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BOUNDS FOR THE CHARACTERISTIC FUNCTIONS OF THE SYSTEM OF MONOMIALS IN RANDOM VARIABLES AND OF ITS TRIGONOMETRIC ANALOGUE

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Abstract. Using a multidimensional analogue of Vinogradov's inequality for a trigonometric integral, the upper bounds are constructed for the moduli of the characteristic functions both of the system of monomials in components of a random vector with an absolutely continuous distribution in $\mathbb{R}^s$ and of the system

$$(\cos j_1\pi\xi_1\cdots\cos j_s\pi\xi_s,\ \ 0\le j_1,\dots,j_s\le k,\ \ j_1+\cdots+j_s\ge 1),$$

where $(\xi_1,\dots,\xi_s)$ is uniformly distributed in $[0,1]^s$.

Introduction. This note continues the series of studies [1–3] carried out on the initiative of Yu. V. Prokhorov and dealing with estimates of the characteristic functions (c.f.) of degenerate multidimensional distributions of the form

$$\left|\int_{\mathbb{R}^s}\exp\{i(t,\varphi(x))\}p(x)\,dx\right|\le C|t|^{-\alpha}\quad\text{for}\quad |t|>t_0, \tag{1}$$

where $\varphi:\mathbb{R}^s\to\mathbb{R}^k$, $s<k$, $t\in\mathbb{R}^k$; $p(x)$ is the distribution density; C, α, and t_0 are positive constants. In what follows we shall use the notation $I(t;\varphi(x),p(x))$ for the integral from (1) and omit the condition $|t|>t_0$.

When $s=1$, Sadikova has shown in [1] that for $\varphi(x)=\varphi_0(x)=(x,x^2,\dots,x^k)$ one can take $\alpha=(1+1/k)^{-(k-1)}/k!$ if the integrals of $|p'(x)|$ and $|x|^{k-1}p(x)$ are finite. Her method is based on van der Corput's lemma whose generalization allowed Yurinskii to assert that "for decreasing of the c.f. like a negative power of the argument modulus it is, roughly speaking, sufficient that the surface carrying the distribution have no tangencies of an

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infinitely high order with $(k-1)$-dimensional hyperplanes, and that the surface density of the distribution be bounded, satisfy the Lipschitz condition in L_1-norm, and have several moments" (see [2]).

The author's paper [3] deals with the case $s=1$, where $\varphi(x)=\varphi_1(x)=(\cos\pi x,\dots,\cos k\pi x)$, $p(x)=\mathbb{1}_{[0,1]}(x)$, where $\mathbb{1}_A(x)$ is the indicator of the set A. Using Vinogradov's famous inequality for a trigonometric integral [4], it is shown in [3] that in that case $\alpha=1/2k$ (for $k=1$ it can be deduced from [2] as well). An order with respect to $|t|$ turned out to be exact (because of the exactness of Vinogradov's inequality), i.e., in some directions of $\mathbb{R}^k$ the c.f. behaves like $|t|^{-1/2k}$ for $|t|\to\infty$. Concomitantly, we obtained bounds with $\alpha=1/k$ for the case $\varphi(x)=\varphi_0(x)$ assuming that $\int_{\mathbb{R}^1}\max(1,|x|)|p'(x)|dx<\infty$ for the density $p(x)$; for analogous estimates see [5].

$\varphi_0(x)$ arises in problems connected with behavior of the joint distribution of sample moments. For $\varphi_1(x)$ the inequality from [3] turns out important in proving the fact that the deviation of the distribution function of ω^2-test statistic from the limiting one has order n^{-1} (see [6], cf. [7]).

1. Multidimensional Analogue of Vinogradov's Inequality. The Results. Let $x=(x_1,\dots,x_s)\in\mathbb{R}^s$ and the the multiindex $\mathbf{j}=(j_1,\dots,j_s)\neq\mathbf{0}=(0,\dots,0)\in\mathbb{R}^s$ vary in $J_{k,s}=\{0,1,\dots,k\}^s\setminus\{\mathbf{0}\}$. Consider the system of monomials in s real variables $\varphi_0^{(s)}(x)=\big(x_1^{j_1}\cdots x_s^{j_s},\,\mathbf{j}\in J_{k,s}\big)$ and also the corresponding system of cosines products

$$\varphi_1^{(s)}(x)=\big(\cos j_1\pi x_1\cdots\cos j_s\pi x_s,\ \ \mathbf{j}\in J_{k,s}\big).$$

Denote $\tau=\tau(t)=\max\big\{|t_{\mathbf{j}}|:\mathbf{j}\in J_{k,s}\big\}$, $\ t=(t_{\mathbf{j}},\mathbf{j}\in J_{k,s})\in\mathbb{R}^{(k+1)^s-1}$.

For $\varphi_j^{(s}(x)$, $j=0,1$, inequalities of form (1), which can be used to study distributions of a system of mixed sample moments and other multivariate statistics, can be derived by the following multidimensional analogue of Vinogradov's inequality [8, p. 39] written for our convenience in the form

$$\big|I(t;\varphi_0^{(s)}(x),\mathbb{1}_{[0,1]^s}(x)\big|\le 32^s(2\pi)^{1/k}\tau^{-1/k}\ln^{s-1}(2+\tau/2\pi); \qquad (2)$$

note that by [8, p. 41] we have for $\gamma>1$ that

$$\left|\int_{[0,1]^s}\exp\{i\gamma x_1^k\cdots x_s^k\}dx\right|\ge[2\pi k^s(s-1)!]^{-1}(2\pi)^{1/k}\gamma^{-1/k}\ln^{s-1}\frac{\gamma}{2\pi}.$$

By virtue of the fact that the inequality $\tau(\widetilde{t})\ge\prod_{j=1}^s\big[\min(1,|y_j|)\big]^k\tau$ holds for $y=(y_1,\dots,y_s)\in\mathbb{R}^s$ and for $\widetilde{t}=(y_1^{j_1}\cdots y_s^{j_s}t_{\mathbf{j}},\,\mathbf{j}\in J_{k,s})$ and the function on the right-hand side of (2) decreases with respect to τ, we have

$$|I_y(t)|\le\prod_{j=1}^s|\operatorname{sgn}y_j|\max(1,|y_j|)32^s(2\pi)^{1/k}\tau^{-1/k}\ln^{s-1}(2+\tau/2\pi) \quad (3)$$

for

$$I_y(t) = \int_0^{y_1} \cdots \int_0^{y_s} \exp\{i(t, \varphi_0^{(s)}(x))\} dx.$$

Let $D = [a_1, b_1] \times \cdots \times [a_s, b_s]$ with positive $h_j = b_j - a_j$, $j = 1, \dots, s$; denote $D(j_1, \dots, j_r) = [a_{j_1}, b_{j_1}] \times \cdots \times [a_{j_r}, b_{j_r}]$ and $D_c(j_1 \dots j_r) = \prod\{[a_j, b_j] : j = 1, \dots, s,\ j \neq j_1, \dots, j_r\}$ for $1 \leq j_1 < \cdots < j_r \leq s$, $1 < r < s$. The notation $x(j_1, \dots, j_s)$ and $x_c(j_1 \dots j_r)$ for $x \in \mathbb{R}^s$ is evident. Denote further by V, $V_c(j_1 \dots j_r)$ the sets of vertices of the parallelepipeds D, $D_c(j_1 \dots j_r)$, respectively, and $\Pi(y) = \prod_{j=1}^{r} \max(1, |y_j|)$ for $y = (y_1, \dots, y_r) \in \mathbb{R}^r$, $1 \leq r \leq s$.

Proposition 1. *If a random vector ξ has the density*

$$u_D(x) = (h_1 \cdots h_s)^{-1} \mathbb{1}_D(x),$$

then for the c.f. of $\varphi_0^{(s)}(\xi)$ the inequality

$$\left|I(t; \varphi_0^{(s)}(x), u_D(x))\right| \leq \Pi(h_1^{-1}, \dots, h_s^{-1}) 32^s (2\pi)^{1/k} \tau^{-1/k} \ln^{s-1}(2 + \tau/2\pi)$$

holds.

To proceed to the general case with density $p(x)$ concentrated on D one should use

Proposition 2. *If for $x \in D$ a function $p(x) \geq 0$ is continuous in D and has, in D, continuous partial derivatives $p_i = p_{x_i}$, $p_{ij} = p_{x_i x_j}, \dots, p_{1 \cdots s} = p_{x_1 \dots x_s}$, then the estimate*

$$\left|I(t; \varphi_0^{(s)}(x), \mathbb{1}_D(x) p(x))\right| \leq C 32^s (2\pi)^{1/k} \tau^{-1/k} \ln^{s-1}(2 + \tau/2\pi)$$

holds, where $C = C_0 + \cdots + C_s$ with

$$C_r = \sum_{1 \leq j_1 < \cdots < j_r \leq s} \ \sum_{x_c(j_1 \dots j_r) \in V_c(j_1 \dots j_r)} \Pi(x_c(j_1 \dots j_r)) \times$$
$$\times \int_{D(j_1 \dots j_r)} \Pi(x(j_1 \dots j_r)) p_{j_1 \dots j_r}(x)\, dx(j_1 \dots j_r), \quad 1 \leq r < s,$$
$$C_0 = \sum_{x \in V} \Pi(x) p(x), \quad C_s = \int_D \Pi(x) p_{1 \dots s}(x)\, dx.$$

If the above functions are unbounded or discontinuous in D or D is unbounded, then the estimate will demand obvious changes (conditions of the existence of some limits and integrals).

Since $|t| \leq [(k+1)^s - 1]^{1/2} \tau$ for $t \in \mathbb{R}^{(k+1)^s - 1}$, Propositions 1–2 can be expressed in terms of $|t|$.

Let now $T_r(x) = \sum_{j=0}^{r} a_{rj} x^j$, $x \in \mathbb{R}^1$, be a Chebyshev polynomial of the first kind and A_k be the matrix with rows $(a_{r0}, \dots, a_{rr}, 0, \dots, 0)$, $r = 0, \dots, k$.

For a matrix $B = (b_{pq})_{p,q=0,\dots,k}$ denote $\rho(B) = \max\{\sum_q |b_{pq}| : p = 0,\dots,k\}$ and $\rho_k = \rho((A_k^{-1})')$ ($'$ means transposition). Let $\lambda_k = \lambda_{\min}(A_k A_k')$ be the least eigenvalue of the matrix $A_k A_k'$.

Proposition 3. *The following inequalities hold:*

(a) $|I(t;\varphi_1^{(s)}(x), \mathbb{1}_{[0,1]^s}(x))| \le 2^{\frac{9ks+1}{k(s+1)}} \pi^{-\frac{2ks-1}{k(s+1)}} s^{\frac{1}{s+1}} \times$
$$\times \rho_k^{\frac{s}{k(s+1)}} \tau^{-\frac{1}{k(s+1)}} \ln^{\frac{s-1}{s+1}} (2 + \tau/2\pi\rho_k^s);$$

(b) $|I(t;\varphi_1^{(s)}(x), \mathbb{1}_{[0,1]^s}(x))| \le 2^{\frac{9ks+1}{k(s+1)}} \pi^{-\frac{2ks-1}{k(s+1)}} s^{\frac{1}{s+1}} [(k+1)^s - 1]^{\frac{1}{2k(s+1)}} \times$
$$\times \lambda_k^{-\frac{1}{2k(s+1)}} |t|^{-\frac{1}{k(s+1)}} \ln^{\frac{s-1}{s+1}} \left(2 + \frac{|t|\lambda_k^{s/2}}{2\pi[(k+1)^s-1]^{1/2}}\right).$$

2. Proofs.

Proposition 1 is evident.

Proof of Proposition 2. If the functions $u : \mathbb{R}^s \to \mathbb{C}$ and $v : \mathbb{R}^s \to \mathbb{C}$ are continuous in D along with the partial derivatives u_{x_i}, v_{x_i}, $u_{x_i x_j}$, $v_{x_i x_j}, \dots, u_{x_1\dots x_s}$, $v_{x_1\dots x_s}$, then

$$\int_D u v_{x_1\dots x_s} dx = \Delta_a^b(uv) - \sum_{1\le i\le s} \Delta_{a_c(i)}^{b_c(i)} \int_{D(i)} v u_{x_i} dx(i) +$$
$$+ \sum_{1\le i<j\le s} \Delta_{a_c(ij)}^{b_c(ij)} \int_{D(ij)} v u_{x_i x_j} dx(ij) - \dots + (-1)^s \int_D v u_{x_1\dots x_s} dx.$$

Substituting here $u = p(x)$ and $v = I_x(t)$, passing to moduli, and applying (3), we complete the proof. $\square$

Proof of Proposition 3. By the change of variables $x_j = \arccos \pi y_j$, $j = 1,\dots,s$, we obtain

$$I(t;\varphi_1^{(s)}(x), \mathbb{1}_{(0,1)^s}(x)) = I(t;\widetilde{\varphi}_0^{(s)}(y), q(y)), \tag{4}$$

where

$$q(y) = \pi^{-s} \prod_{j=1}^s (1-y_j^2)^{-1/2} \mathbb{1}_{(-1,1)^s}(y),$$
$$\widetilde{\varphi}_0^{(s)}(y) = (T_{j_1}(y_1)\cdots T_{j_s}(y_s),\ \mathbf{j} \in J_{k,s}).$$

Apply Proposition 2 for a cube $[-(1-\varepsilon), 1-\varepsilon]^s$ with $0<\varepsilon<1$. Since $(1-(1-\varepsilon)^2)^{1/2} < \varepsilon^{-1/2}$ and $\int\limits_0^{1-\varepsilon} x(1-x^2)^{-3/2} dx < \varepsilon^{-1/2}$, we have

$$C_r < (2/\pi)^s \binom{s}{r} \varepsilon^{-s/2},\ \ 0 \le r \le s, \quad \text{i.e.,} \quad C < (4/\pi)^s \varepsilon^{-s/2}.$$

The remaining part of $|I(t;\varphi_0^{(s)}(y),q(y)|$ is less than $1-(1-z)^s<sz$ with $z=\frac{2}{\pi}(\frac{\pi}{2}-\arcsin(1-\varepsilon))\le\frac{4}{\pi}\sqrt{\varepsilon}$. Finally, we have

$$|I(t;\varphi_0^{(y)}(s),q(y))|\le\Big(\frac{4}{\pi}\Big)^s 32^s(2\pi)^{1/k}\tau^{-1/k}\ln^{s-1}(2+\tau/2\pi)\varepsilon^{-s/2}+\frac{4s}{\pi}\sqrt{\varepsilon},$$

whence, minimizing with respect to ε, we get

$$|I(t;\varphi_0^{(s)}(y),q(y))|\le 2^{\frac{9ks+1}{k(s+1)}}\pi^{-\frac{2ks-1}{k(s+1)}}s^{\frac{1}{s+1}}\tau^{-\frac{1}{k(s+1)}}\ln^{\frac{s-1}{s+1}}(2+\tau/2\pi). \quad (5)$$

Further,

$$(t,\widetilde{\varphi}_0^{(s)}(y))=\sum_{\mathbf{j}\in J_{k,s}}t_{\mathbf{j}}T_{j_1}(y_1)\cdots T_{j_s}(y_s)=(t^0)'A_k^{(s)}(1,\varphi_0^{(s)}(y))$$

with $A_k^{(s)}=\underbrace{A_k\otimes\cdots\otimes A_k}_{s\ \text{times}}$, $A_k^{(1)}=A_k$, where $\otimes$ denotes the Kronecker product, $t^0=(0,t)$ and $\varphi_0^{(s)}(y)$ and t are understood as row vectors with lexicographically arranged components.

From (4) we now obtain

$$I(t;\varphi_1^{(s)}(x),\mathbb{1}_{(0,1)^s}(x))=I(A_k^{(s)'}t^0;(1,\varphi_0^{(s)}(y)),q(y)). \quad (6)$$

Since $\rho(B_1\otimes B_2)=\rho(B_1)\rho(B_2)$ for matrices B_1 and B_2, according to [9, Theorem 6.5.1], $\tau((A_k^{(s)})'t^0)\ge\tau/\rho((A_k^{(s)'})^{-1})\ge\tau/\rho_k^s$, which by virtue of (5) and (6) leads to (a).

To prove (b), note that since $A_k^{(s)}(A_k^{(s)})'=(A_kA_k')^{(s)}$ (see [10]) and for two positive definite matrices $\lambda_{\min}(B_1\otimes B_2)=\lambda_{\min}(B_1)\lambda_{\min}(B_2)$ (see, e.g., Supplement to [11]), we obtain $|A_k^{(s)'}t^0|=((t^0)'A_k^{(s)}(A_k^{(s)})'(t^0))^{1/2}\ge\lambda_k^{s/2}|t|$ and $\tau(A_k^{(s)'}t^0)\ge[(k+1)^s-1]^{-1/2}\lambda_k^{s/2}|t|$. □

Other applications of (2) to obtain bounds for c.f. can be found in [12]. Inequalities similar to the general ones from [2] can be found in [13].

Based on the theory of singularities and asymptotic expansions of oscillating integrals [14] one can study the exactness properties of bounds (a) and (b) with respect to $|t|$ and τ for the case $s>1$, too.

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