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ON THE REPRESENTATION OF NUMBERS BY THE DIRECT SUMS OF SOME BINARY QUADRATIC FORMS

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ABSTRACT. The systems of bases are constructed for the spaces of cusp forms $S_k(\Gamma_0(3),\chi)$ $(k\geq 6)$, $S_k(\Gamma_0(7),\chi)$ $(k\geq 3)$ and $S_k(\Gamma_0(11),\chi)$ $(k\geq 3)$. Formulas are obtained for the number of representation of a positive integer by the sum of k binary quadratic forms of the kind $x_1^2+x_1x_2+x_2^2$ $(6\leq k\leq 17)$, of the kind $x_1^2+x_1x_2+2x_2^2$ $(3\leq k\leq 11)$ and of the kind $x_1^2+x_1x_2+3x_1^2$ $(3\leq k\leq 7)$.

Let F_k denote a direct sum of k binary quadratic forms F_1 with the same negative discriminant $-q$ (q is prime $\equiv 3 \pmod 4$). The question of the representation of natural numbers by the direct sum of some binary quadratic forms was for the first time considered by Petersson [1]. In particular, he constructed the basis for the space of cusp forms $S_k(\Gamma_0(3),\chi)$ for an arbitrary integer $k\geq 6$ using the Dedekind η-function and obtained formulas for the number of representation $r(n,F_k)$ of the positive integer n by F_k when $F_1=x_1^2+x_1x_2+x_2^2$ and $2\leq k\leq 6$. Lomadze [2, 3] obtained formulas for $r(n,F_k)$ when $F_1=x_1^2+x_1x_2+x_2^2$ $(2\leq k\leq 17)$ and $F_1=x_1^2+x_1x_2+2x_2^2$ $(2\leq k\leq 11)$ and showed that they obey a fairly definite law. To this end, he constructed the bases of the spaces $S_k(\Gamma_0(3),\chi)$ and $S_k(\Gamma(7),\chi)$ apart for each k using a generalized multiple theta series. Merzlyakov [4] constructed the basis of the space $S_k(\Gamma_0(3),\chi)$ for an arbitrary integer $k\geq 6$ and obtained formulas for $r(n,F_k)$ when $F_1=x_1^2+x_1x_2+x_2^2$ $(2\leq k\leq 11)$. But these formulas do not obey any law.

In the present paper, using the generalized multiple theta series, the systems of bases for the space of cusp forms $S_k(\Gamma_0(3),\chi)$ with an integer $k\geq 6$ and for the spaces $S_k(\Gamma_0(7),\chi)$ and $S_k(\Gamma_0(11),\chi)$ with an integer $k\geq 3$ are constructed. After that, these bases are used to obtain the formulas for $r(n,F_k)$ when $F_1=x_1^2+x_1x_2+x_2^2$ $(6\leq k\leq 17)$, $F_1=x_1^2+x_1x_2+2x_2^2$

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$(3 \le k \le 11)$ and $F_1 = x_1^2 + x_1x_2 + 3x_2^2$ $(3 \le k \le 7)$. These formulas obey a fairly definite law.

1.1. Let

$$Q(X) = Q(x_1, \dots, x_f) = \sum_{1 \le r \le s \le f} b_{rs} x_r x_s$$

be a positive quadratic form with f (f is even) variables and integral coefficients b_{rs}. Let D denote a determinant of the quadratic form

$$2Q(X) = \sum_{r,s=1}^{f} a_{rs} x_r x_s \quad (a_{rr} = 2b_{rr}; \quad a_{sr} = a_{rs} = b_{rs}, \quad r < s).$$

A_{rs} is a cofactor of a_{rs} in the matrix $A = (a_{ij})$. One calls $\Delta = (-1)^{f/2} D$ the discriminant of $Q(X)$. $N = \Delta/\delta$ is the level of $Q(X)$, where $\delta =$ g.c.d.$(\frac{A_{rs}}{2}, A_{rr})_{(r,s=1,\dots,f)}$ $\cdot$ $\chi(d)$ is the character of $Q(X)$, i.e., if Δ is a perfect square, then $\chi(d) = 1$; if Δ is not a perfect square and $2 \nmid \Delta$, then $\chi(d) = (\frac{d}{|\Delta|})$ for $d > 0$, and $\chi(d) = (-1)^{f/2}\chi(-d)$ for $d < 0$ ($(\frac{\Delta}{d})$ is the Jacobi symbol). A positive quadratic form with f variables of level N with the character χ is called a quadratic form of type $(\frac{f}{2}, N, \chi)$. Further, q denotes an odd prime number, $z = \exp(2\pi i\tau)$, $z_N = \exp(\frac{2\pi i\tau}{N})$, $\operatorname{Im}\tau > 0$.

Let $\Gamma(1)$ denote a full modular group and Γ any subgroup of a finite index in $\Gamma(1)$. In particular, if $N \in \mathbb{N}$, then

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \;\middle|\; c \equiv 0 \pmod{N} \right\},$$
$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \;\middle|\; a \equiv d \equiv 1 \pmod{N} \right\},$$
$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N) \;\middle|\; b \equiv 0 \pmod{N} \right\}.$$

Let $G_k(\Gamma, \chi)$ and $S_k(\Gamma, \chi)$ denote the spaces of entire modular and cusp forms, respectively, of type (k, Γ, χ). If $\Gamma = \Gamma(N)$ or $\Gamma = \Gamma_1(N)$, then $\chi(d) = 1$; hence we shall write $G_k(\Gamma)$ instead of $G_k(\Gamma, \chi)$. As is known ([5], p. 109), if $F(\tau) \in G_k(\Gamma, \chi)$, then in the neighborhood of the cusp $\zeta = i\infty$

$$F(\tau) = \sum_{m=m_0 \ge 0}^{\infty} a_m z^m, \quad a_{m_0} \ne 0.$$

The order of an entire modular form $F(\tau) \ne 0$ of type (k, Γ, χ) at the cusp $\zeta = i\infty$ with respect to Γ is called the number ([5], p. 41)

$$\operatorname{ord}\big(F(\tau), i\infty, \Gamma\big) = m_0. \tag{1.1}$$

Let $F(\tau)$ be any function on $\mathbb{H}=\{\tau\in\mathbb{C}|\operatorname{Im}\tau>0\}$ and $k\in\mathbb{Z}$. Then, for any matrix $L=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma(1)$, let

$$F(\tau)\big|_k L=(c\tau+d)^{-k}F(L\tau)\quad(\tau\in\mathbb{H}).$$

Let

$$\theta(\tau;Q(X),P_\nu(X),h)=\sum_{n_i\equiv h_i \pmod N} P_\nu(n_1,\dots,n_f)z_N^{\frac{1}{N}Q(n_1,\dots,n_f)},$$

$$\theta(\tau;Q(X),P_\nu(X))=\sum_{X\in\mathbb{Z}^f}P_\nu(X)z^{Q(X)}=\sum_{n=1}^{\infty}\Big(\sum_{Q(X)=n}P_\nu(X)\Big)z^n.$$

Here $Q(X)=\frac{1}{2}\sum\limits_{r,s=1}^{f}a_{rs}x_rx_s$ is a quadratic form of type $(\frac{f}{2},N,\chi)$; $P_\nu(X)$ is a spherical function of order ν with respect to $Q(X)$; $n_1,\dots,n_f$ are integers; $h=(h_1,\dots,h_f)$, where h_i are integers satisfying the conditions

$$\sum_{s=1}^{f}a_{rs}h_s\equiv 0 \pmod N\quad(r=1,\dots,f).$$

Lemma 1 ([5], p. 217). *If $\theta(\tau;Q(X),P_\nu(X),h)$ is not identically equal to zero, then*

$$\theta(\tau;Q(X),P_\nu(X),h)\in G_{\nu+f/2}(\Gamma(N)).$$

Lemma 2 ([5], p. 218). *If $L=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma_0(N)$, then*

$$\theta(\tau;Q(X),P_\nu(X),h)\big|_{\nu+f/2}L=$$
$$=\chi(d)\exp\Big(\frac{2\pi iab}{N^2}\,Q(h)\Big)\theta(\tau;Q(X),P_\nu(X),ah).$$

Note ([5], p. 210; [6], p. 856) that

$$\theta(\tau;Q(X),P_\nu(X),h)=\theta(\tau;Q(X),P_\nu(X),g)\ \text{ for }\ h\equiv g\pmod N, \tag{1.2}$$
$$\theta(\tau;Q(X),P_\nu(X),-h)=(-1)^\nu\theta(\tau;Q(X),P_\nu(X),h), \tag{1.3}$$
$$\theta(\tau;Q(X),P_\nu(X),0)=N^\nu\theta(\tau;Q(X),P_\nu(X)). \tag{1.4}$$

Lemma 3 ([6], p. 855).

$$\theta(\tau;Q(X),P_\nu(X))\in G_{\nu+f/2}(\Gamma_0(N),\chi)$$

and when $\nu>0$

$$\theta(\tau;Q(X),P_\nu(X))\in S_{\nu+f/2}(\Gamma_0(N),\chi).$$

Lemma 4 ([5], p. 114; [7], Ch. 3, §1). *Let* $F(\tau) \in G_k(\Gamma_1(N))$ *and*

$$F(\tau) = \sum_{m=0}^{\infty} a_m z^m.$$

If all $a_m = 0$ *when* $m \leq \frac{kN^2}{24} \prod\limits_{p|N} (1 - \frac{1}{p^2})$, *then* $F(\tau)$ *is identically equal to zero.*

Lemma 5 ([6], p. 846). *If* $Q_1(X)$ *and* $Q_2(X)$ *are quadratic forms of types* (k_1, N, χ_1) *and* (k_2, N, χ_2), *respectively, then* $Q_1(X) \oplus Q_2(X)$ *is a quadratic form of type* $(k_1 + k_2, N, \chi_1\chi_2)$.

As is known ([6], pp. 874, 875, 817), if $Q(X)$ is a quadratic form of type $(k, q, 1)$, $2|k$, $k > 2$, then

$$\Delta = q^{2l}, \quad 1 \leq l \leq k-1, \tag{1.5}$$

and the Eisenstein series

$$E(\tau, Q(X)) = 1 + \sum_{n=1}^{\infty} \big(\alpha\sigma_{k-1}(n) z^n + \beta\sigma_{k-1}(n) z^{qn}\big) \tag{1.6}$$

corresponds to it, where

$$\alpha = \frac{i^k}{\rho_k} \cdot \frac{q^{k-l} - i^k}{q^k - 1}, \quad \beta = \frac{1}{\rho_k} \cdot \frac{q^k - i^k q^{k-l}}{q^k - 1}, \tag{1.7}$$

$\sigma_{k-1}(n) = \sum\limits_{d|n} d^{k-1}$, $\rho_k = (-1)^{k/2} \cdot \frac{(k-1)!}{(2\pi)^k}\, \zeta(k)$ ($\zeta(k)$ is the zeta function of Riemann).

If $Q(X)$ is a quadratic form of type (k, q, χ), $k \geq 2$, $\chi(-1) = (-1)^k$, then ([6], pp. 877, 818)

$$\Delta = (-1)^{\frac{q-1}{2}} q^{2l+1}, \quad 0 \leq l \leq k-1, \tag{1.8}$$

and the Eisenstein series

$$E(\tau, Q(X)) = 1 + \frac{1}{A_k(q)} \sum_{n=1}^{\infty} \bigg((-1)^{k/2} q^{k-l-1} \sum_{\delta d = n} \chi(\delta) d^{k-1} + \\ + \sum_{d|n} \chi(d) d^{k-1}\bigg) z^n \tag{1.9}$$

corresponds to it, where

$$A_k(q) = (-1)^{k/2} \cdot \frac{q^{k-\frac{1}{2}}}{(2\pi)^k}\, (k-1)! \sum_{n=1}^{\infty} \chi(n) n^{-k}.$$

To any positive quadratic form $Q(X)$, as is well-known, corresponds the theta series

$$\theta(\tau, Q(X)) = 1 + \sum_{n=1}^{\infty} r(n, Q(X)) z^n = \sum_{X \in \mathbb{Z}^f} z^{Q(X)}, \tag{1.10}$$

where $r(n, Q(X))$ denotes the number of representations of a positive integer n by the form $Q(X)$.

Lemma 6 ([6], pp. 874, 875). *If $Q(X)$ is a quadratic form of type $(k,q,1)$ or of type (k,q,χ), then if $k>2$,*

$$\theta(\tau, Q(X)) - E(\tau, Q(X))$$

is the cusp form of type $(k,\Gamma_0(q),1)$ or of type $(k,\Gamma_0(q),\chi)$, respectively.

Lemma 7 ([8], p.76). *Let $\eta(\tau)$ denote the η-function of Dedekind. Then if $q>3$*

$$f_q(\tau) = \frac{\eta^q(q\tau)}{\eta(\tau)} = z^{\frac{q^2-1}{24}} \prod_{n=1}^{\infty} \frac{(1-z^{qn})^q}{1-z^n} \in G_{\frac{q-1}{2}}(\Gamma_0(q),\chi),$$

where $\chi = \chi(d) = (\frac{d}{q})$.

1.2. **Lemma 8.** *Let*

$$Q^{(q)}_{\frac{q-1}{2}} = Q^{(q)}_{\frac{q-1}{2}}(X) = q \sum_{1\le i\le j\le q-2} x_i x_j + q \sum_{1\le i\le q-2} x_i x_{q-1} + \frac{q-1}{2} x_{q-1}^2,$$
$$h^{(q)} = \Big(1, 2, \ldots, \frac{q-1}{2}, \frac{q+3}{2}, \frac{q+5}{2}, \ldots, q-1, \frac{q(1-q)}{2}\Big).$$

Then

a) *if $q=3$*

$$\theta^3(\tau; Q^{(3)}_1, 1, h^{(3)}) \in G_3(\Gamma_0(3),\chi), \quad \mathrm{ord}(\theta^3(\tau; Q^{(3)}_1, 1, h^{(3)}), i\infty, \Gamma_0(3)) = 1;$$

b) *if $q = 5, 7, 11$ and 13*

$$\theta\big(\tau; Q^{(q)}_{\frac{q-1}{2}}, 1, h^{(q)}\big) \in G_{\frac{q-1}{2}}(\Gamma_0(q),\chi),$$
$$\mathrm{ord}\,\big(\theta(\tau; Q^{(q)}_{\frac{q-1}{2}}, 1, h^{(q)}), i\infty, \Gamma_0(q)\big) = \frac{q^2-1}{24}.$$

Proof. In [9] it is shown that a quadratic form $Q^{(q)}_{\frac{q-1}{2}}$ is of type $(\frac{q-1}{2}, q, \chi)$. It is easy to verify that if $Q^{(q)}_{\frac{q-1}{2}} = \frac{1}{2}\sum\limits_{r,s=1}^{q-1} a_{rs}x_rx_s$, then

$$\sum_{s=1}^{q-1} a_{rs}h^{(q)} \equiv 0 \pmod q \quad \text{and} \quad Q^{(q)}_{\frac{q-1}{2}}(h^{(q)}) = q^2\,\frac{q^2-1}{24}.$$

a) By Lemma 1, $\theta^3(\tau; Q_1^{(3)}, 1, h^{(3)}) \in G_3(\Gamma(3))$. Let $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(3)$. Then $(a,c) = 1$, i.e., $a \equiv 1$ or $-1 \pmod 3$ and, as $\nu = 0$, from Lemma 2 and (1.2), (1.3) we get

$$\theta^3(\tau; Q_1^{(3)}, 1, h^{(3)})\big|_3 L = \chi^3(d)\exp\Big(\frac{6\pi iab}{9}\cdot 3\Big)\theta^3(\tau; Q_1^{(3)}, 1, ah^{(3)}) = \\ = \chi(d)\theta^3(\tau; Q_1^{(3)}, 1, h^{(3)}).$$

Hence,by definition, $\theta^3(\tau; Q_1^{(3)}, 1, h^{(3)}) \in G_3(\Gamma_0(3), \chi)$. Using the computer we get

$$\theta^3(\tau; Q_1^{(3)}, 1, h^{(3)}) = 27z + 81z^2 + 243z^3 + \cdots .$$

Consequently, by (1.1),

$$\operatorname{ord}(\theta^3(\tau; Q_1^{(3)}, 1, h^{(3)}), i\infty, \Gamma_0(3)) = 1.$$

b) By Lemma 1, $\theta(\tau; Q^{(q)}_{\frac{q-1}{2}}, 1, h^{(q)}) \in G_{\frac{q-1}{2}}(\Gamma(q))$. If $L \in \Gamma_1(q) \subset \Gamma_0(q)$, then $a \equiv d \equiv 1 \pmod q$, hence by Lemma 2 and (1.2)

$$\theta(\tau; Q^{(q)}_{\frac{q-1}{2}}, 1, h^{(q)})\big|_{\frac{q-1}{2}} L = \\ = \chi(d)\exp\Big(\frac{2\pi iab}{q^2}\cdot Q^{(q)}_{\frac{q-1}{2}}(h^{(q)})\Big)\theta(\tau; Q^{(q)}_{\frac{q-1}{2}}, 1, ah^{(q)}) = \\ = \theta(\tau; Q^{(q)}_{\frac{q-1}{2}}, 1, h^{(q)}).$$

That is, by definition, $\theta(\tau; Q^{(q)}_{\frac{q-1}{2}}, 1, h^{(q)}) \in G_{\frac{q-1}{2}}(\Gamma_1(q))$. By Lemma 7,

$$f_q(\tau) = \frac{\eta^q(q\tau)}{\eta(\tau)} \in G_{\frac{q-1}{2}}(\Gamma_0(q), \chi) \subset G_{\frac{q-1}{2}}(\Gamma_1(q)).$$

Hence

$$\psi_q(\tau) = f_q(\tau) - \frac{1}{q}\,\theta(\tau; Q^{(q)}_{\frac{q-1}{2}}, 1, h^{(q)}) = \sum_{m=0}^{\infty} c_m^{(q)} z^m \in G_{\frac{q-1}{2}}(\Gamma_1(q)).$$

If $q = 5, 7, 11$ and 13, using the computer we get that all $c_m^{(q)} = 0$, when $m \le \frac{q-1}{2}\cdot\frac{q^2-1}{24}$. Hence,by Lemma 4, $\psi_q(\tau)$ is identically equal to zero, i.e.,

$\theta(\tau; Q^{(q)}_{\frac{q-1}{2}}, 1, h^{(q)}) = qf_q(\tau) \in G_{\frac{q-1}{2}}(\Gamma_0(q), \chi)$. Using the computer we get

$$\theta(\tau; Q^{(q)}_{\frac{q-1}{2}}, 1, h^{(q)}) = \begin{cases} 5z + 5z^2 + 10z^3 + \cdots & \text{when } q = 5, \quad (1.11) \\ 7z^2 + 7z^3 + \cdots & \text{when } q = 7, \quad (1.12) \\ 11z^5 + \cdots & \text{when } q = 11, \quad (1.13) \\ 13z^7 + \cdots & \text{when } q = 13. \quad (1.14) \end{cases}$$

By (1.11)–(1.14) and (1.1)

$$\operatorname{ord}(\theta(\tau; Q^{(q)}_{\frac{q-1}{2}}, 1, h^{(q)}), i\infty, \Gamma_0(q)) = \frac{q^2 - 1}{24}. \qquad \square$$

Further Φ_{km} denotes the direct sum of m quadratic forms Φ_k.

Lemma 9. *Let $l_1, l_2, t_1, t_2 \in \mathbb{N} \cup \{0\}$, $X \in \mathbb{Z}^{l_1}$, $Y \in \mathbb{Z}^{l_2+t_1}$, $V \in \mathbb{Z}^{t_2}$, $W = (X, Y, Z) \in \mathbb{Z}^{l_1+l_2+t_1+t_2}$, F_1 and Φ_1 are positive quadratic forms. If $\theta(\tau; F_{l_1}, P_1(X)) \in S_{k_1}(\Gamma_0(q), \chi)$, $\theta(\tau; F_{l_2} \oplus \Phi_{t_1},\ P_2(Y)) \in S_{k_2}(\Gamma_0(q), \chi)$, $\theta(\tau; \Phi_{t_2}, P_3(V)) \in S_{k_3}(\Gamma_0(q), \chi)$, then*

$$\begin{aligned} &\theta(\tau; F_{l_1}, P_1(X))\theta(\tau; F_{l_2} \oplus \Phi_{t_1}, P_2(Y))\theta(\tau; \Phi_{t_2}, P_3(V)) = \\ &= \theta(\tau; F_{l_1+l_2} \oplus \Phi_{t_1+t_2}, P(W)) \in S_{k_1+k_2+k_3}(\Gamma_0(q), \chi), \end{aligned}$$

where

$$P(W) = P_1(X)P_2(Y)P_3(V).$$

Proof directly follows from (1.4) and Lemma 3. $\square$

In the sequel let

$$\sigma_t^*(n) = \begin{cases} \sigma_t(n) & \text{if } q \nmid n, \\ \sigma_t(n) + (-q)^{\frac{t+1}{2}} \sigma_t(\frac{n}{q}) & \text{if } q | n; \end{cases}$$

$$\rho_t^*(n) = q^{\frac{t}{2}} \sum_{\delta d = n} \Big(\frac{\delta}{q}\Big) d^t + (-1)^{\frac{t}{2}} \sum_{d|n} \Big(\frac{d}{q}\Big) d^t.$$

If F_k is a quadratic form of type (k, q, χ), then by Lemma 3,

$$\theta(\tau, F_k) = \theta(\tau; F_k, 1) \in G_k(\Gamma_0(q), \chi). \qquad (1.15)$$

It follows from (1.10) and (1.1) that

$$\operatorname{ord}(\theta(\tau, F_k), i\infty, \Gamma_0(q)) = 0. \qquad (1.16)$$

Using the formula ([6], pp. 824, 825)

$$A_k(q) = \frac{\chi(2)(k-1)!}{1 - \chi(2)2^k} \cdot q^{k-1} \sum_{1 \le l \le \frac{q-1}{2}} \chi(l) \Big\{ 2^k g_k\Big(\frac{l}{q}\Big) - g_k\Big(\frac{2l}{q}\Big) \Big\},$$

where

$$g_k(x) = \frac{x^k}{k!} - \frac{x^{k-1}}{2(k-1)!} - 4 \sum_{1 \le r \le \frac{k}{2}} \frac{r\rho_{2r} x^{k-2r}}{(k-2r)!(2r)!}$$

are the Bernoulli polynomials defined on interval $0 < x < 1$, we get

$$A_3(11) = -3, \quad A_5(11) = \frac{3 \cdot 25 \cdot 17}{11}, \quad A_7(11) = -9 \cdot 17 \cdot 71. \tag{1.17}$$

2. Let

$$F_1 = x_1^2 + x_1 x_2 + x_2^2.$$

In [2] it is shown, that F_k is a quadratic form of type $(k, 3, \chi)$ ($\chi = \chi(d) = (\frac{d}{3})$ when $2 \nmid k$ and $\chi = 1$ when $2|k$) and

$$\theta(\tau, F_1) = 1 + 6z + 0 \cdot z^2 + \cdots . \tag{2.1}$$

Applyng the Jacobi method it is easy to show that the linear transformation

$$\begin{aligned} x_1 &= y_1 - \frac{1}{\sqrt{3}}\, y_2, \quad x_2 = \frac{2}{\sqrt{3}}\, y_2, \\ x_3 &= y_3 - \frac{1}{\sqrt{3}}\, y_4, \quad x_4 = \frac{2}{\sqrt{3}}\, y_4 \end{aligned} \tag{2.2}$$

reduces the quadratic form F_2 into the normal form.

Lemma 10. a) $\varphi = \varphi(x_1, \dots, x_4) = x_2^4 + x_4^4 - 6x_2^2 x_4^2$ *is a spherical function of order* 4 *with respect to* F_2;

b) $\theta(\tau; F_2, \varphi) = 8z - 48z^2 + 72z^3 + 32z^4 + \cdots \in S_6(\Gamma_0(3), 1)$;

c) $\operatorname{ord}(\theta(\tau; F_2, \varphi), i\infty, \Gamma_0(3)) = 1$.

Proof. Using transformation (2.2) we get

$$\varphi(x_1, \dots, x_4) = \frac{16}{9}\, y_2^4 + \frac{16}{9}\, y_4^4 - \frac{32}{3}\, y_2^2 y_4^2 = \varphi(y_1, \dots, y_4),$$

$$\sum_{i=1}^{4} \frac{\partial^2 \varphi}{\partial y_i^2} = \Big(\frac{16 \cdot 4 \cdot 3}{9} - \frac{32 \cdot 2}{3}\Big)(y_2^2 + y_4^2) = 0.$$

Consequently, by definition, φ is a spherical function of order 4 with respect to F_2 and, by Lemma 3, $\theta(\tau; F_2, \varphi) \in S_6(\Gamma_0(3), 1)$.

Using the computer we get

$$\begin{aligned} \theta(\tau; F_2, \varphi) &= \sum_{n=1}^{\infty} \Bigg(\sum_{F_2 = n} x_2^4 + x_4^4 - 6x_2^2 x_4^2 \Bigg) z^n = \\ &= 8z - 48z^2 + 72z^3 + 32z^4 + \cdots . \end{aligned} \tag{2.3}$$

It follows from (2.3) and (1.1) that

$$\operatorname{ord}(\theta(\tau; F_2, \varphi), i\infty, \Gamma_0(3)) = 1. \quad \square$$

Theorem 11. *Let $C = (c_{sr})$ $(s = 1, 2, 3; r = 1, \ldots, [\frac{k}{3}]-1)$ be the matrix whose elements are non-negative integers satisfying the conditions*

$$\left.\begin{array}{l} 6c_{1r} + 3c_{2r} + c_{3r} = k \\ c_{1r} + c_{2r} = r, \quad c_{1r} > 0 \end{array}\right\} \quad \Big(r = 1, \ldots, \Big[\frac{k}{3}\Big] - 1\Big). \tag{2.4, 2.5}$$

Then for $k \geq 6$ the system of functions

$$\theta^{c_{1r}}(\tau; F_2, \varphi)\theta^{3c_{2r}}(\tau; Q_1^{(3)}, 1, h^{(3)})\theta^{c_{3r}}(\tau, F_1) \quad \Big(r = 1, \ldots, \Big[\frac{k}{3}\Big] - 1\Big), \tag{2.6}$$

where $\theta(\tau; Q_1^{(3)}, 1, h^{(3)})$ is defined by Lemma 8, is the basis of the space $S_k(\Gamma_0(3), \chi)$.

Proof. By Lemmas 8–10 and (1.15), (2.4), functions (2.6) are cusp forms of type $(k, \Gamma_0(3), \chi)$. By Lemmas 10, 8 and (1.16), (2.5), we have

$$\operatorname{ord}(\theta^{c_{1r}}(\tau; F_2, \varphi)\theta^{3c_{2r}}(\tau; Q_1^{3)}, 1, h^{(3)})\theta^{c_{3r}}(\tau, F_1), i\infty, \Gamma_0(3)) = r$$
$$\Big(r = 1, \ldots, \Big[\frac{k}{3}\Big] - 1\Big).$$

Functions (2.6) are linearly independent because their orders at the cusp $i\infty$ are different. Hence the theorem is proved as it is known ([6], pp. 815, 816) that $\dim S_k(\Gamma_0(3), \chi) = [\frac{k}{3}] - 1$. $\square$

Corollary 12. *The system of functions*

$$\theta(\tau; F_2, \varphi)\theta^{k-6}(\tau, F_1) \quad \textit{when} \quad 6 \leq k \leq 8 \quad \textit{and}$$
$$\theta^r(\tau; F_2, \varphi)\theta^{k-6r}(\tau, F_1) \quad \Big(r = 1, \ldots \Big[\frac{k}{6}\Big]\Big),$$
$$\theta^{[\frac{k}{3}]-r}(\tau; F_2, \varphi)\theta^{3(2r-[\frac{k}{3}])}(\tau; Q_1^{(3)}, 1, h^{(3)})\theta^{3\{\frac{k}{3}\}}(\tau, F_1)$$
$$\Big(r = \Big[\frac{k}{6}\Big] + 1, \ldots, \Big[\frac{k}{3}\Big] - 1\Big), \quad \textit{when} \quad k \geq 9,$$

is the basis of the space $S_k(\Gamma_0(3), \chi)$.

By Lemma 6, $\theta(\tau, F_k) - E(\tau, F_k) \in S_k(\Gamma_0(3), \chi)$. Hence,by Corollary 12, there are constants $\alpha_r^{(k)}$ such that

$$\theta(\tau, F_k) = E(\tau, F_k) + \sum_{r=1}^{[\frac{k}{6}]} \alpha_r^{(k)}\theta^r(\tau; F_2, \varphi)\theta^{k-6r}(\tau, F_1) +$$
$$+ \sum_{r=[\frac{k}{6}]+1}^{[\frac{k}{3}]-1} \alpha_r^{(k)}\theta)^{[\frac{k}{3}]-r}(\tau; F_2, \varphi)\theta^{3(2r-[\frac{k}{3}])}(\tau; Q_1^{(3)}, 1, h^{(3)})\theta^{3\{\frac{k}{3}\}}(\tau, F_1). \tag{2.7}$$

Using the expansions of $\theta(\tau, F_k)$, $E(\tau, F_k)$ [2] and the expansions of $\theta(\tau; F_2, \varphi)$ and $\theta(\tau; Q_1^{(3)}, 1, h^{(3)})$ and equating, in both parts of (2.7), the coefficients of $z, z^2, \dots, z^l$, where $l = \dim S_k(\Gamma_0(3), \chi)$, we get the systems of linear equations. By solving these systems we find the constants $\alpha_r^{(k)}$. Equating, in both parts of (2.7), the coefficients of z^n, we obtain formulas for the arithmetical function $r(n, F_k)$.

Let $G_i(x_1, \dots, x_{4i}) = \prod_{t=1}^{i}(x_{4t-2}^4 + x_{4t}^4 - 6x_{4t-2}^2 x_{4t}^2)$. Then formulas for $r(n, F_k)$ when $k = 6, 7, \dots, 17$ have the following form:

$$
\begin{aligned}
r(n,F_6) &= \frac{252}{13}\,\sigma_5^*(n) + \frac{27}{13}\sum_{F_2=n} G_1(x_1,\dots,x_4),\\
r(n,F_7) &= \frac{3}{7}\,\rho_6^*(n) + \frac{27}{7}\sum_{F_3=n} G_1(x_1,\dots,x_4),\\
r(n,F_8) &= \frac{240}{41}\,\sigma_7^*(n) + \frac{216}{41}\sum_{F_4=n} G_1(x_1,\dots,x_4),\\
r(n,F_9) &= \frac{27}{809}\,\rho_8^*(n) + \frac{5184}{809}\sum_{F_5=n} G_1(x_1,\dots,x_4),\\
r(n,F_{10}) &= \frac{12}{11}\,\sigma_9^*(n) + \frac{81}{11}\sum_{F_6=n} G_1(x_1,\dots,x_4),\\
r(n,F_{11}) &= \frac{3}{1847}\,\rho_{10}^*(n) + \frac{15147}{1847}\sum_{F_7=n} G_1(x_1,\dots,x_4),\\
r(n,F_{12}) &= \frac{6552}{73\cdot 691}\,\sigma_{11}^*(n) + \frac{453168}{73\cdot 691}\sum_{F_8=n} G_1(x_1,\dots,x_4) -\\
&\quad - \frac{36450}{73\cdot 691}\sum_{F_4=n} G_2(x_1,\dots,x_8),\\
r(n,F_{13}) &= \frac{3}{7\cdot 13^2\cdot 47}\,\rho_{12}^*(n) + \frac{541836}{7\cdot 13^2\cdot 47}\sum_{F_9=n} G_1(x_1,\dots,x_4) -\\
&\quad - \frac{138510}{7\cdot 13^2\cdot 47}\sum_{F_5=n} G_2(x_1,\dots,x_8),\\
r(n,F_{14}) &= \frac{12}{1093}\,\sigma_{13}^*(n) + \frac{11475}{1093}\sum_{F_{10}=n} G_1(x_1,\dots,x_4) -\\
&\quad - \frac{5832}{1093}\sum_{F_6=n} G_2(x_1,\dots,x_8),
\end{aligned}
$$

$$r(n,F_{15})=\frac{9}{419\cdot 16519}\,\rho^*_{14}(n)+\frac{77863977}{419\cdot 16519}\sum_{F_{11}=n}G_1(x_1,\dots,x_4)-$$
$$-\frac{63425916}{419\cdot 16519}\sum_{F_7=n}G_2(x_1,\dots,x_8),$$
$$r(n,F_{16})=\frac{480}{193\cdot 3617}\,\sigma^*_{15}(n)+\frac{8376912}{193\cdot 3617}\sum_{F_{12}=n}G_1(x_1,\dots,x_4)-$$
$$-\frac{9669456}{193\cdot 3617}\sum_{F_8=n}G_2(x_1,\dots,x_8),$$
$$r(n,F_{17})=\frac{3}{23\cdot 401\cdot 13687}\,\rho^*_{16}(n)+\frac{1609496352}{23\cdot 401\cdot 13687}\sum_{F_{13}=n}G_1(x_1,\dots,x_4)-$$
$$-\frac{2434381776}{23\cdot 401\cdot 13687}\sum_{F_9=n}G_2(x_1,\dots,x_8).$$

3. Let
$$F_1=x_1^2+x_1x_2+2x_2^2.$$
In [3] it is shown that the quadratic form F_k is of type $(k,7,\chi)$ ($\chi=\chi(d)=(\frac{d}{7})$ when $2\nmid k$ and $\chi=1$ when $2|k$).

The following Lemma is proved exactly as Lemma 10.

Lemma 13. a) $\varphi=\varphi(x_1,x_2)=x_1^2-2x_2^2$ *is a spherical function of order* 2 *with respect to* F_1*;*

b) $\theta(\tau;F_1,\varphi)=2z-6z^2+0\cdot z^3+10z^4+0\cdot z^5+\dots\in S_3(\Gamma_0(7),\chi)$*;*

c) $\operatorname{ord}(\theta(\tau;F_1,\varphi),i\infty,\Gamma_0(7))=1$.

Theorem 14. *Let* $C=(c_{sr})$ $(s=1,2,3;\ r=1,\dots,2[\frac{k}{3}]-1)$ *be the matrix whose elements are non-negative integers satisfying the conditions*
$$\left.\begin{array}{l}3c_{1r}+3c_{2r}+c_{3r}=k \\ c_{1r}+2c_{2r}=r,\quad c_{1r}>0\end{array}\right\}\quad\Big(r=1,\dots,2\Big[\frac{k}{3}\Big]-1\Big).\qquad\begin{array}{r}(3.1)\\(3.2)\end{array}$$
Then for $k\ge 3$ *the system of functions*
$$\theta^{c_{1r}}(\tau;F_1,\varphi)\theta^{c_{2r}}(\tau;Q_3^{(7)},1,h^{(7)})\theta^{c_{3r}}(\tau,F_1)\ \ \Big(r=1,\dots,2\Big[\frac{k}{3}\Big]-1\Big),\qquad(3.3)$$
where $\theta(\tau;Q_3^{(7)},1,h^{(7)})$ *is defined by Lemma* 8*, is the basis of the space* $S_k(\Gamma_0(7),\chi)$.

Proof. By Lemmas 8, 9, 13 and (1.15),(3.1), functions (3.3) are cusp forms of type $(k,\Gamma_0(7),\chi)$. By Lemmas 8, 13 and (1.16), (3.2), we have
$$\operatorname{ord}(\theta^{c_{1r}}(\tau;F_1,\varphi)\theta^{c_{2r}}(\tau;Q_3^{(7)},1,h^{(7)})\theta^{c_{3r}}(\tau,F_1),i\infty,\Gamma_0(7))=r$$
$$\Big(r=1,\dots,2\Big[\frac{k}{3}\Big]-1\Big).$$

Functions (3.3) are linearly independent because their orders at the cusp $i\infty$ are different. Hence the theorem is proved as it is known ([6], pp. 815, 816) that $\dim S_k(\Gamma_0(7),\chi)=2[\frac{k}{3}]-1$. □

Corollary 15. *The system of functions*

$$\theta(\tau;F_1,\varphi)\theta^{k-3}(\tau,F_1) \quad \textit{when} \quad 3\le k\le 5 \quad \textit{and}$$
$$\theta^r(\tau;F_1,\varphi)\theta^{k-3r}(\tau,F_1) \quad \Big(r=1,\dots,\Big[\frac{k}{3}\Big]\Big),$$
$$\theta^{2[\frac{k}{3}]-r}(\tau;F_1,\varphi)\theta^{r-[\frac{k}{3}]}(\tau;Q_3^{(7)},1,h^{(7)})\theta^{3\{\frac{k}{3}\}}(\tau,F_1)$$
$$\Big(r=\Big[\frac{k}{3}\Big]+1,\dots,2\Big[\frac{k}{3}\Big]-1\Big), \quad \textit{when} \quad k\ge 6,$$

is the basis of the space $S_k(\Gamma_0(7),\chi)$.

By Lemma 6, $\theta(\tau,F_k)-E(\tau,F_k)\in S_k(\Gamma_0(7),\chi)$. Hence by Corollary 15 there are constants $\alpha_r^{(k)}$ such that

$$\theta(\tau,F_k)=E(\tau,F_k)+\sum_{r=1}^{[\frac{k}{3}]}\alpha_r^{(k)}\theta^r(\tau;F_1,\varphi)\theta^{k-3r}(\tau,F_1)+$$
$$+\sum_{r=[\frac{k}{3}]+1}^{2[\frac{k}{3}]-1}\alpha_r^{(k)}\theta^{2[\frac{k}{3}]-r}(\tau;F_1,\varphi)\theta^{r-[\frac{k}{3}]}(\tau;Q_3^{(7)},1,h^{(7)})\theta^{3\{\frac{k}{3}\}}(\tau,F_1).$$

Then, using the expansions of $\theta(\tau,F_k)$, $E(\tau,F_k)$ [3], the expansions of $\theta(\tau;F_1,\varphi)$ and $\theta(\tau;Q_3^{(7)},1,h^{(7)})$, exactly as above, we get formulas for $r(n,F_k)$.

Let $G_i(x_1,\dots,x_{2i})=\prod_{t=1}^{i}(x_{2t-1}^2-x_{2t}^2)$. Then formulas for $r(n,F_k)$ when $k=3,4,\dots,11$ have the form

$$r(n,F_3)=\frac{7}{8}\,\rho_2^*(n)+\frac{3}{8}\sum_{F_1=n}G_1(x_1,x_2),$$
$$r(n,F_4)=\frac{24}{5}\,\sigma_3^*(n)+\frac{8}{5}\sum_{F_2=n}G_1(x_1,x_2),$$
$$r(n,F_5)=\frac{1}{16}\,\rho_4^*(n)+\frac{55}{16}\sum_{F_3=n}G_1(x_1,x_2),$$
$$r(n,F_6)=\frac{28}{19}\,\sigma_5^*(n)+\frac{100}{19}\sum_{F_4=n}G_1(x_1,x_2)+\frac{18}{19}\sum_{F_2=n}G_2(x_1,\dots,x_4),$$
$$r(n,F_7)=\frac{1}{584}\,\rho_6^*(n)+\frac{3917}{584}\sum_{F_5=n}G_1(x_1,x_2)+\frac{501}{292}\sum_{F_3=n}G_2(x_1,\dots,x_4),$$

$$r(n,F_8)=\frac{240}{1201}\,\sigma_7^*(n)+\frac{9488}{1201}\sum_{F_6=n}G_1(x_1,x_2)+$$
$$+\frac{2288}{1201}\sum_{F_4=n}G_2(x_1,\dots,x_4),$$
$$r(n,F_9)=\frac{7}{2^5\cdot 8831}\,\rho_8^*(n)+\frac{2534921}{2^5\cdot 8831}\sum_{F_7=n}G_1(x_1,x_2)+$$
$$+\frac{57299}{2^3\cdot 8831}\sum_{F_5=n}G_2(x_1,\dots,x_4)-\frac{27675}{2^3\cdot 8831}\sum_{F_3=n}G_3(x_1,\dots,x_6),$$
$$r(n,F_{10})=\frac{44}{2801}\,\sigma_9^*(n)+\frac{27988}{2801}\sum_{F_8=n}G_1(x_1,x_2)-$$
$$-\frac{5522}{2801}\sum_{F_6=n}G_2(x_1,\dots,x_4)-\frac{4320}{2801}\sum_{F_4=n}G_3(x_1,\dots,x_6),$$
$$r(n,F_{11})=\frac{1}{2^3\cdot 11\cdot 73\cdot 701}\,\rho_{10}^*(n)+\frac{49527061}{2^3\cdot 11\cdot 73\cdot 701}\sum_{F_9=n}G_1(x_1,x_2)-$$
$$-\frac{1319075}{2^2\cdot 73\cdot 701}\sum_{F_7=n}G_2(x_1,\dots,x_4)-$$
$$-\frac{4070853}{2\cdot 11\cdot 73\cdot 701}\sum_{F_5=n}G_3(x_1,\dots,x_6).$$

4. Let

$$F_1=x_1^2+x_1x_2+3x_2^2,$$
$$\widetilde{F}_2=2(x_1^2+x_2^2+x_3^2+x_4^2)+2x_1x_3+x_1x_4+x_2x_3-2x_2x_4.$$

It is easy to verify that the quadratic form F_k is of type $(k,11,\chi)$ ($\chi=\chi(d)=(\frac{d}{11})$ when $2\nmid k$ and $\chi=1$ when $2|k$) and the quadratic form $\widetilde{F}_2$ is of type $(2,11,1)$.

The following Lemmas are proved exactly as Lemma 10.

Lemma 16. a) $\varphi_1=\varphi_1(x_1,x_2)=x_1^2-3x_2^2$ *is a spherical function of order* 2 *with respect to* F_1*;*

b) $\theta(\tau;F_1,\varphi_1)=2z-10z^3+8z^4-2z^5+0\cdot z^6+\cdots\in S_3(\Gamma_0(11),\chi)$*;*

c) $\operatorname{ord}(\theta(\tau;F_1,\varphi_1),i\infty,\Gamma_0(11))=1$.

Lemma 17. a) $\varphi_2=\varphi_2(x_1,\dots,x_4)=x_2^2+4x_2x_3$ *is a spherical function of order* 2 *with respect to* $\widetilde{F}_2$*;*

b) $\theta(\tau;\widetilde{F}_2,\varphi_2)=4z^2-16z^3+8z^4+32z^5-20z^6+\cdots\in S_4(\Gamma_0(11),1)$*;*

c) $\operatorname{ord}(\theta(\tau;\widetilde{F}_2,\varphi_2),i\infty,\Gamma_0(11))=2$.

Lemma 18. a) $\varphi_3 = \varphi_3(x_1,\dots,x_6) = x_6^2 + 2x_2x_3$ *is a spherical function of order* 2 *with respect to* $\widetilde{F}_2 \oplus F_1$*;*

b) $\theta(\tau; \widetilde{F}_2 \oplus F_1, \varphi_3) = -8z^3 - 24z^4 + 64z^5 + 0\cdot z^6 + \dots \in S_5(\Gamma_0(11),\chi)$*;*

c) $\operatorname{ord}(\theta(\tau; \widetilde{F} \oplus F_1, \varphi_3), i\infty, \Gamma_0(11)) = 3.$

Lemma 19. a) $\varphi_4 = \varphi_4(x_1,\dots,x_4) = 2x_2^4 - 3x_4^4 - 6x_2^2x_3^2 + 4x_3x_4^3 + 6x_3^2x_4^2 + 6x_2^2x_4^2$ *is a spherical function of order* 4 *with respect to* F_2*;*

b) $\theta(\tau; F_2, \varphi_4) = -48z^4 + 48z^5 + 96z^6 + \dots \in S_6(\Gamma_0(11), 1)$*;*

c) $\operatorname{ord}(\theta(\tau; F_2, \varphi_4), i\infty, \Gamma_0(11)) = 4.$

Lemma 20. a) $\varphi_5 = \varphi_5(x_1,\dots,x_6) = 4x_2x_3^3 + 3x_3^2x_5^2 - 24x_2x_3x_6^2 - 3x_3^2x_6^2 - 6x_5^2x_6^2 - 4x_5x_6^3 + x_6^4$ *is a spherical function of order* 4 *with respect to* $\widetilde{F}_2 \oplus F_1$*;*

b) $\theta(\tau; \widetilde{F}_2 \oplus F_1, \varphi_5) = 528z^6 + \dots \in S_7(\Gamma_0(11),\chi)$*;*

c) $\operatorname{ord}(\theta(\tau; \widetilde{F}_2 \oplus F_1, \varphi_5), i\infty, \Gamma_0(11)) = 6.$

Theorem 21. *Let* $C = (c_{sr})$ $(s = 1,\dots,7;\ r = 1,\dots,k-2)$ *be the matrix whose elements are non-negative integers satisfying the following conditions:*

$$\left.\begin{array}{l} 7c_{1r}+4c_{2r}+5c_{3r}+6c_{4r}+3c_{5r}+5c_{6r}+c_{7r}=k \\ 6c_{1r}+2c_{2r}+3c_{3r}+4c_{4r}+c_{5r}+5c_{6r}=m_r, \\ \sum\limits_{s=1}^{5} c_{sr} > 0;\ \ c_{1r}\cdot c_{3r}=0;\ \ c_{1r}, c_{3r}=0 \ \ or \ \ 1 \end{array}\right\} \ (r=1,\dots,k-2), \qquad \begin{array}{r}(4.1)\\(4.2)\end{array}$$

$m_r = r$ *when* $r = 1,\dots,k-3$*;* $m_{k-2} = k-2$ *if* $k \neq 5l+2$ *and* $m_{k-2} = k-1$ *if* $k = 5l+2$, $l \in \mathbb{N}$*. Further, let* $\varphi^{(r)} = \varphi_5^{c_{1r}}\varphi_2^{c_{2r}}\varphi_3^{c_{3r}}\varphi_4^{c_{4r}}\varphi_1^{c_{5r}}$ *be the direct product of functions* φ_s *defined by Lemmas* 16–20 $(\varphi_s^0 \equiv 1)$*. Then for each* $k \geq 3$, $k \neq 8$*, the system of functions*

$$\theta(\tau; \widetilde{F}_{2v_r} \oplus F_{t_r}, \varphi^{(r)}))\theta^{c_{6r}}(\tau; Q_5^{(11)}, 1, h^{(11)})\theta^{c_{7r}}(\tau, F_1) \\ (r = 1,\dots,k-2), \tag{4.3}$$

where $\theta(\tau; Q_5^{(11)}, 1, h^{(11)})$ *is defined by Lemma* 8 *and*

$$v_r = c_{1r} + c_{2r} + c_{3r}, \quad t_r = c_{1r} + c_{3r} + c_{4r} + c_{5r} \tag{4.4}$$

is the basis of the space $S_k(\Gamma_0(11),\chi)$*.*

Proof. It follows from the definition of $\varphi^{(r)}$, (4.4) and Lemma 9 that

$$\theta(\tau; \widetilde{F}_{2v_r} \oplus F_{t_r}, \varphi^{(r)}) = \theta^{c_{1r}}(\tau; \widetilde{F}_2 \oplus F_1, \varphi_5)^{c_{2r}}\theta(\tau; \widetilde{F}_2, \varphi_2) \times \\ \times \theta^{c_{3r}}(\tau; \widetilde{F}_2 \oplus F_1, \varphi_3)\theta^{c_{4r}}(\tau; F_2, \varphi_4)\theta^{c_{5r}}(\tau; F_1, \varphi_1). \tag{4.5}$$

Therefore (4.5), (1.15), (4.1) and Lemmas 8, 16–20 imply that functions (4.3) are cusp forms of type $(k, \Gamma_0(11), \chi)$. By Lemmas 16–20, 8 and (1.16), (4.2), we have

$$\operatorname{ord}(\theta(\tau; \widetilde{F}_{2v_r} \oplus F_{t_r}, \varphi^{(r)}))\theta^{c_{6r}}(\tau; Q_5^{(11)}, 1, h^{(11)})\theta^{c_{7r}}(\tau, F_1), i\infty, \Gamma_0(11)) = m,$$

$m = 1, 2, \ldots, k-2$ when $k \neq 5l+2$ and $m = 1, 2, \ldots, k-3, k-1$ when $k = 5l+2$, $l \in \mathbb{N}$. Functions (4.3) are linearly independent because their orders at the cusp $i\infty$ are different. Hence the theorem is proved as it is known ([6], p. 815, 816) that $\dim S_k(\Gamma_0(11), \chi) = k-2$. □

By Lemma 6, $\theta(\tau, F_k) - E(\tau, F_k) \in S_k(\Gamma_0(11), \chi)$. Hence by Theorem 21 there are constants $\alpha_r^{(k)}$ such that

$$\theta(\tau, F_k) = E(\tau, F_k) +$$
$$+ \sum_{r=1}^{k-2} \alpha_r^{(k)} \theta(\tau; \widetilde{F}_{2v_r} \oplus F_{t_r}, \varphi^{(r)})\theta(\tau; Q_5^{(11)}, 1, h^{(11)})\theta^{c_{7r}}(\tau, F_1).$$

From (1.5)–(1.10) and (1.17) when $q = 11$, using the computer we get the expansions of $\theta(\tau, F_k)$ and $E(\tau, F_k)$. Then exactly as above we get formulas for $r(n, F_k)$. When $k = 3, \ldots, 7$ they have the form

$$r(n, F_3) = \frac{1}{3}\rho_2^*(n) + \frac{4}{3}\sum_{F_1=n} x_1^2 - 3x_2^2,$$

$$r(n, F_4) = \frac{120}{61}\sigma_3^*(n) + \frac{184}{61}\sum_{F_2=n} x_1^2 - 3x_2^2 - \frac{88}{61}\sum_{\widetilde{F}_2=n} x_2^2 + 4x_2x_3,$$

$$r(n, F_5) = \frac{11}{3\cdot 5^2\cdot 17}\rho_4^*(n) + \frac{5704}{3\cdot 5^2\cdot 17}\sum_{F_3=n} x_1^2 - 3x_2^2 -$$
$$- \frac{3608}{3\cdot 5^2\cdot 17}\sum_{\widetilde{F}_2\oplus F_1=n} x_2^2 + 4x_2x_3,$$

$$r(n, F_6) = \frac{36}{5\cdot 19}\sigma_5^*(n) + \frac{552}{5\cdot 19}\sum_{F_4=n} x_1^2 - 3x_2^2 -$$
$$- \frac{528}{5\cdot 19}\sum_{\widetilde{F}_2\oplus F_2=n} x_2^2 + 4x_2x_3 - \frac{121}{5\cdot 19}\sum_{\widetilde{F}_2\oplus F_2=n} x_6^2 + 2x_2x_3 +$$
$$+ \frac{1331}{2\cdot 3\cdot 5\cdot 19}\sum_{F_2=n} 2x_2^4 - 3x_4^4 - 6x_2^2x_3^2 + 4x_3x_4^3 + 6x_3^2x_4^2 + 6x_2^2x_4^2,$$

$$r(n, F_7) = \frac{1}{3^2\cdot 17\cdot 71}\rho_6^*(n) + \frac{75376}{3^2\cdot 17\cdot 71}\sum_{F_5=n} x_1^2 - 3x_2^2 -$$

$$-\frac{94360}{3^2\cdot 17\cdot 71}\sum_{\widetilde{F}_2\oplus F_3=n} x_2^2+4x_2x_3-$$
$$-\frac{33187}{3^2\cdot 17\cdot 71}\sum_{\widetilde{F}_2\oplus F_3=n} x_6^2+2x_2x_3+$$
$$+\frac{365057}{2\cdot 3^3\cdot 17\cdot 71}\sum_{F_3=n} 2x_2^4-3x_4^4-6x_2^2x_3^2+4x_3x_4^3+6x_3^2x_4^2+6x_2^2x_4^2.$$

References

1. H. Petersson, Modulfunktionen und quadratische Formen. *Springer–Verlag, Berlin–Heidelberg–New York,* 1982.

2. G. A. Lomadze, On the representation of numbers by the sums of quadratic forms $x_1^2+x_1x_2+x_2^2$. (Russian) *Acta Arith.* **54**(1989), 9–36.

3. G. A. Lomadze, On the number of representations of numbers by a direct sum of quadratic forms of the kind $x_1^2+x_1x_2+2x_2^2$. (Russian) *Proc. Tbilisi Univ. Math. Mekh. Astron.* **299**(1990), 12–59.

4. A. S. Merzlyakov, The problem of G. A. Lomadze on the representation of numbers by the sum of binary quadratic forms of the kind x^2+xy+y^2. (Russian) *Deposited in VINITI* (*Vsesoyuzn. Inst. Nauch. Tekh. Inform. Acad. Nauk SSSR*) 26.04.89, No. 2753–B89.

5. B. Shoeneberg, Elliptic modular functions. *Springer-Verlag, Berlin–Heidelberg–New York,* 1974.

6. E. Hecke, Mathematische Werke. Zweite Aufl. *Vandenhoeck und Ruprecht, Göttingen,* 1970.

7. N. Koblitz, Introduction to elliptic curves and modular forms. *Graduate texts in mathematics,* 97. *Springer-Verlag, New York–Berlin,* 1984.

8. A. A. Klyachko, Modular forms and representations of symmetric groups. (Russian) *Zapiski Nauchn. Semin. LOMI* **116**(1982), 74–85.

9. G. L. Erkomaishvili, On the simultaneous representation of two integers as a sum of integers and their squares. *Bull. Acad. Sci. Georgia* **145**(1992), 17–20.

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