GEORGIAN MATHEMATICAL JOURNAL: Vol. 4, No. 6, 1997, 567-578

# ON THE BIFURCATION OF FLOWS OF A HEAT-CONDUCTING FLUID BETWEEN TWO ROTATING PERMEABLE CYLINDERS

L. SHAPAKIDZE

Abstract. Sufficient conditions are found for the bifurcation of flow of a viscous heat-conducting fluid between two rotating permeable cylinders.

This paper deals with second stationary flows generated in a heat-conducting fluid contained between two permeable cylinders rotating in the same direction. Among other papers where similar problems are treated mention should be made of [1–4] for the case of a noncompressible fluid and [5] for the case of a heat-conducting fluid. The permeability of the cylinders changes the character of the obtained operator equations, which results in nonsymmetricity of the kernels of the corresponding integral equations. This fact necessitates to another method of investigation of this problem and this is what we do here.

**1.** Let a homogeneous viscous heat-conducting fluid fill up the hollow space between two rotating permeable cylinders heated up to different temperatures. The radii, angular velocities, and temperatures of the internal and outer cylinders are denoted by $R_1$, $\Omega_1$, $\theta_1$ and $R_2$, $\Omega_2$, $\theta_2$, respectively. It is assumed that there are no external mass forces, the velocity of the flow across the cross-section of the hollow space between the cylinders is zero, and the fluid inflow through one cylinder is equal to the fluid outflow through the other. The scales of length, velocity, and temperature will be denoted by $R_1$, $\Omega_1 R_1$, $\theta_1$, while the density scale will be understood as the fluid density at the temperature $\theta_1$. Under these assumptions, if we write the Navier–Stokes equations and heat conductivity equation in terms of cylindrical coordinates $(r,\varphi,z)$ with the axis $z$ coinciding with the axis

1991 *Mathematics Subject Classification.* 76E15.
*Key words and phrases.* Flow, bifurcation, perturbation, axisymmetric flow, permeable cylinders.



1072-947X/97/1100-0567$12.50/0 

of the cylinders, then they will admit the following exact solution with the velocity vector $\vec{V_0}(v_{0r}, v_{0\varphi}, v_{0z})$, temperature $T_0$, and pressure $\Pi_0$:

$$
\begin{gathered}
v_{0r} = \frac{\varkappa_0}{r}, \quad v_{0\varphi} = \begin{cases} ar^{\varkappa+1} + b/r, & \varkappa \neq -2, \\ \dfrac{a_1 \ln r + 1}{r}, & \varkappa = -2, \end{cases} \\
v_{0z} = 0, \quad T_0 = c_1 + c_2 r^{\varkappa_1}, \qquad (1.1) \\
\Pi_0 = \int\limits_1^r \Big\{ \big[1 - \beta\theta_1(c_1 + c_2 r^{\varkappa_1})\big] \Big( ar^{\varkappa} + \frac{b}{r^2} \Big)^2 r + \frac{\varkappa_0^2}{r^3} \Big\} dr;
\end{gathered}
$$

here

$$
\begin{gathered}
a = \frac{\Omega R^2 - 1}{R^{\varkappa+2} - 1}, \quad b = 1 - a, \quad a_1 = \frac{\Omega R^2 - 1}{\ln R}, \\
c_1 = \frac{\theta - R^{\varkappa_1}}{1 - R^{\varkappa_1}}, \quad c_2 = \frac{1 - \theta}{1 - R^{\varkappa_1}}, \quad \varkappa_0 = \frac{s}{\Omega_1 R_1^2}, \quad \varkappa = \frac{s}{\nu}, \\
\varkappa_1 = \frac{s}{\chi}, \quad \theta = \frac{\theta_2}{\theta_1}, \quad R = \frac{R_2}{R_1},
\end{gathered}
$$

$s$ is the radial flow per cylinder length unit; $\beta$, $\nu$ and $\chi$ are, respectively, the thermal expansion, kinematic viscosity, and heat conductivity coefficients.

Our task here consists in finding axisymmetric stationary flows which differ from (1.1), are periodic with respect to $z$ with period $2\pi/\alpha_0$, and are such that the velocity flow across the cross-section of the cylinder cavity is zero.

**2.** To find solutions $V'$, $\Pi'$, $T'$ of our problem in the form $\vec{V}' = \vec{V_0} + \vec{v}(v_r, v_\varphi, v_z)$, $T' = T_0 + c_2 PT$, $\Pi' = \Pi_0 + \Pi/\lambda$, we obtain the following system of perturbation equations:

$$
\begin{aligned}
&\Delta v_r - \frac{v_r}{r^2} - \frac{\partial \Pi}{\partial r} = \lambda \Big[ (\vec{v}, \nabla) v_r - \frac{v_\varphi^2}{r} - 2\omega_1 v_\varphi + \\
&\qquad\qquad + \frac{\varkappa_0}{r} \Big( \frac{\partial v_r}{\partial r} - \frac{v_r}{r} \Big) + Ra\, \omega_2 T \Big], \\
&\Delta v_\varphi - \frac{v_\varphi}{r^2} = \lambda \Big[ (\vec{v}, \nabla) v_\varphi + \frac{v_r v_\varphi}{r} - g_1(r) v_r + \frac{\varkappa_0}{r} \Big( \frac{\partial v_\varphi}{\partial r} + \frac{v_\varphi}{r} \Big) \Big], \\
&\Delta v_z - \frac{\partial \Pi}{\partial z} = \lambda \Big[ (\vec{v}, \nabla) v_z + \frac{\varkappa_0}{r} \frac{\partial v_z}{\partial r} \Big], \\
&\Delta T = \lambda P \Big[ (\vec{v}, \nabla) T + \frac{\varkappa_0}{r} \frac{\partial T}{\partial r} + g_2(r) v_r \Big], \\
&\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = 0;
\end{aligned}
\qquad (2.1)
$$

$$
\vec{v}\big|_{r=1,R} = 0, \quad T\big|_{r=1,R} = 0, \qquad (2.2)
$$

where $Ra = \beta c_2 \theta_1 P$ is the Rayleigh number, $P = \frac{\nu}{\chi}$ is the Prandtl number, $\lambda = \frac{\Omega_1 R_1^2}{\nu}$ is the Reynolds number, $\omega_1 = \frac{v_{0\varphi}}{r}$, $\omega_2 = \omega_1^2 r$, $\varkappa_1 = \varkappa P$,

$$g_1(r) = \begin{cases} -(\varkappa+2)ar^{\varkappa}, & \varkappa \neq -2, \\ -\dfrac{a_1}{r^2}, & \varkappa = -2, \end{cases} \qquad g_2(r) = \varkappa r^{\varkappa P - 1},$$

$$(\vec{V}, \nabla) = v_r \frac{\partial}{\partial r} + v_z \frac{\partial}{\partial z}, \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2},$$

and the components $v_r$, $v_\varphi$, $v_z$,$T$ must satisfy the following conditions: $\int\limits_1^r v_z(r,z) r\,dr = 0$, $\vec{V}$, $T$ are periodic with respect to $z$ with period $2\pi/\alpha_0$; $v_r$, $v_\varphi$, $T$ are odd functions, and $v_z$ is an even function with respect to $z$.

Problem (2.1)–(2.2) is written in terms of the Boussinesq approximation [6] assuming that the flow velocity through the cylinder walls is such that it is not influenced by perturbations arising in the fluid between the two cylinders.

To flow (1.1) there corresponds a trivial solution of problem (2.1)–(2.2) and we assume that for small $\lambda$ this system has a unique solution $\vec{v} = T = 0$.

The linearized problem corresponding to system (2.1)–(2.2)

$$\begin{aligned}
&\Delta u_r - \frac{u_r}{r^2} - \frac{\partial \Pi_1}{\partial r} = \lambda\Big[-2\omega_1 u_\varphi + \frac{\varkappa_0}{r}\Big(\frac{\partial u_r}{\partial r} - \frac{u_r}{r}\Big) + Ra\,\omega_2 T_1\Big], \\
&\Delta u_\varphi - \frac{u_\varphi}{r^2} = \lambda\Big[-g_1(r)u_r + \frac{\varkappa_0}{r}\Big(\frac{\partial u_\varphi}{\partial r} + \frac{u_\varphi}{r}\Big)\Big], \\
&\Delta u_z - \frac{\partial \Pi_1}{\partial z} = \lambda\,\frac{\varkappa_0}{r}\frac{\partial u_z}{\partial r}, \\
&\Delta T_1 = \lambda P\Big[\frac{\varkappa_0}{r}\frac{\partial T_1}{\partial r} + g_2(r)u_r\Big], \\
&\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = 0;
\end{aligned} \tag{2.3}$$

$$\vec{u}(u_r, u_\varphi, u_z)\big|_{r=1,R} = 0, \quad T_1\big|_{r=1,R} = 0 \tag{2.4}$$

and the conjugate problem of (2.3)–(2.4) with respect to the scalar product

$$[\vec{u}, \vec{\psi}] = \int\limits_1^R \int\limits_{-\pi/\alpha_0}^{\pi/\alpha_0} \vec{u}\cdot\vec{\psi}\, r\,dr\,dz$$

can be respectively written as

$$
\begin{aligned}
&\Delta\psi_r-\frac{\psi_r}{r^2}=\frac{\partial Q}{\partial r}+\lambda\Big[-g_1(r)\psi_\varphi-\frac{\varkappa_0}{r}\Big(\frac{\partial\psi_r}{\partial r}+\frac{\psi_r}{r}\Big)+Pg_2(r)T_2\Big],\\
&\Delta\psi_\varphi-\frac{\psi_\varphi}{r^2}=\lambda\Big[-2\omega_1\psi_r-\frac{\varkappa_0}{r}\Big(\frac{\partial\psi_\varphi}{\partial r}-\frac{\psi_\varphi}{r^2}\Big)\Big],\\
&\Delta\psi_z=\frac{\partial Q}{\partial z}-\lambda\,\frac{\varkappa_0}{r}\,\frac{\partial\psi_z}{\partial r},\\
&\Delta T_2=\lambda P\Big[-\frac{\varkappa_0}{r}\,\frac{\partial T_2}{\partial r}+\frac{Ra}{P}\,\omega_2\psi_r\Big],\\
&\frac{\partial\psi_r}{\partial r}+\frac{\psi_r}{r}+\frac{\partial\psi_z}{\partial z}=0;
\end{aligned}
\tag{2.5}
$$

$$
\vec{\psi}(\psi_r,\psi_\varphi,\psi_z)\big|_{r=1,R}=0,\quad T_2\big|_{r=1,R}=0. \tag{2.6}
$$

Let us consider the set $M$ of twice continuously differentiable solenoidal pairs $\vec{V}\{\vec{v}(v_r,v_\varphi,v_z),T\}$ which are defined in the closed domain $\{1\le r\le R$, $-\infty<z<+\infty\}$ and which are axisymmetric, vanish for $r=1,R$, have a flow across the cross-section of the hollow space between the cylinders equal to zero, and are such that $v_r$, $v_\varphi$, $T$ are even functions and $v_z$ is an odd function with respect to $z$. Denote by $H_1$ the Hilbert space obtained by completion of the set $M$ with respect to the norm generated by the scalar product

$$
\begin{aligned}
(\vec{V}\cdot\vec{V}^{\mathrm{I}})_{H_1}=-\int\limits_{-\pi/\alpha_0}^{\pi/\alpha_0}dz\int\limits_1^R\Big\{\Big(\Delta v_r-\frac{v_r}{r^2}\Big)v_r^{\mathrm{I}}+\\
+\Big(\Delta v_\varphi-\frac{v_\varphi}{r^2}\Big)v_\varphi^{\mathrm{I}}+\Delta v_z\cdot v_z^{\mathrm{I}}+\Delta T\cdot T^{\mathrm{I}}\Big\}rdr,\ \ \vec{V}^{\mathrm{I}}\in M.
\end{aligned}
$$

Following [7], problem (2.1)–(2.2) can be reduced to the nonlinear operator equation

$$
\vec{V}=\lambda K\vec{V}. \tag{2.7}
$$

The linearized problem (2.3)–(2.4) and its conjugate problem will respectively satisfy the operator equations

$$
\vec{U}=\lambda A\vec{U}, \tag{2.8}
$$

$$
\vec{\Psi}=\lambda A^*\vec{\Psi}. \tag{2.9}
$$

Applying the results of [7, 8], we easily ascertain that the operators $K$, $A$, and $A^*$ are completely continuous in the space $H_1$. The operator $A$ is the Frechet differential of the operator $K$ at the point $\vec{V}=0$, and $A^*$ is the conjugate operator of $A$ in the space $H_1$.

To apply the bifurcation theory of nonlinear operator equations it is necessary to investigate the spectrum of the linear operator $A$, since, as follows from Krasnoselskii's results [9], the bifurcation points of the nonlinear operator $K$ can be only having the odd multiplicity (in particular, simple ones) characteristic numbers of its Frechet differential at the point $\vec{V}=0$.

**3. Theorem.** *Let the following conditions be fulfilled: $\varkappa Ra>0$ and the functions $\omega_k(r)$, $g_k(r)$ $(k=1,2)$ are positive throughout the interval $(1,R)$. Then for all $\alpha_0$, except some countable set, the operator $A$ has at least one positive simple characteristic number $\lambda_0$ which is the bifurcation point of the nonlinear operator $K$. This characteristic number is less than the moduli of all other characteristic numbers of the operator $A$.*

*Proof.* Using a Fourier series expansion, the solution of the linear problem (2.3)–(2.4) can be represented as a linear combination of solutions of the form

$$\{u_r,u_\varphi,\Pi_1,T_1\}=\{u(r),v(r),p_1(r),\tau(r)\}\cos\alpha z,$$
$$u_z=w\sin\alpha z,\quad \alpha=n\alpha_0\quad (n=1,2,\dots),$$

which leads us to the spectral problem

$$\begin{aligned}&\Big[L-\frac{\varkappa}{r}\Big(\frac{d}{dr}-\frac{1}{r}\Big)-\alpha^2\Big](L-\alpha^2)u=\lambda(2\alpha^2\omega_1 v-\alpha^2 Ra\,\omega_2\tau),\\ &-\Big[L-\frac{\varkappa}{r}\Big(\frac{d}{dr}+\frac{1}{r}\Big)-\alpha^2\Big]v=\lambda g_1(r)u,\\ &-\Big(L-\frac{\varkappa P}{r}\frac{d}{dr}+\frac{1}{r^2}-\alpha^2\Big)\tau=-\lambda P g_2(r)u;\end{aligned}\tag{3.1}$$

$$u\big|_{r=1,R}=v\big|_{r=1,R}=\frac{du}{dr}\Big|_{r=1,R}=\tau\big|_{r=1,R}=0,\tag{3.2}$$

where

$$L=\frac{d}{dr}\Big(\frac{d}{dr}+\frac{1}{r}\Big),\quad w(r)=-\frac{1}{\alpha r}\frac{d}{dr}(ru),$$
$$p_1=-\frac{1}{\alpha}\Big(\frac{d^2}{dr^2}+\frac{1-\varkappa}{r}\frac{d}{dr}-\alpha^2\Big)w.$$

We introduce the integral operators

$$G_kf=\int_1^R G_\varkappa^k(r,\rho)f(\rho)\rho\,d\rho\quad (k=1,2,3),$$

where $G_\varkappa^k$ are the Green functions of the operators on the left-hand sides of system (3.1) at the boundary conditions (3.2).

Denote by $H_1^0$ the Hilbert space $L_2$ with the weight $\sigma(r) = r$ on the segment $[1, R]$ with the scalar product

$$(\psi_1, \psi_2)_{H_1^0} = \int_1^R \psi_1(r)\psi_2(r) r \, dr.$$

**Lemma 1.** *The kernels $G_\varkappa^k$ $(k = 1, 2, 3)$ are nonsymmetric and oscillatory.*

The lemma can be easily proved by the methods of Krein [10]. The kernels $G_\varkappa^1$ and $G_\varkappa^2$ are proved to be oscillatory in [11]. As for $G_\varkappa^3$, the fact that it is oscillatory follows from the representation

$$-\Big(L - \frac{\varkappa P}{r}\frac{d}{dr} + \frac{1}{r}\Big)\tau = \frac{r^{\varkappa P}}{\omega_0}\frac{d}{dr}\, r^{1-\varkappa P}\omega_0^2 \frac{d}{dr}\frac{\tau}{\omega_0},$$

where $\omega_0 = I_{\frac{\varkappa P}{2}}$ is the modified Bessel function which is a solution of the equation

$$\Big(L - \frac{\varkappa P}{r}\frac{d}{dr} + \frac{1}{r^2} - \alpha^2\Big)\omega_0 = 0.$$

By inverting the operators on the left-hand sides of system (3.1) we obtain

$$\begin{aligned} u &= \lambda(2\alpha^2 G_1\omega_1(r)v - Ra\,\alpha^2 G_1\omega_2(r)\tau), \\ v &= \lambda G_2 g_1(r)u, \\ \tau &= -\lambda P G_3 g_2(r)u. \end{aligned} \tag{3.3}$$

The spectral problem (3.3) is equivalent to an integral equation

$$u = \mu B u, \tag{3.4}$$

where $\mu = 2\alpha^2\lambda^2$, $B = B_1 + B_2$,

$$B_1 = G_1\omega_1(r)G_2 g_1(r), \quad B_2 = \frac{1}{2}\, Ra\, P G_1\omega_2(r)G_3 g_2(r).$$

Lemma 1 implies that the kernels of the integral operators $B_1$ and $B_2$ are nonsymmetric oscillatory ones.

Similarly, in finding a solution of the conjugate problem (2.5)–(2.6) in the form

$$\begin{gathered} \{\psi_r, \psi_\varphi, Q, T_2\} = \{u_1(r), v_1(r), q(r), \tau_1(r)\}\cos\alpha z, \\ \psi_z = w_1(r)\sin\alpha z, \end{gathered}$$

we come to the problem of defining the eigenfunctions:

$$
\begin{aligned}
&(L-\alpha^2)\Big[L+\frac{\varkappa}{r}\Big(\frac{d}{dr}+\frac{1}{r}\Big)-\alpha^2\Big]u_1=\lambda\alpha^2 g_1(r)v_1-\lambda\alpha^2 P g_2(r)\tau_1,\\
&\Big[L+\frac{\varkappa}{r}\Big(\frac{d}{dr}-\frac{1}{r}\Big)-\alpha^2\Big]v_1=-2\lambda\omega_1(r)u_1,\\
&\Big(L+\frac{\varkappa P}{r}\,\frac{d}{dr}+\frac{1}{r^2}\Big)\tau_1=\lambda Ra\,\omega_2(r)u_1,
\end{aligned}
\tag{3.5}
$$

$$
u_1\big|_{r=1,R}=\frac{du_1}{dr}\Big|_{r=1,R}=v_1\big|_{r=1,R}=\tau_1\big|_{r=1,R}=0,
\tag{3.6}
$$

where

$$
q=-\frac{1}{\alpha}\Big(\frac{d^2}{dr^2}+\frac{1+\varkappa}{r}\,\frac{d}{dr}-\alpha^2\Big)w_1,\quad w_1=-\frac{1}{\alpha r}\,\frac{d}{dr}\,(ru_1).
$$

Thus the linearized problem (2.3)–(2.4), equivalent to the operator equation (2.8) in the Hilbert space $H_1$, can be reduced, after separation of variables, to the integral equation (3.4). The characteristic numbers of the operators $A$ and $B$ are related by the relation $\mu=2\alpha^2\lambda^2$.

**Lemma 2.** *If $\varkappa Ra>0$ and the functions $\omega_k(r)$, $g_k(r)$ $(k=1,2)$ are positive throughout the interval $(1,R)$, then the operator $B$ is $u_0$-positive in the cone of non-negative functions.*

The proof of this lemma follows from the results of [12] and Lemma 1. Similar statements for the corresponding operator represented as the sum of oscillatory operators can be found in [13].

Lemma 2 implies that for any value of $\alpha_0$ the operator $B$ has at least one positive simple characteristic number $\mu_0$ [12]. In particular, this means that the rank of $\mu_0$ (i.e., $\dim(\operatorname{Ker}(B-\mu_0 I))$, where $I$ is the identical operator) is equal to unity (see [7]).

**Lemma 3.** *Let $\mu>0$ be the characteristic number of the operator $B$ whose rank is equal to unity. Then $\lambda=\pm\sqrt{\mu/2\alpha^2}$ is the characteristic number of the operator $A$ whose rank is also equal to unity.*

To prove a similar lemma for the case of solid cylinders and a noncompressible fluid [1, 2] it is essential to assume that the operator $B$ is symmetric, since the operators contained in it are symmetric. Then the corresponding operator $B$ is a symmetric oscillatory operator. In the presence of the parameter $s$, i.e., when the cylinder walls are permeable, the symmetricity of the operator $B$ is violated and the corresponding operator $B$ is a nonsymmetric oscillatory one [11]. In the case of a heat-conducting fluid and permeable cylinder walls the corresponding operator $B$ is, as shown above, a nonsymmetric, $u_0$-positive one in the cone of non-negative functions.

*Proof.* We calculate the scalar product $(\vec{U}\cdot\vec{\Psi})$, where $\vec{U}$, $\vec{\Psi}$ are the eigenvectors of the operators $A$ and $A^*$. Multiplying the equations of system (2.3) by $\psi_r$, $\psi_\varphi$, $\psi_z$, $T_2$, respectively, and taking into account (3.1) and (3.5), also performing integration by parts and some simple transformations we obtain

$$
\begin{aligned}
(\vec{U}\cdot\vec{\Psi})_{H_1} &= \lambda\,\frac{\pi}{\alpha_0}\int\limits_1^R \Big\{\Big(g_1(r)u-\frac{\varkappa_0}{r}\Big(\frac{dv}{dr}+\frac{v}{r}\Big)\Big)v_1+\\
&+\big(2\omega_1(r)v-Ra\,\omega_2(r)\tau\big)u_1-\frac{1}{\alpha^2}\,\frac{\varkappa_0}{r}\Big(\frac{du_1}{dr}+\frac{u_1}{r}\Big)(L-\alpha^2)u-\\
&-\Big(\frac{\varkappa_0}{r}\,\frac{d\tau}{dr}+g_2(r)\Big)P\tau_1\Big\}r\,dr=\\
&=\lambda\,\frac{\pi}{\alpha_0}\int\limits_1^R \Big\{\Big(g_1(r)u-\frac{\varkappa_0}{r}\Big(\frac{dv}{dr}+\frac{v}{r}\Big)\Big)v_1+\big(2\omega_1(r)v-Ra\,\omega_2(r)\tau\big)u_1+\\
&+\frac{1}{\alpha^2}\,\frac{\varkappa_0}{r}\Big(\frac{d}{dr}-\frac{1}{r}\Big)(L-\alpha^2)u\cdot u_1-\Big(\frac{\varkappa_0}{r}\,\frac{d\tau}{dr}+g_2(r)u\Big)P\tau_1\Big\}r\,dr=\\
&=\frac{\pi}{\alpha_0}\int\limits_1^R\Big\{\frac{1}{\alpha^2}(L-\alpha^2)^2u\cdot u_1-(L-\alpha^2)v\cdot v_1-\Big(L+\frac{1}{r^2}-\alpha^2\Big)\tau\cdot\tau_1\Big\}r\,dr.
\end{aligned}
$$

Denote by $H_2^0$ the Hilbert space of square-summable vector-functions $\vec{V}(u,v,\tau)$ with the scalar product

$$
(\vec{V}\cdot\vec{V}^{\mathrm{I}})_{H_2^0}=\int\limits_1^R (u\cdot u_1+v\cdot v_1+\tau\cdot\tau_1)r\,dr,\quad \vec{V}^{\mathrm{I}}(u_1,v_1,\tau_1)\in H_2^0.
$$

Since the characteristic number $\mu>0$ of the operator $B$ is simple, one can easily verify that $\vec{V}(u,v,\tau)\in H_2^0$, where $(u,v,\tau)$ is a solution of problem (3.1)–(3.2), is also a simple eigenvector of this system.

Let us consider the linear space $N$ of the vector-functions defined on the segment $[1,R]$ and satisfying the following conditions: $u$ are continuously differentiable functions on the segment $[1,R]$ up to the fourth order inclusive with the condition $u|_{r=1,R}=\frac{du}{dr}|_{r=1,R}=0$; $v$, $\tau$ are continuously differentiable functions up to the second order inclusive with the boundary condition $v|_{r=1,R}=\tau|_{r=1,R}=0$.

Since the operators $r(L-\alpha^2)^2$, $-r(L-\alpha^2)$, $-r(L+\frac{1}{r^2}-\alpha^2)$ are positive definite, by closing the linear space $N$ in the norm generated by the scalar product

$$(\vec{V}\cdot\vec{V}^{\mathrm{I}})_{H_2}=\int\limits_1^R\Big[\frac{1}{\alpha^2}(L-\alpha^2)^2u\cdot u^{\mathrm{I}}-(L-\alpha^2)v\cdot v^{\mathrm{I}}-\Big(L+\frac{1}{r^2}-\alpha^2\Big)\tau\cdot\tau^{\mathrm{I}}\Big]r\,dr,$$

$$\vec{V}(u,v,\tau),\quad \vec{V}^{\mathrm{I}}(u^{\mathrm{I}},v^{\mathrm{I}},\tau^{\mathrm{I}})\in H_2,$$

we obtain the complete energetic Hilbert space $H_2$ (see [14]).

Rewrite problems (3.1)–(3.2) and (3.3)–(3.4) in the space $H_2$ in the operator form:

$$\vec{V}=\lambda K_1\vec{V},\quad \vec{V}_1=\lambda K_1^*\vec{V}_1,\quad \vec{V},\vec{V}_1\in H_2,$$

where $K_1$ and $K_1^*$ are completely continuous operators acting in the space $H_2$ and satisfying an additional requirement that the integral identities

$$\begin{aligned}(K_1\vec{V}\cdot\vec{\Phi})_{H_2}=\lambda\int\limits_1^R\Big\{&\Big[2\omega_1v+\frac{\varkappa_0}{r\alpha^2}\Big(\frac{d}{dr}-\frac{1}{r}\Big)(L-\alpha^2)u-\\ &-Ra\,\omega_2\tau\Big]\Phi_r-\Big[\frac{\varkappa_0}{r}\Big(\frac{d}{dr}+\frac{1}{r}\Big)v-g_1(r)u\Big]\Phi_\varphi-\\ &-\Big[Pg_2(r)u+\frac{\varkappa P}{r}\frac{d}{dr}\Big]\Phi_z\Big\}r\,dr,\end{aligned}$$

$$\begin{aligned}(K_1^*\vec{V}_1\cdot\vec{\Phi})_{H_2}=\lambda\int\limits_1^R\Big\{&\Big[g_1(r)v_1-\frac{\varkappa_0}{\alpha^2}(L-\alpha^2)\frac{1}{r}\Big(\frac{d}{dr}+\frac{1}{r}\Big)u_1-\\ &-Pg_2(r)\tau_1\Big]\Phi_r+\Big[2\omega_1(r)u_1+\frac{\varkappa_0}{r}\Big(\frac{d}{dr}-\frac{1}{r}\Big)\Big]\Phi_\varphi+\\ &-\Big[Ra\,\omega_2(r)u_1-\frac{\varkappa P}{r}\frac{d\tau_1}{dr}\Big]\Phi_z\Big\}r\,dr\end{aligned}$$

be fulfilled for any vectors $\vec{V}$, $\vec{V}_1$, $\vec{\Phi}\in H_2$ (see [7]).

Performing integration by parts, we readily obtain the equality

$$(K_1^*\vec{V}_1,\vec{\Phi})_{H_2}=(\vec{V}_1,K_1\vec{\Phi})_{H_2}.$$

Therefore $K_1$ is the conjugate operator of $K_1^*$ in the space $H_2$.

We use the results of [14], in particular the theorem stating that to each element from $H_2$ there may correspond only one element from $H_2^0$. Note that in that case to different elements from $H_2$ there correspond different elements from $H_2^0$. Hence it is not difficult to show that if the equations

$$\vec{V}=\lambda K_1\vec{V},\quad \vec{W}=\lambda K_1\vec{W}+\vec{V},\quad \vec{V},\vec{W}\in H_2,$$

where $K_1$ is a completely continuous operator in $H_2$, are fulfilled, then in the space $H_2^0$ the equations

$$\tilde{\vec{V}} = \lambda K_1 \tilde{\vec{V}}, \quad \tilde{\vec{W}} = \lambda K_1 \tilde{\vec{W}} + \tilde{\vec{V}}, \quad \tilde{\vec{V}}, \tilde{\vec{W}} \in H_2^0,$$

will also have solutions.

Indeed, let us be given the equation

$$\vec{V} = \lambda K_1 \vec{V}, \quad \vec{V} \in H_2.$$

Let us consider a sequence $\vec{V}_n \in H_2$ such that

$$\|\vec{V} - \vec{V}_n\|_{H_2} \to 0, \quad \|\tilde{\vec{V}} - \tilde{\vec{V}}_n\|_{H_2^0} \to 0.$$

The existence of such a sequence follows from the proof of the above-mentioned theorem from [14].

We write the equality

$$\vec{V} - \vec{V}_n = \lambda K_1(\vec{V} - \vec{V}_n) + \delta_n, \tag{3.7}$$

where $\vec{V}_n \in H_2$, $\delta_n = \lambda K_1 \vec{V}_n - \vec{V}_n$.

Then we have

$$\|\delta_n\|_{H_2} \leq \|\vec{V} - \vec{V}_n\|_{H_2} + \lambda \|K_1(\vec{V} - \vec{V}_n)\|_{H_2}.$$

Using the inequality from [14]

$$\|\vec{V}\|_{H_2^0} \leq \|\vec{V}\|_{H_2}$$

we find that $\|\delta_n\|_{H_2} \to 0$ implies $\|\delta_n\|_{H_2^0} \to 0$.

Passing in (3.7) to the limit in $H_2^0$, we obtain

$$\vec{V} - \tilde{\vec{V}} = \lambda(K_1 \vec{V} - K_1 \tilde{\vec{V}})$$

which gives

$$\tilde{\vec{V}} = \lambda K_1 \tilde{\vec{V}},$$

where $\tilde{\vec{V}} \in H_2^0$.

By a similar reasoning one can ascertain that if the equation

$$\vec{W} = \lambda K_1 \vec{W} + \vec{V}, \quad \vec{V}, \vec{W} \in H_2,$$

has a solution, then the corresponding equation

$$\tilde{\vec{W}} = \lambda K_1 \tilde{\vec{W}} + \tilde{\vec{V}}, \quad \tilde{\vec{V}}, \tilde{\vec{W}} \in H_2^0,$$

will be fulfilled in $H_2^0$, which is impossible because $\vec{V} \in H_2^0$ is a simple eigenvector and, accordingly, $\lambda$ is a simple characteristic number of problem (3.1)–(3.2). Hence it follows that $(\vec{V} \cdot \vec{V}_1)_{H_2} \neq 0$ [7]. Now we obtain

$$(\vec{U} \cdot \vec{\Psi})_{H_1} = \frac{\pi}{\alpha_0}(\vec{V} \cdot \vec{V}_1)_{H_2} \neq 0.$$

Therefore the rank of the characteristic number $\lambda$ of the operator $A$ is equal to unity. □

Next, using the arguments from [1], we show that $\lambda_0 = \sqrt{\dfrac{\mu_0}{2\alpha_0^2}}$ is the simple characteristic number of the operator $A$.

Since the operator $B$ is $u_0$-positive, the characteristic number $\mu_0$ is less than the moduli of all other characteristic numbers of the operator $B$ [9]. But in that case $\lambda_0$ is less than the moduli of all other characteristic numbers $\lambda = \sqrt{\dfrac{\mu}{2\alpha^2}}$ of the operator $A$. □

Thus we have shown that under the conditions of the theorem the operator $A$ has at least one simple characteristic number which is the bifurcation point of the operator $K$. In that case the main flow (1.1) gives rise to secondary axisymmetric stationary flow bifurcations.

One can easily verify that the conditions of the theorem are fulfilled when the temperature of the internal cylinder exceeds the temperature of the external cylinder ($\theta < 1$) in the case of fluid inflow through the external cylinder ($\varkappa < 0$), and, conversely, when the temperature of the external cylinder exceeds the temperature of the internal cylinder ($\theta > 1$) in the case of fluid inflow through the internal cylinder ($\varkappa > 0$), while the angular velocities and radii are related through the relation $0 < \Omega < \frac{1}{R^2}$.

Note that if $Ra = 0$ then for any $\varkappa$ the condition $0 < \Omega < \frac{1}{R^2}$ is the sufficient one for secondary axisymmetric stationary flows to arise in the noncompressible fluid between two rotating permeable cylinders. Moreover, for each $\alpha_0$ we have a sequence of simple characteristic numbers of the operator $A$, each of which is the bifurcation point of the corresponding nonlinear operator [11].

## References

1. V. I. Yudovich, Secondary flows and fluid instability between rotating cylinders. (Russian) *Prikl. Mat. Mekh.* **30**(1966), No. 4, 688–698; *English translation: J. Appl. Math. Mech.* **30(1966)**(1967), 822–833.

2. V. I. Yudovich, The bifurcation of a rotating flow of liquid. (Russian) *Dokl. Akad Nauk SSSR* **169**(1966), No. 2, 306–309; *English translation: Soviet Phys. Doklady* **11(1966)**(1967), 566–568.

3. Yu. P. Ivanilov and G. N. Yakovlev, The bifurcation of fluid flow between rotating cylinders. (Russian) *Prikl. Mat. Mekh.* **30**(1966), No. 4, 688–698; *English translation: J. Appl. Math. Mech.* **30(1966)**(1967), 910–916.

4. W. Velte, Stabilität und Verzweigung stationärer Lösungen der Navier-Stokesschen Gleichungen beim Taylor problem. *Arch. Rat. Mech. Anal.* **22**(1966), No. 1, 1–14.

5. V. V. Kolesov, Occurrence of Taylor vortices between heated rotating cylinders. (Russian) *Prikl. Mekh. Tekn. Fiz.* **6**(1981), 87–93.

6. L. D. Landau and E. M. Lifshits, Fluid dynamics, 3rd ed. (Russian) *Nauka, Moscow,* 1986.

7. V. I. Yudovich, Example of the generation of a secondary stationary or periodic flow when there is loss of stability of the laminar flow of a viscous incompressible fluid. (Russian) *Prikl. Mat. Mekh.* **29**(1965), No. 3, 453–467; *English translation: J. Appl. Math. Mech.* **29**(1965), 527–544.

8. I. I. Vorovich and V. I. Yudovich, Steady flow of a viscous incompressible fluid. (Russian) *Mat. Sb.* **53(95)**(1961),393–428.

9. M. A. Krasnoselskii, Topological methods in the theory of non-linear integral equations. (Russian) *Gosudarstv. Izdat Tekhn. Teor. Lit., Moscow,* 1956.

10. M. G. Krein, On nonsymmetric oscillatory Green's functions of ordinary differential operators. (Russian) *Dokl. Akad. Nauk SSSR* **25**(1939), No. 8, 643–646.

11. L. D. Shapakidze, On the bifurcation of fluid flow between two rotating permeable cylinders. (Russian) *Soobshch. Akad. Nauk Gruz. SSR* **99**(1980), No. 2, 325–328.

12. M. A. Krasnoselskii, Positive solutions of operator equations. (Russian) *Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow,* 1962.

13. V. V. Kolesov, On stability conditions for the nonisothermic Couette flow. (Russian) *Prikl. Mat. Mech.* **44**(1980), No. 3, 441–447; *English translation: J. Appl. Math. Mech.* **44(1980)**(1981), No. 3, 311–315.

14. S. G. Mikhlin, Variational methods in mathematical physics. (Russian) *Nauka, Moscow,* 1970.

(Received 16.08.1995)

Author's address:
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 380093
Georgia