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BOUNDARY PROPERTIES OF FIRST-ORDER PARTIAL DERIVATIVES OF THE POISSON INTEGRAL FOR THE HALF-SPACE $\mathbb{R}^+_{k+1}$ $(k > 1)$

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ABSTRACT. Boundary properties of first-order partial derivatives of the Poisson integral are studied in the half-space $\mathbb{R}^+_{k+1}$ $(k > 1)$.

The boundary properties of the Poisson integral for a circle were thoroughly studied by Fatou [1]. In particular, he showed that the following theorems are valid:

Theorem A. *If there exists a finite $f'(x_0)$, then*

$$\lim_{re^{ix} \overset{\wedge}{\to} e^{ix_0}} \frac{\partial u(f;r,x)}{\partial x} = f'(x_0),$$

where $u(f;r,x)$ is the Poisson integral for a circle, and the symbol $re^{ix} \overset{\wedge}{\to} e^{ix_0}$ means that the point re^{ix} tends to e^{ix_0} along the paths which are non-tangential to the circumference (*see* [2], *p.* 100, *and* [3], *p.* 156).

Theorem B. *If there exists a finite or infinite $\mathcal{D}_1 f(x_0)$ which is a first symmetric derivative of f at the point x_0* (*see* [2], *p.* $99 - 100$), *i.e.,*

$$\mathcal{D}_1 f(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0-h)}{2h},$$

then

$$\lim_{r\to 1-} \frac{\partial u(f;r,x_0)}{\partial x} = \mathcal{D}_1 f(x_0).$$

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In [4] a continuous 2π-periodic function $f(x)$ is constructed such that $\mathcal{D}_1 f(x_0)=0$, but

$$\lim_{re^{ix}\overset{\wedge}{\to}e^{ix_0}} \frac{\partial u(f;r,x)}{\partial x}$$

does not exist. Thus it is shown that Theorem B cannot be strengthened in the sense of the existence of an angular limit.

An analogue of Theorem A for a half-plane $\mathbb{R}^2_+$ is proved in [5, Theorem 4], while an analogue of Theorem B given in [6, Theorem 1] shows that this theorem cannot be strengthened in the sense of the existence of an angular limit.

The question as to the validity of Fatou's theorem for a bicylinder was considered in [7], where it is proved that in the neighborhood of some point the density of the Poisson integral can have no smoothness that would ensure the existence of a boundary value of partial derivatives of the Poisson integral at the considered point. Furthermore, in this paper sufficient conditions are found for the convergence of first- and second- order partial derivatives of the Poisson integral for a bicylinder, and it is shown that the results obtained cannot be strengthened (in the definite sense).

The boundary properties of the integral $\mathcal{D}_k u(f;r,\vartheta_1,\vartheta_2,\dots,\vartheta_{k-2},\varphi)$ were studied in [8] (see also [9], p. 118), where $u(f;r,\vartheta_1,\vartheta_2,\dots,\vartheta_{k-2}\varphi)$ is the Poisson integral for the unit sphere in $\mathbb{R}^k$ $(k>2)$, and $\mathcal{D}_k$ is the Laplace operator on the sphere, i.e., the angular part of the Laplace operator written in terms of spherical coordinates (see [9], p. 14). The boundary properties of first- and second- order partial derivatives of the Poisson integral for the unit sphere in $\mathbb{R}^3$ are given a detailed consideration in [10, 11, 12], but for the half-space $\mathbb{R}^3_+$ in [13], [14], [15]. In [14] it is shown that there exists a continuous function of two variables $f(x,y)\in L(\mathbb{R}^2)$ which, at the point (x_0,y_0), has the partial derivatives $f'_x(x_0,y_0)$ and $f'_y(x_0,y_0)$, but the integrals $\frac{\partial u(f;x,y,z)}{\partial x}$ and $\frac{\partial u(f;x,y,z)}{\partial y}$ ($u(f;x,y,z)$ is the Poisson integral for $\mathbb{R}^3_+$) of this function have no values at the point (x_0,y_0) even along the normal.

Hence the question arises how to generalize the notion of derivatives of a function of many variables so that a Fatou type theorem would hold for the integral $u(f;x,x_{k+1})$ ($u(f;x,x_{k+1})$ is the Poisson integral for $\mathbb{R}^{k+1}_+$ $(k>1)$).

In this paper, the notion of a generalized partial derivative is introduced for a function of many variables and Fatou type theorems are proved on boundary properties of first-order partial derivatives of the Poisson integral for a half-space. These results complement and generalize the author's studies in [13], [14], [15]. In particular, in this paper it is shown that the boundary properties of derivatives of the Poisson integral for a half-space essentially depend on the sense in which the integral density is differentiable. Examples are constructed testifying to the fact that the results obtained are unimprovable (in the definite sense).

1. Notation, Definitions, and Auxiliary Propositions

The following notation is used in this paper:

$\mathbb{R}^k$ is a k-dimensional Euclidean space ($\mathbb{R} = \mathbb{R}^1$);

$x = (x_1, x_2, \dots, x_k)$, $t = (t_1, t_2, \dots, t_k)$, $x^0 = (x_1^0, x_2^0, \dots, x_k^0)$ are the points (vectors) of the space $\mathbb{R}^k$;

$(x,t) = \sum\limits_{i=1}^{k} x_i t_i$ is the scalar product;

$|x| = \sqrt{(x,x)}$; $x + t = (x_1 + t_1, x_2 + t_2, \dots, x_k + t_k)$;

e_i $(i = 1, 2, \dots, k)$ is the coordinate vector.

Let (see [16], p. 174) $M = \{1, 2, \dots, k\}$ $(k \in N,\ k \geq 2)$, B be an arbitrary subset from M and $B' = M \setminus B$. For any $x \in \mathbb{R}^k$ and an arbitrary set $B \subset M$, the symbol $x_{_B}$ denotes a point from $\mathbb{R}^k$ whose coordinates with indices from the set B coincide with the corresponding coordinates of the point x, while coordinates with indices from the set B' are zeros ($x_{_M} = x$, $B \setminus i = B \setminus \{i\}$); if $B = \{i_1, i_2, \dots, i_s\}$, $1 \leq s \leq k$ ($i_l < i_r$ for $l < r$), then $\overline{x}_{_B} = (x_{i_1}, x_{i_2}, \dots, x_{i_s}) \in \mathbb{R}^s$; $m(B)$ is the number of elements of the set B;

$\widetilde{L}(\mathbb{R}^k)$ is the set of functions $f(x) = f(x_1, x_2, \dots, x_k)$ such that

$$\frac{f(x)}{(1+|x|^2)^{\frac{k+1}{2}}} \in L(\mathbb{R}^k);$$

$\mathbb{R}_+^{k+1} = \{(x, x_{k+1}) \in \mathbb{R}^{k+1};\ \ x_{k+1} > 0\}$;

$u(f; x, x_{k+1})$ is the Poisson integral of the function $f(x)$ for the half-space $\mathbb{R}_+^{k+1}$, i.e.,

$$u(f; x, x_{k+1}) = \frac{x_{k+1}\Gamma(\frac{k+1}{2})}{\pi^{\frac{k+1}{2}}} \int\limits_{\mathbb{R}^k} \frac{f(t)\,dt}{(|t-x|^2 + x_{k+1}^2)^{\frac{k+1}{2}}}.$$

In investigating the boundary properties of the partial derivatives $\frac{\partial}{\partial\vartheta}u_f(r,\vartheta,\varphi)$ and $\frac{\partial}{\partial\varphi}u_f(r,\vartheta,\varphi)$ of the spherical Poisson integral $u_f(r,\vartheta,\varphi)$ for the summable function $f(\vartheta,\varphi)$ on the rectangle $[0,\pi]\times[0,2\pi]$, Dzagnidze introduced the notion of a dihedral-angular limit [10] which is applicable to $\mathbb{R}_+^{k+1}$ in the manner as follows: if the point $N \in \mathbb{R}_+^{k+1}$ converges to the point $\mathcal{P}(x^0, 0)$ under the condition $x_{k+1}(\sum\limits_{i\in B}(x_i - x_i^0)^2)^{-1/2} \geq C > 0$,[1] then we shall write $N(x, x_{k+1}) \underset{x_B}{\overset{\wedge}{\to}} \mathcal{P}(x^0, 0)$. When $B = M$, we have an angular convergence and thus we write $N(x, x_{k+1}) \overset{\wedge}{\to} \mathcal{P}(x^0, 0)$. Finally, the notation $N(x, x_{k+1}) \to \mathcal{P}(x^0, 0)$ means that the point $N(x, x_{k+1})$ remaining in $\mathbb{R}_+^{k+1}$ converges to $\mathcal{P}(x^0, 0)$ without any restrictions.

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[1] Here and further C denotes absolute positive constants which, generally speaking, may be different in different relations.

It is known that $\frac{\partial}{\partial\vartheta}u_f(r,\vartheta,\varphi)$ and $\frac{\partial}{\partial\varphi}u_f(r,\vartheta,\varphi)$ have dihedral-angular limits if partial derivatives of the function $f(\vartheta,\varphi)$ exist in a strong sense [10], [12]. This notion admits various generalizations when the function depends on three and more variables and we shall also discuss them below.

Let $u \in \mathbb{R}$. We shall consider the following derivatives of the function $f(x)$:

1. Denote the limit

$$\lim_{(u,\overline{x}_B)\to(0,\overline{x}_B^0)} \frac{f(x_B + x_{B'}^0 + ue_i) - f(x_B + x_{B'}^0)}{u}$$

by:

(a) $f'_{x_i}(x^0)$ for $B \neq \varnothing$;
(b) $\mathcal{D}_{x_i(\overline{x}_B)}f(x^0)$ for $i \in B'$;
(c) $\overline{\mathcal{D}}_{x_i(\overline{x}_B)}f(x^0)$ for $i \in B$.

2. Denote the limit

$$\lim_{(u,\overline{x}_B)\to(0,\overline{x}_B^0)} \frac{f(x_B + x_{B'}^0 + ue_i) - f(x_B + x_{B'}^0 - ue_i)}{2u}$$

by:

(a) $\mathcal{D}^*_{x_i}(x^0)$ for $B \neq \varnothing$;
(b) $\mathcal{D}^*_{x_i(\overline{x}_B)}f(x^0)$ for $i \in B'$;
(c) $\overline{\mathcal{D}}_{x_i(\overline{x}_B)}f(x^0)$ for $i \in B$.

The following propositions are valid:

(1) If $B_2 \subset B_1$, the existence of $\mathcal{D}_{x_i(\overline{x}_{B_1})}f(x^0)$ implies the existence of $\mathcal{D}_{x_i(\overline{x}_{B_2})}f(x^0)$ and $\mathcal{D}_{x_i(\overline{x}_{B_1})}f(x^0) = f'_{x_i}(x^0)$. The converse does not hold.

(2) The existence of $\overline{\mathcal{D}}_{x_i(\overline{x}_{B_1})}f(x^0)$ implies the existence of $\overline{\mathcal{D}}_{x_i(\overline{x}_{B_2})}f(x^0)$ and their equality.

(3) The existence of $\overline{\mathcal{D}}_{x_i(\overline{x}_B)}f(x^0)$ implies the existence of $\overline{\mathcal{D}}_{x_i(\overline{x}_{B_1})}f(x^0)$ and $\overline{\mathcal{D}}_{x_i(\overline{x}_B)}f(x^0) = \overline{\mathcal{D}}_{x_i(\overline{x}_{B\setminus i})}f(x^0) = f'_{x_i}(x^0)$.

(4) If $f'_{x_i}(x)$ is a continuous function at x^0, then for any $B \subset M$ all derivatives $\overline{\mathcal{D}}_{x_i(\overline{x}_B)}f(x^0)$ exist and

$$\overline{\mathcal{D}}_{x_i(\overline{x}_B)}f(x^0) = f'_{x_i}(x^0).$$

Indeed, by virtue of the Lagrange theorem

$$\frac{f(x_B + x_{B'}^0 + ue_i) - f(x_B + x_{B'}^0)}{u} = \frac{f'_{x_i}[x_B + x_{B'}^0 + \theta(x)ue_i]u}{u} = \\ = f'_{x_i}[x_B + x_{B'}^0 + \theta(x)ue_i], \quad 0 < \theta < 1.$$

Hence we conclude that statement (4) is valid.

(5) There exists a function $f(x)$ for which $\overline{\mathcal{D}}_{x_i(x)}f(x^0)$ exist, but on an everywhere dense set in the neighborhood of the point x^0 there are no $f'_{x_i}(x)$.

(6) If the function $f(x)$ has finite derivatives

$$\mathcal{D}_{x_1(x_2,\dots,x_k)}f(x^0),\ \mathcal{D}_{x_2(x_3,\dots,x_k)}f(x^0),\dots,\mathcal{D}_{x_{k-1}(x_k)}f(x^0)$$

at the point x^0, then its continuity at x^0 with respect to the argument x_k is a necessary and sufficient condition for $f(x)$ to be continuous at x^0 (see [12], p.15).

(7) The existence of the derivatives $\mathcal{D}_{x_1(x_2,\dots,x_k)}f(x^0)$, $\mathcal{D}_{x_2(x_3,\dots,x_k)}f(x^0)$, $\dots$, $\mathcal{D}_{x_1(x_2,\dots,x_k)}f(x^0)$ and $f'_{x_k}(x^0)$ implies the existence of the differential $df(x^0)$ (see [12], p. 16).

In what follows it will be assumed that $f\in\widetilde{L}(\mathbb{R}^k)$.

Lemma. *The equalities*

$$I_1=\int\limits_{\mathbb{R}^k}\frac{(t_i-x_i)f(t-t_ie_i+x_ie_i)dt}{(|t-x|^2+x_{k+1}^2)^{\frac{k+3}{2}}}=0,$$

$$I_2=\frac{(k+1)x_{k+1}\Gamma(\frac{k+1}{2})}{\pi^{\frac{k+1}{2}}}\int\limits_{\mathbb{R}^k}\frac{(t_i-x_i)^2dt}{(|t-x|^2+x_{k+1}^2)^{\frac{k+3}{2}}}=1$$

hold for any (x,x_{k+1}).

Proof. We have

$$I_1=\int\limits_{\mathbb{R}^{k-1}}f(x+t-t_ie_i)\,dS(\overline{t}_{M\setminus i})\int\limits_{-\infty}^{\infty}\frac{t_i\,dt_i}{(|t|^2+x_{k+1}^2)^{\frac{k+3}{2}}}=0.$$

Further, if we use the spherical coordinates, $\rho,\theta_1,\dots,\theta_{k-2},\varphi$, we shall have

$$I_2=\frac{(k+1)x_{k+1}\Gamma(\frac{k+1}{2})}{\pi^{\frac{k+1}{2}}}\int\limits_{\mathbb{R}^k}\frac{t_i\,dt_i}{(|t|^2+x_{k+1}^2)^{\frac{k+3}{2}}}=$$

$$=\frac{(k+1)x_{k+1}\Gamma(\frac{k+1}{2})}{\pi^{\frac{k+1}{2}}}\int\limits_{\mathbb{R}^k}\frac{\rho^2\sin^2\vartheta_1\sin^2\vartheta_2\cdots\sin^2\vartheta_{i-1}\cos^2\vartheta_i}{(\rho^2+x_{k+1}^2)^{\frac{k+3}{2}}}\times$$

$$\times\rho^{k-1}\sin^{k-2}\vartheta_1\cdots\sin^{k-i}\vartheta_{i-1}\cdots\sin\vartheta_{k-2}d\rho d\vartheta_1\cdots d\vartheta_{k-2}d\varphi=$$

$$=\frac{2(k+1)\Gamma(\frac{k+1}{2})}{\pi^{\frac{k-1}{2}}}\int\limits_0^{\infty}\frac{\rho^{k+1}d\rho}{(1+\rho^2)^{\frac{k+3}{2}}}\overbrace{\int\limits_0^{\pi}\cdots\int\limits_0^{\pi}}^{k-2}\sin^k\vartheta_1\sin^{k-1}\vartheta_2\cdots\sin^{k-i+2}\vartheta_{i-1}\times$$

$$\times\cos^2\vartheta_i\sin^{k-i-1}\vartheta_i\cdots\sin\vartheta_{k-2}d\vartheta_1 d\vartheta_2\cdots d\vartheta_{k-2}=$$

$$=\frac{2(k+1)\Gamma(\frac{k+1}{2})}{\pi^{\frac{k-1}{2}}}\cdot\frac{k\Gamma(\frac{k}{2})\sqrt{\pi}}{2(k+1)\Gamma(\frac{k+1}{2})}\cdot\frac{\pi^{\frac{k-2}{2}}}{k\Gamma(\frac{k}{2})}=1.\qquad\square$$

2. Boundary Properties of the Integral $\frac{\partial u(f;x,x_{k+1})}{\partial x_i}$

The following theorem is valid.

Theorem 1.

(a) *If a finite derivative $\overline{\mathcal{D}}_{x_i(x)}f(x^0)$ exists at the point x^0, then*

$$\lim_{(x,x_{k+1})\to(x^0,0)} \frac{\partial u(f;x,x_{k+1})}{\partial x_i} = \frac{\partial f(x^0)}{\partial x_i}. \tag{1}$$

(b) *There is a continuous function $f \in L(\mathbb{R}^k)$ such that for any $B \subset M$, $m(B) < k$, at the point $x^0 = (0,0,\dots,0) = 0$ all derivatives $\overline{\mathcal{D}}_{x_i(\overline{x}_B)}f(0) = 0$, $i = \overline{1,k}$, but the limits*

$$\lim_{x_{k+1}\to 0+} \frac{\partial u(f;0,x_{k+1})}{\partial x_i}, \quad i = \overline{1,k},$$

do not exist.

Proof.

(a) Let $x^0 = 0$, $C_k = \frac{(k+1)\Gamma(\frac{k+1}{2})}{\pi^{\frac{k+1}{2}}}$. It is easy to check that

$$\frac{\partial u(f;x,x_{k+1})}{\partial x_i} = C_k x_{k+1} \int\limits_{\mathbb{R}^k} \frac{(t_i - x_i) f(t)\, dt}{(|t-x|^2 + x_{k+1}^2)^{\frac{k+3}{2}}}.$$

By virtue of the lemma we have

$$\frac{\partial u(f;x,x_{k+1})}{\partial x_i} - \overline{\mathcal{D}}_{x_i(x)}f(0) =$$

$$= C_k x_{k+1} \int\limits_{\mathbb{R}^k} \frac{t_i^2}{(|t|^2 + x_{k+1}^2)^{\frac{k+3}{2}}} \Big[\frac{f(x+t) - f(x+t-t_ie_i)}{t_i} - \overline{\mathcal{D}}_{x_i(x)}f(0)\Big] dt =$$

$$= I_1 + I_2,$$

where

$$I_1 = C_k x_{k+1} \int\limits_{V_\delta}, \quad I_2 = C_k x_{k+1} \int\limits_{CV_\delta},$$

V_δ is the ball with center at 0 and radius δ. Let $\varepsilon > 0$. Choose $\delta > 0$ such that

$$\Big|\frac{f(x+t) - f(x+t-t_ie_i)}{t_i} - \overline{\mathcal{D}}_{x_i(x)}f(0)\Big| < \varepsilon$$

for $|x| < \delta$, $|t| < 2dl$.

Hence

$$|I_1| < C_k x_{k+1} \varepsilon \int\limits_{V_\delta} \frac{t_i^2 dt}{(|t|^2 + x_{k+1}^2)^{\frac{k+3}{2}}} <$$

$$< C_k x_{k+1} \varepsilon \int\limits_{\mathbb{R}^k} \frac{t_i^2 dt}{(|t|^2 + x_{k+1}^2)^{\frac{k+3}{2}}} = \varepsilon. \tag{2}$$

It is likewise easy to show that

$$\lim_{(x, x_{k+1}) \to (x^0, 0)} I_2 = 0. \tag{3}$$

Equalities (2) and (3) imply the validity of (1).

(b) Let $D = (0 \le t_1 < \infty;\ 0 \le t_2 < \infty; \dots, 0 \le t_k < \infty)$. Define the function f as follows:

$$f(t) = \begin{cases} \sqrt[k+1]{t_1 t_2 \cdots t_k} & \text{if } (t_1, t_2, \dots, t_k) \in D, \\ 0 & \text{if } (t_1, t_2, \dots, t_k) \in CD. \end{cases}$$

Clearly, $f(t)$ is continuous in $\mathbb{R}^k$ and $\overline{\mathcal{D}}_{x_i(\overline{x}_B)} f(0) = 0$, $i = \overline{1,k}$, for any B when $m(B) < k$.

If in the integral

$$\frac{\partial u(f; x, x_{k+1})}{\partial x_i} = C_k x_{k+1} \int\limits_{\mathbb{R}^k} \frac{(t_i - x_i) f(t)\, dt}{(|t - x|^2 + x_{k+1}^2)^{\frac{k+3}{2}}}$$

we use spherical coordinates, then for the considered function we shall have

$$\frac{\partial u(f; 0, x_{k+1})}{\partial x_i} = C_k x_{k+1} \int\limits_{\mathbb{R}^k} \frac{t_i f(t)\, dt}{(|t|^2 + x_{k+1}^2)^{\frac{k+3}{2}}} =$$

$$= C x_{k+1} \int\limits_0^\infty \frac{\rho \sqrt[k-1]{\rho^k}}{(\rho^2 + x_{k+1}^2)^{\frac{k+3}{2}}} \rho^{k-1} d\rho =$$

$$= C x_{k+1} \int\limits_0^\infty \frac{\rho^{k + \frac{k}{k+1}} d\rho}{(\rho^2 + x_{k+1}^2)^{\frac{k+3}{2}}} > C x_{k+1} \int\limits_0^{x_{k+1}} \frac{\rho^{k + \frac{k}{k+1}} d\rho}{(\rho^2 + x_{k+1}^2)^{\frac{k+3}{2}}} \rho^{k-1} d\rho >$$

$$> C x_{k+1} \int\limits_0^{x_{k+1}} \frac{\rho^{k + \frac{k}{k+1}} d\rho}{x_{k+1}^{k+3}} = \frac{C}{\sqrt[k+1]{x_{k+1}}}.$$

Hence

$$\lim_{x_{k+1} \to 0+} \frac{\partial u(f; 0, x_{k+1})}{\partial x} = +\infty. \qquad \square$$

Corollary 1. *If finite derivatives $\overline{\mathcal{D}}_{x_i(x)} f(x^0)$, $i = \overline{1,k}$, exist at the point x^0, then*

$$\lim_{(x, x_{k+1}) \to (x^0, 0)} d_x u(f; x, x_{k+1}) = df(x^0).$$

Corollary 2. *If f has a continuous partial derivative at the point x^0, then*

$$\lim_{(x,x_{k+1})\to(x^0,0)} \frac{\partial u(f;x,x_{k+1})}{\partial x_i} = f'_{x_i}(x^0).$$

Corollary 3.
(a) *If f is a continuously differentiable function at the point x^0, then*

$$\lim_{(x,x_{k+1})\to(x^0,0)} d_x u(f;x,x_{k+1}) = df(x^0).$$

(b) *There exists a differentiable function $f(t_1,t_2)$ at the point $(0,0)$ such that $df(0,0)=0$ but the limits*

$$\lim_{(x_1,x_2,x_3)\to(0,0,0)} \frac{\partial u(f;x_1,x_2,x_3)}{\partial x_1}, \qquad \lim_{(x_1,x_2,x_3)\to(0,0,0)} \frac{\partial u(f;x_1,x_2,x_3)}{\partial x_2}$$

do not exist.

Proof of assertion (b) of Corollary 3. We set $D=[0,1;0,1]$. Let

$$f(t_1,t_2) = \begin{cases} \sqrt[5]{t_1^3 t_2^3} & \text{for } (t_1,t_2)\in D, \\ 0 & \text{for } (t_1,t_2)\in[-\infty,0;0,\infty[\cup]-\infty,\infty;-\infty,0[, \end{cases}$$

and continue f onto the set $]0,\infty;0,\infty[\setminus D$ so that $f\in L(\mathbb{R}^2)$. It is easy to check that $f(t_1,t_2)$ is differentiable at the point $(0,0)$ and

$$f'_{t_1}(0,0) = f'_{t_2}(0,0) = 0.$$

Let $(x_1,x_2,x_3)\to(0,0,0)$ for $x_1=0$, $x_3=x_2^2$, $x_2>0$. Then for the considered function

$$\begin{aligned}
\frac{\partial u(f;0,x_2,x_3)}{\partial x_1} &= \frac{3x_3}{2\pi}\int_0^\infty\int_0^\infty \frac{t_1 f(t_1,t_2)\,dt_1dt_2}{[t_1^2+(t_2-x_2)^2+x_2^4]^{5/2}} = \\
&= \frac{3x_2^2}{2\pi}\int_0^\infty\int_{-x_2}^\infty \frac{t_1 f(t_1,t_2+x_2)\,dt_1dt_2}{(t_1^2+t_2^2+x_2^4)^{5/2}} = \\
&= \frac{3x_2^2}{2\pi}\int_0^\infty\int_{-x_2}^\infty \frac{t_1\sqrt[5]{t_1^3(t_2+x_2)^3}\,dt_1dt_2}{(t_1^2+t_2^2+x_2^4)^{5/2}} > \\
&> \frac{3x_2^2}{2\pi}\int_{x_2^2}^{2x_2^2}\int_{x_2^2}^{2x_2^2} \frac{t_1\sqrt[5]{t_1^3}\,\sqrt[5]{x_2^3}\,dt_1dt_2}{(t_1^2+t_2^2+x_2^4)^{5/2}} >
\end{aligned}$$

$$> \frac{3x_2^2 x_2^{3/5}}{2\pi} \int_{x_2^2}^{2x_2^2} \int_{x_2^2}^{2x_2^2} \frac{x_2^2 \cdot x_2^{6/5}\, dt_1 dt_2}{(4x)2^4 + 4x_2^4 + x_2^4)^{5/2}} =$$

$$= \frac{1}{162\pi \sqrt[5]{x_2}} \to \infty \quad \text{for} \quad x_2 \to 0+. \qquad \square$$

Theorem 2. *If a finite derivative* $\mathcal{D}_{x_i(\overline{x}_{M\setminus i})} f(x^0)$ *exists at the point* x^0, *then*

$$\lim_{(x,x_{k+1}) \underset{x_i}{\overset{\wedge}{\to}} (x^0,0)} \frac{\partial u(f;x,x_{k+1})}{\partial x_i} = \frac{\partial f(x^0)}{\partial x_i}.$$

Proof. Let $x^0 = 0$. By virtue of the lemma we have the equality

$$\frac{\partial u(f;x,x_{k+1})}{\partial x_i} - \mathcal{D}_{x_i(\overline{x}_{M\setminus i})} f(0) =$$

$$= C_k x_{k+1} \int_{\mathbb{R}^k} \frac{t_i(t_i - x_i)}{(|t-x|^2 + x_{k+1}^2)^{\frac{k+3}{2}}} \Big[\frac{f(t) - f(t - t_i e_i)}{t_i} - \mathcal{D}_{x_i(\overline{x}_{M\setminus i})} f(0)\Big] dt =$$

$$= I_1 + I_2,$$

where $I_1 = C_k x_{k+1} \int_{V_\delta}$, $I_2 = C_k x_{k+1} \int_{CV_\delta}$.

Let $\varepsilon > 0$ and choose $\delta > 0$ such that

$$\Big|\frac{f(t) - f(t - t_i e_i)}{t_i} - \mathcal{D}_{x_i(\overline{x}_{M\setminus i})} f(0)\Big| < \varepsilon \quad \text{for} \quad |r| < \delta.$$

Now,

$$|I_1| < C_k x_{k+1} \varepsilon \int_{\mathbb{R}^k} \frac{|t_i(t_i - x_i)|\, dt}{(|t-x|^2 + x_{k+1}^2)^{\frac{k+3}{2}}} <$$

$$< C_k x_{k+1} \varepsilon \int_{\mathbb{R}^k} \frac{t_i^2\, dt}{(|t|^2 + x_{k+1}^2)^{\frac{k+3}{2}}} +$$

$$+ C_k x_{k+1} \varepsilon |x_i| \int_{\mathbb{R}^k} \frac{|t_i|\, dt}{(|t|^2 + x_{k+1}^2)^{\frac{k+3}{2}}} =$$

$$= \varepsilon + C x_{k+1} \varepsilon |x_i| \int_0^\infty \frac{\rho^k\, d\rho}{(\rho^2 + x_{k+1}^2)^{\frac{k+3}{2}}} =$$

$$= \varepsilon + \frac{C x_{k+1}^{k+2} |x_i| \varepsilon}{x_{k+1}^{k+3}} \int_0^\infty \frac{\rho^k\, d\rho}{(1+\rho^2)^{\frac{k+3}{2}}} = \Big(1 + \frac{C|x_i|}{x_{k+1}}\Big)\varepsilon.$$

Hence we obtain

$$\lim_{(x,x_{k+1})\underset{x_i}{\overset{\wedge}{\to}}(x^0,0)} I_1 = 0.$$

In a similar manner we prove that

$$\lim_{(x,x_{k+1})\underset{x_i}{\overset{\wedge}{\to}}(x^0,0)} I_2 = 0. \qquad \square$$

Theorem 3. *If at the point x^0 there exist finite derivatives $\mathcal{D}_{x_i(\overline{x}_{M\setminus i})}f(x^0)$ and $\mathcal{D}_{x_j(\overline{x}_B)}f(x^0)$, $i \neq j$, $B = M \setminus \{i,j\}$, then*

$$\lim_{(x,x_{k+1})\underset{x_i}{\overset{\wedge}{\to}}(x^0,0)} \frac{\partial u(f;x,x_{k+1})}{\partial x_i} = \frac{\partial f(x^0)}{\partial x_i},$$

$$\lim_{(x,x_{k+1})\underset{x_i}{\overset{\wedge}{\to}}(x^0,0)} \frac{\partial u(f;x,x_{k+1})}{\partial x_j} = \frac{\partial f(x^0)}{\partial x_j}.$$

Proof. Let $x^0 = 0$. By virtue of the lemma we have

$$\frac{\partial u(f;x,x_{k+1})}{\partial x_j} = C_k x_{k+1} \int\limits_{\mathbb{R}^k} \frac{(t_j - x_j) f(t)\, dt}{(|t-x|^2 + x_{k+1}^2)^{\frac{k+3}{2}}} =$$

$$= C_k x_{k+1} \int\limits_{\mathbb{R}^k} \frac{(t_j - x_j)\{[f(t) - f(t-t_i e_i)] + [f(t-t_i e_i) - f(t-t_i e_i - t_j e_j)]\} dt}{(|t-x|^2 + x_{k+1}^2)^{\frac{k+3}{2}}} =$$

$$= C_k x_{k+1} \int\limits_{\mathbb{R}^k} \frac{(t_j - x_i)[f(t) - f(t-t_i e_i)]}{(|t-x|^2 + x_{k+1}^2)^{\frac{k+3}{2}}}\, dt +$$

$$+ C_k x_{k+1} \int\limits_{\mathbb{R}^k} \frac{(t_j - x_i)[f(t-t_i e_i) - f(t-t_i e_i - t_j e_j)]}{(|t-x|^2 + x_{k+1}^2)^{\frac{k+3}{2}}}\, dt = I_1 + I_2,$$

where

$$I_1 = C_k x_{k+1} \int\limits_{\mathbb{R}^k} \frac{t_i(t_j - x_j)}{(|t-x|^2 + x_{k+1}^2)^{\frac{k+3}{2}}} \cdot \frac{f(t) - f(t-t_i e_i)}{t_i}\, dt =$$

$$= C_k x_{k+1} \int\limits_{\mathbb{R}^k} \frac{t_i(t_j - x_j)}{(|t-x|^2 + x_{k+1}^2)^{\frac{k+3}{2}}} \Big[\frac{f(t) - f(t-t_i e_i)}{t_i} - \mathcal{D}_{x_i(\overline{x}_{M\setminus i})} f(0)\Big] dt +$$

$$+ C_k x_{k+1} \mathcal{D}_{x_i(\overline{x}_{M\setminus i})} f(0) \int\limits_{\mathbb{R}^k} \frac{t_i(t_j - x_j)\, dt}{(|t-x|^2 + x_{k+1}^2)^{\frac{k+3}{2}}} = I_1' + I_1''.$$

It is easy to see that I_1'' and

$$|I_1'|<C_k x_{k+1}\int\limits_{\mathbb{R}^k}\frac{|t_i(t_j-x_j)|}{(|t-x|^2+x_{k+1}^2)^{\frac{k+3}{2}}}\left|\frac{f(t)-f(t-t_ie_i)}{t_i}-\mathcal{D}_{x_i(\overline{x}_{M\setminus i})}f(0)\right|dt.$$

Hence we obtain

$$\lim_{(x,x_{k+1})\underset{x_i}{\overset{\wedge}{\to}}0} I_1' = \lim_{(x,x_{k+1})\underset{x_i}{\overset{\wedge}{\to}}0} I_1 = 0.$$

Now we shall show that

$$\lim_{(x,x_{k+1})\underset{x_j}{\overset{\wedge}{\to}}0} I_2 = \mathcal{D}_{x_j(\overline{x}_{M\setminus\{i,j\}})}f(0).$$

Indeed,

$$I_2 = C_k x_{k+1}\int\limits_{\mathbb{R}^k}\frac{|t_j(t_j-x_j)|}{(|t-x|^2+x_{k+1}^2)^{\frac{k+3}{2}}}\times$$

$$\times\left[\frac{f(t-t_ie_i)-f(t-t_ie_i-t_je_j)}{t_j}-\mathcal{D}_{x_j(\overline{x}_{M\setminus\{i,j\}})}f(0)\right]dt+\mathcal{D}_{x_j(\overline{x}_{M\setminus\{i,j\}})}f(0).$$

This readily implies

$$\lim_{(x,x_{k+1})\underset{x_j}{\overset{\wedge}{\to}}0} I_2 = \mathcal{D}_{x_j(\overline{x}_{M\setminus\{i,j\}})}f(0) = \frac{\partial f(0)}{\partial x_j}.$$

Finally, we obtain

$$\lim_{(x,x_{k+1})\underset{x_i x_j}{\overset{\wedge}{\to}}0}\frac{\partial u(f;x,x_{k+1})}{\partial x_j} = \frac{\partial f(0)}{\partial x_j}. \qquad \square$$

By a similar reasoning we prove

Theorem 4. *If at the point x^0 there exist finite derivatives $\mathcal{D}_{x_1(x_2,x_3,\dots,x_k)}f(x^0)$, $\mathcal{D}_{x_2(x_3,\dots,x_k)}f(x^0)$, $\dots$, $\mathcal{D}_{x_{k-1}(x_k)}f(x^0)$, $f'_{x_k}(x^0)$, then*

$$\lim_{(x,x_{k+1})\underset{x_1}{\overset{\wedge}{\to}}(x^0,0)}\frac{\partial u(f;x,x_{k+1})}{\partial x_1} = \frac{\partial f(x^0)}{\partial x_1},$$

$$\lim_{(x,x_{k+1})\underset{x_1 x_2}{\overset{\wedge}{\to}}(x^0,0)}\frac{\partial u(f;x,x_{k+1})}{\partial x_2} = \frac{\partial f(x^0)}{\partial x_2},$$

....

$$\lim_{(x,x_{k+1})\overset{\wedge}{\to}(x^0,0)} \frac{\partial u(f;x,x_{k+1})}{\partial x_k} = \frac{\partial f(x^0)}{\partial x_k}.$$

Corollary. *If at the point x^0 there exist finite derivatives $\mathcal{D}_{x_1(\overline{x}_{M\setminus 1})}f(x^0)$, $\mathcal{D}_{x_2(\overline{x}_{M\setminus\{1,2\}})}f(x^0)$, $\dots$, $\mathcal{D}_{x_{k-1}(x_k)}f(x^0)$, $f'_{x_k}(x^0)$, then*

$$\lim_{(x,x_{k+1})\overset{\wedge}{\to}(x^0,0)} d_x u(f;x,x_{k+1}) = df(x^0).$$

Theorem 5.

(a) *If at the point x^0 there exists a finite derivative $\mathcal{D}^*_{x_i(\overline{x}_{M\setminus i})}f(x^0)$, then*

$$\lim_{(x-x_ie_i+x_i^0e_i,x_{k+1})\to(x^0,0)} \frac{\partial u(f;x-x_ie_i+x_i^0e_i,x_{k+1})}{\partial x_i} = \mathcal{D}^*_{x_i}f(x^0).$$

(b) *There exists a continuous function $f(x)$ such that $\mathcal{D}^*_{x_i(\overline{x}_{M\setminus i})}f(x^0)=0$, but the limit*

$$\lim_{(x,x_{k+1})\overset{\wedge}{\to}(x^0,0)} \frac{\partial u(f;x,x_{k+1})}{\partial x_i}$$

does not exist.

Proof. (a) Let $x^0=0$. The validity of (a) follows from the equality

$$\frac{\partial u(f;x-x_ie_i+x_i^0e_i,x_{k+1})}{\partial x_i} - \mathcal{D}^*_{x_i(\overline{x}_{M\setminus i})}f(x^0) =$$
$$= C_k x_{k+1}\int_{\mathbb{R}^k} \frac{t_i^2}{(|t|^2+x_{k+1}^2)^{\frac{k+3}{2}}}\Big[\frac{f(t+x-x_ie_i)-f(t+x-x_ie_i-2t_ie_i)}{2t_i} -$$
$$-\mathcal{D}_{x_i(\overline{x}_{M\setminus i})}f(x^0)\Big]dt.$$

(b) We set $D_1=[0,1;0,1]$, $D_2=p-1,0;0,1]$. Let

$$f(t_1,t_2)=\begin{cases}\sqrt{t_1\sqrt{t_2}} & \text{for } (t_1,t_2)\in D_1,\\ \sqrt{-t_1\sqrt{t_2}} & \text{for } (t_1,t_2)\in D_2,\\ 0 & \text{for } t_2\le 0\end{cases}$$

and continue $f(t_1,t_2)$ onto the set $\mathbb{R}^2_+\setminus(D_1\cup D_2)$ so that $f\in L(\mathbb{R}^2)$. It is easy to check that $\mathcal{D}^*_{t_1(t_2)}f(0)=0$. Let $x_1^0=x_2^0=0$ and $(x_1,x_2,x_3)\to(0,0,0)$ so that $x_2=0$ and $x_3=x_1$. Then for the constructed function we have

$$\frac{\partial u(f;x_1,x_2,x_3)}{\partial x_1} = \frac{3x_3}{2\pi}\int_{\mathbb{R}^2}\frac{(t_1-x_1)f(t_1,t_2)\,dt_1\,dt_2}{[(t_1-x_1)^2+(t_2-x_2)^2+x_3^2]^{5/2}} =$$

$$= Cx_3\Big\{\int_{-1}^{0}\int_{0}^{1}\frac{(t_1-x_1)\sqrt{-t_1\sqrt{t_2}}\,dt_1\,dt_2}{[(t_1-x_1)^2+t_2^2+x_3^2]^{5/2}}+$$

$$+\int_{0}^{1}\int_{0}^{1}\frac{(t_1-x_1)\sqrt{t_1\sqrt{t_2}}\,dt_1\,dt_2}{[(t_1-x_1)^2+t_2^2+x_3^2]^{5/2}}\Big\}+o(1)=$$

$$= Cx_1\Big[\int_{x_1}^{1+x_1}\int_{0}^{1}\frac{t_1\sqrt{(t_1-x_1)\sqrt{t_2}}}{(t_1^2+t_2^2+x_1^2)^{5/2}}\,dt_1dt_2+$$

$$+\int_{-x_1}^{1-x_1}\int_{0}^{1}\frac{t_1\sqrt{(t_1+x_1)\sqrt{t_2}}}{(t_1^2+t_2^2+x_1^2)^{5/2}}\,dt_1dt_2\Big]+o(1)=$$

$$= Cx_1\Big\{\int_{-x_1}^{x_1}\int_{0}^{1}\frac{t_1\sqrt{(t_1+x_1)\sqrt{t_2}}}{(t_1^2+t_2^2+x_1^2)^{5/2}}\,dt_1dt_2+$$

$$+\int_{x_1}^{1-x_1}\int_{0}^{1}\frac{t_1[\sqrt{(t_1+x_1)\sqrt{t_2}}-\sqrt{(t_1-x_1)\sqrt{t_2}}]}{(t_1^2+t_2^2+x_1^2)^{5/2}}\,dt_1dt_2\Big\}=$$

$$= Cx_1(I_1+I_2)+o(1),$$

where

$$I_1=\int_{0}^{x_1}\int_{0}^{1}\frac{t_1[\sqrt{(t_1+x_1)\sqrt{t_2}}-\sqrt{(x_1-t_1)\sqrt{t_2}}]}{(t_1^2+t_2^2+x_1^2)^{5/2}}\,dt_1dt_2>0,$$

$$I_2=\int_{x_1}^{1-x_1}\int_{0}^{1}\frac{t_1\sqrt[4]{t_2}(\sqrt{t_1+x_1}-\sqrt{t_1-x_1})}{(t_1^2+t_2^2+x_1^2)^{5/2}}\,dt_1dt_2>$$

$$>\int_{x_1}^{2x_1}\int_{x_1}^{2x_1}\frac{t_1\sqrt[4]{t_2}(\sqrt{t_1+x_2}-\sqrt{t_1-x_1})}{(t_1^2+t_2^2+x_1^2)^{5/2}}\,dt_1dt_2>$$

$$>\int_{x_1}^{2x_1}\int_{x_1}^{2x_1}\frac{x_1\sqrt[4]{x_1}(\sqrt{2x_1}-\sqrt{x_1})}{(9x_1^2)^{5/2}}\,dt_1dt_2=\frac{\sqrt{2}-1}{128}\cdot\frac{1}{\sqrt[4]{x_1^5}}.$$

Thus, by the chosen path, we obtain

$$\frac{\partial u(f;x_1,0,x_1)}{\partial x_1}>\frac{C}{\sqrt[4]{x_1}},$$

which yields $\frac{\partial u(f;x_1,0,x_1)}{\partial x_1} \to +\infty$ when $(x_1,x_2,x_3) \to (0,0,0)$ by the chosen path. $\square$

By a similar reasoning we prove

Theorem 6.

(a) *If at the point x^0 there exists a finite derivative $\overline{\mathcal{D}}^*_{x_i(x)} f(x^0)$, $i=\overline{1,k}$, then*

$$\lim_{(x,x_{k+1})\to(x^0,0)} \frac{\partial u(f;x,x_{k+1})}{\partial x_i} = \mathcal{D}^*_{x_i} f(x^0).$$

(b) *There exists a continuous function $f(x)$ such that for any $B \subset M$, $m(B) < k$, all derivatives $\overline{\mathcal{D}}^*_{x_i(\overline{x}_B)} f(0) = 0$, $i=\overline{1,k}$, but the limits*

$$\lim_{x_{k+1}\to 0+} \frac{\partial u(f;0,x_{k+1})}{\partial x_i}$$

do not exist.

Statement (a) of Theorem 1 is a corollary of statement (a) of Theorem 6.

The validity of (b) follows from statement (b) of Theorem 1.

Theorem 7.

(a) *If f has a total differential $df(x^0)$ at the point x^0, then*

$$\lim_{(x,x_{k+1})\overset{\wedge}{\to}(x^0,0)} d_x u(f;x,x_{k+1}) = df(x^0). \tag{4}$$

(b) *there exists a continuous function f which has partial derivatives of any order, but the limits*

$$\lim_{x_{k+1}\to 0+} \frac{\partial u(f;x^0,x_{k+1})}{\partial x_i}$$

do not exist.

Proof.

(a) By virtue of the lemma we have $(x^0=0)$

$$\frac{\partial u(f;x,x_{k+1})}{\partial x_i} - \frac{\partial f(0)}{\partial x_i} =$$

$$= C_k x_{k+1} \int\limits_{\mathbb{R}^k} \frac{(t_i - x_i)\sum\limits_{\nu=1}^k |t_\nu|}{(|t-x|^2 + x_{k+1}^2)^{\frac{k+3}{2}}} \cdot \frac{f(t) - f(0) - \sum\limits_{\nu=1}^k \frac{\partial f(0)}{\partial x_i}\, t_i}{\sum\limits_{\nu=1}^k |t_\nu|}\, dt.$$

This equality implies

$$\lim_{(x,x_{k+1})\overset{\wedge}{\to}0} \frac{\partial u(f;x,x_{k+1})}{\partial x_i} = \frac{\partial f(0)}{\partial x_i}, \quad i=\overline{1,k}.$$

Thus equality (4) is valid.

(b) Consider the function

$$f(t_1,t_2) = \begin{cases} \sqrt[4]{(2t_1-t_2)(t_2-\frac{1}{2}t_1)} & \text{for } (t_1,t_2)\in D = \{(t_1,t_2): \\ & \quad 0\le t_1<\infty;\ \frac{1}{2}t_1\le t_2\le 2t_1\}, \\ 0 & \text{for } (t_1,t_2)\in CD. \end{cases}$$

This function is continuous in $\mathbb{R}^2$, has partial derivatives of any order at the point $(0,0)$ which are equal to zero, but

$$\begin{aligned} \frac{\partial u(f;0,0,x_3)}{\partial x_1} &= \frac{3x_3}{2\pi}\int_0^\infty dt_1 \int_{\frac{1}{2}t_1}^{2t_1} \frac{t_1\sqrt[4]{(2t_1-t_2)(t_2-\frac{1}{2}t_1)}}{(t_1^2+t_2^2+x_3^2)^{5/2}}\,dt_2 > \\ &> Cx_3\int_{x_3}^{2x_3} t_1\,dt_1 \int_{t_1}^{\frac{3}{2}t_1} \frac{\sqrt[4]{(2t_1-t_2)(t_2-\frac{1}{2}t_1)}}{(t_1^2+t_2^2+x_3^2)^{5/2}}\,dt_2 > \\ &> Cx_3\int_{x_3}^{2x_3} t_1\,dt_1 \int_{t_1}^{\frac{3}{2}t_1} \frac{\sqrt[4]{(2t_1-\frac{3}{2}t_1)(t_1-\frac{1}{2}t_1)}}{(\frac{13}{4}t_1^2+x_3^2)^{5/2}}\,dt_2 > \\ &> Cx_3\int_{x_3}^{2x_3} t_1\,dt_1 \int_{t_1}^{\frac{3}{2}t_1} \frac{\sqrt[4]{t_1^2}\,dt_2}{x_3^5} = \frac{C}{x_3^4}\int_{x_3}^{2x_3} t_1^{5/2}dt_1 = \\ &= \frac{C}{\sqrt{x_3}} \to +\infty \quad \text{for} \quad x_3\to 0+. \quad \square \end{aligned}$$

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