will apply.

Theorem 3. *Assume that* (2.1) *has a sequence* $\{t_n\}$ *of ZAPs satisfying* $\lim_{n\to\infty} t_n = \infty$. *Then, under the conditions of Theorem* 1, *every solution of* (2.1) *has the asymptotic representation* (2.7).

Proof. We refer to the proof of Theorem 1. Take any $t_k \geq T_1$. Since $t - r(t) \leq t_k$ for all $t \in [\tau, t_k]$, from (3.11) we derive

$$|z(t)| \leq \xi(t)\|z\|_{[t-2\rho,t_k]} \quad (T_1 \leq t \leq t_k) \tag{4.3}$$

instead of (3.12). Then Lemma 4 implies (3.14). Therefore, every solution of (2.1) on $[T-\rho,\infty)$ has representation (2.7). □

Theorem 4. *Assume that* (1.1) *has a sequence* $\{t_n\}$ *of ZAPs satisfying* $\lim_{n\to\infty} t_n = \infty$ *and that the conditions of Theorem* 2 *are met. Then every solution of* (1.1) *has the asymptotic representation* (2.7) *with* X *composed of* N *solutions* (2.10).

Proof. By the proof of Theorem 2, (1.1) is transformed to (2.15). Then the conclusion follows from Theorem 3. □

Corollary. *Assume that* $r \in L^p(\tau,\infty)$ *for some* $p \in (1,\infty)$ *and that* (1.1) *has a sequence* $\{t_n\}$ *of ZAPs satisfying* $\lim_{n\to\infty} t_n = \infty$. *Then, for* $M \geq p$, $r_M \in L(\tau,\infty)$ *and* (1.1) *has* N *solutions of form* (2.10). *Moreover, every solution on* $[T-\rho,\infty)$ *can be represented by* (2.7) *with* X *composed of* N *solutions* (2.10).

§ 5. Asymptotic Representation for Solutions of (1.3)

As a consequence of Theorems 1 and 3, some results for equation (1.3) can be easily obtained. Extend A and B to $\mathbb{R}$ by letting $A(t) = B(t) = 0$ for $t < \tau$ and, for $t \in \mathbb{R}$, let

$$B_{k+1}(t) = B(t)\exp\left\{\int_t^{t-r(t)} (B_k(s) + A(s))\,ds\right\} \quad (k \geq 0) \tag{5.1}$$

with $B_0(t) = B(t)$.

Theorem 5. *Let* A *and* B *have the forms stated at the end of* §1. *Assume that* A *and* B *are bounded,* (2.2) *holds, and* $r_M \in L(\tau,\infty)$. *Then, for sufficiently large* $T \geq \tau$, (1.3) *has* N *solutions* x_n *on* $[T-\rho,\infty)$ *satisfying*

$$x_n(t) = \exp\left\{\int_T^t (B_M(s) + A(s))\,ds\right\}\{e_n + o(1)\} \quad (1 \leq n \leq N) \tag{5.2}$$

as $t \to \infty$, *where* B_M *is determined by* (5.1). *Moreover, every exponentially bounded solution of* (1.3) *has the asymptotic representation* (2.7) *for arbitrary* $\beta > 0$, *where* X *is now composed of* x_n *in* (5.2). *If, in addition, there* *is a sequence* $\{t_n\}$ *of ZAPs of* (1.3) *satisfying* $\lim_{n\to\infty} t_n = \infty$, *then every solution on* $[T-\rho,\infty)$ *has form* (2.7).

The proof of Theorem 5 is similar to that of Theorems 2 and 4.

When A is not bounded, the conclusion of Theorem 5 may not be true. However, it is possible to determine the asymptotic behavior of solutions of (1.3) under other conditions. Let $\tilde{A}(t) = \int_t^{t-r(t)} A(s)ds$.

Corollary. *Assume that* B *and* $\tilde{A}$ *are bounded,* (2.2) *holds, and* $r_M \in L(\tau,\infty)$. *Then, for sufficiently large* $T \geq \tau$, (1.3) *has* N *solutions* x_n *on* $[T-\rho,\infty)$ *satisfying*

$$x_n(t) = \exp\left\{\int_T^t (B_{M+1}(s) + A(s))\,ds\right\}\{e_n + o(1)\} \ \ (1 \leq n \leq N) \qquad (5.3)$$

as $t \to \infty$, *where* B_{M+1} *is given by* (5.1). *Moreover, if* x *is a solution of* (1.3) *of order* $\exp\left\{\int_T^t A(s)ds + \lambda t\right\}$ *for some* $\lambda > 0$, *then* x *has the asymptotic form*

$$x(t) = X(t)c + o\left(\exp\left\{\int_T^t A(s)ds - \beta t\right\}\right) \qquad (5.4)$$

as $r \to \infty$, *where* X *consists of* N *solutions* (5.3), $c \in \mathbb{C}^N$ *depends on* x *and* $\beta > 0$ *is arbitrary. Further, if there is a sequence* $\{t_n\}$ *of ZAPs of* (1.3) *satisfying* $\lim_{n\to\infty} t_n = \infty$, *then every solution on* $[T-\rho,\infty)$ *has form* (5.4).

Proof. Let

$$x(t) = \exp\left\{\int_T^t (B(s) + A(s))\,ds\right\}y(t).$$

Then (1.3) is changed to

$$\frac{d}{dt}y(t) = -B(t)y(t) + B_1(t)y(t-r(t)). \qquad (5.5)$$

The conclusion follows by applying Theorem 5 to (5.5). □

Remarks. (i) The condition $r_M \in L(\tau,\infty)$ in Theorem 5 and its corollary will be met for some M if $r \in L^p(\tau,\infty)$ for some $p \in [1,\infty)$; M can be chosen as in the corollary of Theorem 2 or the remark following that corollary.

(ii) We note that, if Λ is a diagonal matrix whose elements satisfy a suitable dichotomy condition and if R and S are small at infinity, then the equation

$$\frac{d}{dt}x(t) = \{\Lambda(t) + R(t)\}x(t) + S(t)x(t-r(t)) \qquad (5.6)$$

can be treated by the methods of Cassell and Hou [5].

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