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INTERNAL CATEGORIES IN A LEFT EXACT COSIMPLICIAL CATEGORY

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ABSTRACT. The notion of an internal category in a left exact cosimplicial category is introduced. For any topos over sets a certain left exact cosimplicial category is constructed functorially and the category of internal categories in it is investigated. The notion of a fundamental group is defined for toposes admitting the notion of "a discrete category."

INTRODUCTION

Our primary interest in this paper is to introduce the notion of an internal category in a left exact cosimplicial category, generalizing ordinary internal categories. We call a cosimplicial category left exact when all of its components are categories with finite limits and all coface and codegeneracy functors preserve them.

In Section 1 the definitions are given of an internal category in a category (which is wellknown) as well as in a left exact cosimplicial category.

In Section 2 for any topos over **Sets** a certain left exact cosimplicial category is constructed functorially and the category of internal categories in it is investigated. Two examples of these constructions are considered. These examples correspond to the case where the topos is the category of sheaves over a locally compact topological space, and where it is a topos of presheaves.

In Section 3 we consider toposes which admit the notion of "a discrete internal category." For such toposes we determine the notion of fundamental group by means of the "discrete" category corresponding to the terminal object. When the topos is the category of sheaves over a locally compact and locally simply connected space, then its fundamental group is the same as the fundamental group of the underlying space.

Some words about the notation:

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Sets denotes the category of sets.

Cat denotes the 2-category of categories, functors, and natural transformations.

The 2-category of toposes, geometric morphisms, and natural transformations will be denoted by $\mathfrak{Top}$. Let **Top** be the full sub-2-category of $\mathfrak{Top}$, consisting of all toposes over **Sets**, i.e., objects of **Top** are such toposes, from whic there exists a geometric morphism in **Sets** (note that if such a morphism exists, it is unique up to a natural isomorphism). The product in the category **Top** will be denoted by $\underset{\mathbf{Set}}{\times}$.

Internal categories and internal functors in the category of topological spaces will be called continuous categories and continuous functors, respectively.

I denotes the topological space of real numbers between 0 and 1, $I = [0;1]$.

Internal categories in **Cat** will be called double categories [1].

For any category C with finite limits and any finite diagram f in it, we will denote by pr_x, or simply by pr, the natural projection from the limit of the diagram f (i.e., from $\varprojlim(f)$) to the term x (provided that this causes no ambiguity). For example, if f is of the shape $x \to y \leftarrow z$, then pr_x is the projection: $pr_x : \varprojlim(x \to y \leftarrow z) \to x$. For any product $x \times y \times z \times \cdots$ in C we denote by pr_i the projection of this product to the $i+1$-th term.

Let $\mathrm{Cat}(C)$ be the category of internal categories in C (we mean C with finite limits). Consider the forgetful functor from $\mathrm{Cat}(C)$ to C, which sends each internal category in C to its object of objects. We denote the right adjoint functor of this forgetful functor by $ad(-)$. Then for any object $x \in C$, the object of objects of the internal category $ad(x)$ will be x, the object of morphisms $x \times x$, and morphisms of source and target will be respectively the projections pr_0, pr_1 from $x \times x$ to x. (Note that $ad(x)$ is the antidiscrete internal category over x.)

1. The Notion of an Internal Category in a Left Exact Cosimplicial Category

First we introduce some useful notions.

The category Δ has as objects all finite ordinal numbers $[n] = 0, 1, \ldots, n$, and as arrows $f : [n] \to [m]$ all (weakly) monotone functions.

In the category Δ we choose arrows:

$\delta_n^i : [n-1] \to [n]$ for $0 \le i \le n$ is the injective monotone function whose image omits i;

$\sigma_n^i : [n+1] \to [n]$ for $0 \le i \le n$ is the surjective monotone function having the same value on i and $i+1$;

$\rho_n^{i,j} : [1] \to [n]$ for $0 \le i \le j \le n$ is the function: $\rho_n^{i,j}(0) = i$, $\rho_n^{i,j}(1) = j$;

$\rho_n^i : [0] \to [n]$ for $0 \le i \le n$ is the function for which $\rho_n^i(0) = i$.

Sometimes we will write these arrows omitting the subscript n.

As is wellknown, a cosimplicial (resp., simplicial) object in the category $\mathbf{C}$ is determined as a covariant (resp., contravariant) functor from Δ to $\mathbf{C}$. For any cosimplicial object F, we will write as usual F_n (resp., δ^i_n, σ^i_n, $\rho^{i,j}_n$, ρ^i_n) instead of $F[n]$ (resp., instead of $F(\delta^i_n)$, $F(\sigma^i_n)$, $F(\rho^{i,j}_n)$, $F(\rho^i_n)$).

We call a cosimplicial object in the category of categories $\mathbf{Cat}$ a cosimplicial category.

Definition 1.

(1) A cosimplicial category F is called left exact if for any $n\geq 0$, F_n is a category with finite limits and all functors $\delta^i : F_{n-1} \to F_n$, $\sigma^i : F_{n+1} \to F_n$ preserve them.

(2) A natural transformation α between two left exact cosimplicial categories F and F' is called left exact if for any $n\geq 0$ the functor $\alpha : F_n \to F'_n$ preserves all finite limits.

Remark 2. It is easy to check that any simplicial topos determines a left exact cosimplicial category by taking inverse image functors.

Now we will give the definitions of an internal category in a category and in a left exact cosimplicial category. The first definition is the wellknown one, but we need both definitions to compare them.

Definition 3. Let $\mathbf{C}$ be a category with finite limits. By an internal category X in $\mathbf{C}$ we mean a sixtuple $(X_1, X_0, \partial_1, \partial_0, s, m)$, where X_1 and X_0 are the objects of $\mathbf{C}$ (called the object of objects and the object of morphisms, respectively);

$\partial_1 : X_1 \to X_0$, $\partial_0 : X_1 \to X_0$, $s : X_0 \to X_1$,

$m : \varprojlim(X_1 \xrightarrow{\partial_0} X_0 \xleftarrow{\partial_1} X_1) \to X_1$

are morphisms in $\mathbf{C}$ (called morphisms of domain, codomain, identity, and composition, respectively), such that the following diagrams are commutative:

$$\begin{tikzcd} & X_1 & \\ X_0 \arrow[ur, "s"] \arrow[rr, equal, "id_{X_0}"] \arrow[dr, "s"'] & & X_0 \arrow[ul, "\partial_1"'] \arrow[dl, "\partial_0"] \\ & X_1 & \end{tikzcd} , \tag{1}$$

$$
\begin{tikzcd}[column sep=small]
& \varprojlim(X_1 \xrightarrow{\partial_1} X_0 \xleftarrow{\partial_0} X_1) \arrow[dl, "pr_1"'] \arrow[dr, "m"] & \\
X_1 \arrow[dr, "\partial_1"'] & & X_1 \arrow[dl, "\partial_1"] \\
& X_0 &
\end{tikzcd}
\tag{2}
$$

(here, pr_1 is the projection of $\varprojlim(X_1 \to X_0 \leftarrow X_1)$ to the left-hand X_1).

$$
\begin{tikzcd}[column sep=small]
& \varprojlim(X_1 \xrightarrow{\partial_1} X_0 \xleftarrow{\partial_0} X_1) \arrow[dl, "m"'] \arrow[dr, "pr_2"] & \\
X_1 \arrow[dr, "\partial_0"'] & & X_1 \arrow[dl, "\partial_0"] \\
& X_0 &
\end{tikzcd}
\tag{3}
$$

(here pr_2 is the projection of $\varprojlim(X_1 \to X_0 \leftarrow X_1)$ to the right-hand X_1).

$$
\begin{tikzcd}[column sep=tiny]
& \varprojlim\left(X_1 \xrightarrow{\partial_0} X_0 \xleftarrow{\partial_1} X_1 \xrightarrow{\partial_0} X_0 \xleftarrow{\partial_1} X_1\right) \arrow[dl, equal] \arrow[dr, equal] & \\
\varprojlim\left(X_1 \xrightarrow{\partial_0} X_0 \xleftarrow{\partial_1} X_1 \xrightarrow{pr_1} X_2\right) \arrow[d, equal, "\text{by the diagram (2)}"] & & \varprojlim\left(X_2 \xleftarrow{pr_2} X_1 \xrightarrow{\partial_0} X_0 \xleftarrow{\partial_1} X_1\right) \arrow[d, equal, "\text{by the diagram (3)}"] \\
\varprojlim\left(X_1 \xrightarrow{\partial_0} X_0 \xleftarrow{\partial_1} X_1 \xrightarrow{m} X_2\right) \arrow[d, "pr"] & & \varprojlim\left(X_2 \xleftarrow{m} X_1 \xrightarrow{\partial_0} X_0 \xleftarrow{\partial_1} X_1\right) \arrow[d, "pr"] \\
\varprojlim\left(X_1 \xrightarrow{\partial_0} X_0 \xleftarrow{\partial_1} X_1\right) \arrow[dr, "m"'] & & \varprojlim\left(X_1 \xrightarrow{\partial_0} X_0 \xleftarrow{\partial_1} X_1\right) \arrow[dl, "m"] \\
& X_1 &
\end{tikzcd}
\tag{4}
$$

$$\begin{tikzcd}[column sep=small]
 & \varprojlim\left(X_1 \xrightarrow{\partial_0} X_0 \xleftarrow{\partial_1} X_1\right) \arrow[dr, "m"] & \\
X_1 \arrow[ur, "\left(\substack{s\circ\partial_1 \quad id_{X_1} \\ \partial_1}\right)"] \arrow[rr, "id_{X_1}"] & & X_1\ ,
\end{tikzcd} \tag{5}$$

$$\begin{tikzcd}[column sep=small]
 & \varprojlim\left(X_1 \xrightarrow{\partial_0} X_0 \xleftarrow{\partial_1} X_1\right) \arrow[dl, "m"'] & \\
X_1 \arrow[rr, "id_{X_1}"] & & X_1\ . \arrow[ul, "\left(\substack{id_{X_1} \quad s\circ\partial_0 \\ \partial_0}\right)"']
\end{tikzcd} \tag{6}$$

Definition 4. Let F be a left exact cosimplicial category.

By an internal category $\mathbf{X}$ in F we mean a sixtuple $(X_1, X_0, \partial_1, \partial_0, s, m)$ where

X_1 is the object of F_0 (called the object of objects),

X_0 is the object of F_1 (called the object of morphisms),

$\partial_1 : X_1 \to \delta^1(X_0)$, $\partial_0 : X_1 \to \delta^0(X_0)$ are the morphisms in F_1 (called the morphisms of domain and codomain, respectively),

$s : X_0 \to \sigma(X_1)$ is the morphism of F_0 (called the identity morphism),

$m : \varprojlim(\rho_2^{0,1}(X_1) \to \rho_2^1(X_0) \leftarrow \rho_2^{1,2}(X_1)) \to \rho_2^{0,1}(X_1)$ is the morphism in F_2 (called the morphism of composition), such that the following diagrams are commutative:

$$\begin{tikzcd}
 & X_1 \arrow[r, "\sigma(\partial_1)"] & \sigma\circ\delta^1(X_0) \arrow[d, equal] \\
X_0 \arrow[ur, "s"] \arrow[dr, "s"'] \arrow[rr, equal, "id_{X_0}"] & & X_0\ , \\
 & X_1 \arrow[r, "\sigma(\partial_0)"'] & \sigma\circ\delta^0(X_0) \arrow[u, equal]
\end{tikzcd} \tag{1'}$$

$$\begin{tikzcd}
 & X_2 \arrow[dl, "pr"'] \arrow[dr, "m"] & & \\
{\rho_2}^{0,1}(X_1) \arrow[d, "{\rho_2}^{0,1}(\partial_1)"] & & {\rho_2}^{0,2}(X_1) \arrow[d, "{\rho_2}^{0,2}(\partial_1)"] \\
{\rho_2}^{0,1}(\delta^1(X_0)) = {\rho_2}^0(X_0) & & = {\rho_2}^{0,2}(\delta_1(X_0))\ ,
\end{tikzcd} \tag{2'}$$

$$
\begin{tikzcd}
 & X_2 \arrow[dl, "m"'] \arrow[dr, "pr"] & \\
{\rho_2}^{0,2}(X_1) \arrow[d, "{\rho_2}^{0,2}(\partial_0)"] & & {\rho_2}^{1,2}(X_1) \arrow[d, "{\rho_2}^{1,2}(\partial_0)"] \\
{\rho_2}^{0,2}(\delta^0(X_0)) & = {\rho_2}^{0}(X_0) = & {\rho_2}^{1,2}(\delta_0(X_0))\ ,
\end{tikzcd}
\tag{3'}
$$

$$
\varprojlim \left(
\begin{tikzcd}
\rho_3^{0,1}(X_1) \arrow[dr, "\rho_3^{0,1}(\partial_0)"'] & & \rho_3^{1,2}(X_1) \arrow[dl, "\rho_3^{1,2}(\partial_1)"'] \arrow[dr, "\rho_3^{1,2}(\partial_0)"] & & \rho_3^{2,3}(X_1) \arrow[dl, "\rho_3^{2,3}(\partial_1)"] \\
 & \rho_3^1(X_0) & & \rho_3^2(X_0) &
\end{tikzcd}
\right)
$$

$$
= \varprojlim \left(
\begin{tikzcd}
\rho_3^{0,1}(X_1) \arrow[dr, "\rho_3^{0,1}(\partial_0)"'] & & \rho_3^{1,2}(X_1)=(\delta^0\circ\rho_2^{0,1})(X_1) \arrow[dl, "\rho_3^{1,2}(\partial_1)"] & \\
 & \rho_3^1(X_0)=(\delta^0\circ\rho_2^{0})(X_0) & & \delta^0(X_2) \arrow[ul, "\delta^0(pr)"']
\end{tikzcd}
\right)
$$

$$
\overset{\text{by diagram } (2')}{=} \varprojlim \left(
\begin{tikzcd}
\rho_3^{0,1}(X_1) \arrow[dr, "\rho_3^{0,1}(\partial_0)"'] & & (\delta^0\circ\rho_2^{0,1})(X_1) \arrow[dl, "(\delta^0\circ\rho_2^{0,2})(\partial_1)"] & \\
 & \rho_3^1(X_0)=(\delta^0\circ\rho_2^{0})(X_0) & & \delta^0(X_2) \arrow[ul, "\delta^0(m)"']
\end{tikzcd}
\right)
$$

$$
\xrightarrow{pr} \varprojlim \left(
\begin{tikzcd}
\rho_3^{0,1}(X_1) \arrow[dr, "\rho_3^{0,1}(\partial_0)"'] & & (\delta^0\circ\rho_2^{0,1})(X_1) \arrow[dl, "\delta^0\circ\rho_2^{0,2}(\partial_1)"] \\
 & \rho_3^1(X_0)=(\delta^0\circ\rho_2^{0})(X_0) &
\end{tikzcd}
\right) = \delta^2(X_2)
$$

$$
\varprojlim \left(
\begin{tikzcd}
\rho_3^{0,1}(X_1) \arrow[dr, "\rho_3^{0,1}(\partial_0)"'] & & \rho_3^{1,2}(X_1) \arrow[dl, "\rho_3^{1,2}(\partial_1)"'] \arrow[dr, "\rho_3^{1,2}(\partial_0)"] & & \rho_3^{2,3}(X_1) \arrow[dl, "\rho_3^{2,3}(\partial_1)"] \\
 & \rho_3^1(X_0) & & \rho_3^2(X_0) &
\end{tikzcd}
\right)
$$

$$
= \varprojlim \left(
\begin{tikzcd}
 & (\delta^3\circ\rho_2^{1,3})(X_1)=\rho_3^{1,3}(X_1) \arrow[dr, "\rho_3^{1,2}(\partial_0)"] & & \rho_3^{2,3}(X_1) \arrow[dl, "\rho_3^{2,3}(\partial_1)"] \\
\delta^0(X_2) \arrow[ur, "\delta^3(pr)"] & & (\delta^3\circ\rho_2^{2})(X_0)=\rho_3^2(X_0) &
\end{tikzcd}
\right)
$$

$$
\overset{\text{by diagram } (3')}{=} \varprojlim \left(
\begin{tikzcd}
 & (\delta^3\circ\rho_2^{1,3})(X_1) \arrow[dr, "(\delta^3\circ\rho_2^{0,2})(\partial_0)"'] & & \rho_3^{2,3}(X_1) \arrow[dl, "\rho_3^{2,3}(\partial_1)"] \\
\delta^0(X_2) \arrow[ur, "\delta^3(m)"] & & (\delta^3\circ\rho_2^{2})(X_0)=\rho_3^2(X_0) &
\end{tikzcd}
\right)
$$

$$
\xrightarrow{pr} \varprojlim \left(
\begin{tikzcd}
(\delta^3\circ\rho_2^{1,3})(X_1) \arrow[dr, "\delta^3\circ\rho_2^{0,2}(\partial_0)"'] & & \rho_3^{2,3}(X_1) \arrow[dl, "\rho_3^{2,3}(\partial_1)"] \\
 & (\delta^3\circ\rho_2^{2})(X_0)=\rho_3^2(X_0) &
\end{tikzcd}
\right) = \delta^1(X_2)
\tag{4'}
$$

$$
\begin{tikzcd}
\delta^2(X_2) \arrow[dr, "\delta^1(m)"'] & & \delta^1(X_2) \arrow[dl, "\delta^2(m)"] \\
 & (\delta^2\circ\delta^1)(X_1) = (\delta^1\circ\delta^1)(X_1)\ , &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\varprojlim\left(\begin{smallmatrix} (\delta^1\circ\sigma)(X_1) & & X_1 \\ & \searrow^{\delta^1\circ\sigma(\partial_1)} \qquad {}_{\partial_1}\swarrow & \\ & (\delta^1\circ^1)(X_0)=\delta^1(X_0) & \end{smallmatrix}\right) \arrow[r, equal] & \sigma^0(X_2) \arrow[d, "\sigma^0(X_2)"] \\
X_1 \arrow[u, "{\left(\begin{smallmatrix}\delta^1(s)\circ\partial_1 & id_{X_1} \\ \partial_1 & \end{smallmatrix}\right)}"] \arrow[r, "id_{X_1}"] & (\sigma^0\circ\delta^1)(X_1),
\end{tikzcd}
\tag{5'}
$$

$$
\begin{tikzcd}
\sigma^1(X_2) \arrow[r, equal] \arrow[d, "\sigma^1(X_2)"'] & \varprojlim\left(\begin{smallmatrix} (\delta^1\circ\sigma)(X_1) & & X_1 \\ & \searrow^{\delta^1\circ\sigma(\partial_1)} \qquad {}_{\partial_0}\swarrow & \\ & (\delta^0\circ^0)(X_0)=\delta^0(X_0) & \end{smallmatrix}\right) \\
(\sigma^1\circ\delta^1)(X_1) & X_1\ . \arrow[l, "id_{X_1}"'] \arrow[u, "{\left(\begin{smallmatrix}\delta^0(s)\circ\partial_0 & id_{X_1} \\ \partial_0 & \end{smallmatrix}\right)}"']
\end{tikzcd}
\tag{6'}
$$

An internal functor between two internal categories $\mathbf{X}$ and $\mathbf{Y}$ in the left exact cosimplicial category F consists of two morphisms:

$f_0 : X_0 \to Y_0$ in F_0 and $f_1 : X_1 \to Y_1$ in F_1 commuting with ∂_1, ∂_0, s and m.

We denote the category of internal categories and functors in F by $\mathbf{Cat}(F)$.

Any left exact natural transformation α between two left exact cosimplicial categories F and F' induces the functor $\mathbf{Cat}(\alpha)$ from $\mathbf{Cat}(F)$ to $\mathbf{Cat}(F')$;

$$\mathbf{Cat}(\alpha) : \mathbf{Cat}(F) \to \mathbf{Cat}(F'),$$
$$\mathbf{X} = (X_0, X_1, \partial_1, \partial_0, s, m) \mapsto (\alpha_0(X_0), \alpha_1(X_1), \alpha_1(\partial_1), \alpha_1(\partial_0), \alpha_0(s), \alpha_2(m)).$$

2. Constructing a Left Exact Cosimplicial Category from a Topos

Definition 5. Let $\mathbf{E}$ be a topos over **Sets**.

(1) A simplicial topos $\widetilde{E} : \Delta^{op} \to \mathbf{Top}$ is defined as follows:

$\widetilde{E}_0$ is the topos $\mathbf{E}$,

$\widetilde{E}_1$ is the topos $\mathbf{E} \underset{\mathbf{Sets}}{\times} \mathbf{E}$,
$\widetilde{E}_2$ is the topos $\mathbf{E} \underset{\mathbf{Sets}}{\times} \mathbf{E} \underset{\mathbf{Sets}}{\times} \mathbf{E}$,
$\dots$,
$\widetilde{E}_n$ is the topos $\underbrace{\mathbf{E} \underset{\mathbf{Sets}}{\times} \cdots \underset{\mathbf{Sets}}{\times} \mathbf{E}}_{n+1}$,
$\dots$,
for each $0 \leq i \leq n$

$$\partial_i = \widetilde{E}(\delta_n^i) = (pr_0, pr_1, \dots, pr_{i-1}, pr_{i+1}, \dots, pr_n) :$$
$$: \underbrace{\mathbf{E} \underset{\mathbf{Sets}}{\times} \cdots \underset{\mathbf{Sets}}{\times} \mathbf{E}}_{n+1} \to \underbrace{\mathbf{E} \underset{\mathbf{Sets}}{\times} \cdots \underset{\mathbf{Sets}}{\times} \mathbf{E}}_{n} \ \text{(omitting } i)$$

and

$$s_i = \widetilde{E}(\sigma_n^i) = (pr_0, pr_1, \dots, pr_{i-1}, pr_i, pr_i, pr_{i+1}, \dots, pr_n) :$$
$$: \underbrace{\mathbf{E} \underset{\mathbf{Sets}}{\times} \cdots \underset{\mathbf{Sets}}{\times} \mathbf{E}}_{n+1} \to \underbrace{\mathbf{E} \underset{\mathbf{Sets}}{\times} \cdots \underset{\mathbf{Sets}}{\times} \mathbf{E}}_{n+2}.$$

One can easily see that ∂_i and s_i are the geometric morphisms which satisfy the usual axioms of simplicial objects [2].

(2) The left exact cosimplicial category $\mathbf{\Delta}(E)$ is defined as follows:

for $n \geq 0$ let $\mathbf{\Delta}(E)_n$ be the same category as $\widetilde{E}_n$ and for each $0 \leq i \leq n$ let $\delta_n^i : \mathbf{\Delta}(E)_{n-1} \to \mathbf{\Delta}(E)_n$ (resp. $\sigma_n^i : \mathbf{\Delta}(E)_{n+1} \to \mathbf{\Delta}(E)_n$) be the inverse image of the geometric morphism $\partial_i^n : \widetilde{E}_n \to \widetilde{E}_{n-1}$ (resp. $s_i^n : \widetilde{E}_n \to \widetilde{E}_{n+1}$).

It readily follows that for any $n \geq 0$ the category $\mathbf{\Delta}(E)_n$ has finite limits (because it is a topos) and any coface and codegeneracy functor preserves them (because it is the inverse image functor of some geometric morphism).

The category of internal categories in $\mathbf{\Delta}(E)$ is denoted by $\mathbf{Cat}\,\mathbf{\Delta}(E)$.

Definition 6. For any topos $\mathbf{E}$ over $\mathbf{Sets}$ let $S(E)$ be the category, whose objects are all pairs (P, ϕ), where P is an internal category in $\mathbf{Top}$, ϕ is an internal functor $\phi : P \to ad(\mathbf{E})$ such that $\phi_0 : P_0 \to \mathbf{E}$, and $\phi_1 : P_1 \to \mathbf{E} \underset{\mathbf{Sets}}{\times} \mathbf{E}$ are local homeomorphisms of toposes.

A morphism between the pairs (p, ϕ) and (P', ϕ') is an internal functor $f : P \to P'$, such that $\phi' \circ f = \phi$.

Remark 7.

(1) For any $\mathbf{E}_1, \mathbf{E}_2 \in \mathbf{Top}$, the geometric morphism $f : \mathbf{E}_1 \to \mathbf{E}_2$ with the functor $f_! \dashv f^*$, where $f_!$ preserves equalizers, will be called as usual a local homeomorphism. In that case (see [3]) the category $\mathbf{E}_1$ is equivalent to the comma category $\mathbf{E}_2/f_!(1)$ (here 1 means the terminal object).

(2) $S(-)$ is a contravariant functor from **Top** to **Cat**. Any geometric morphism $f:\mathbf{E}\to\mathbf{E}'$ induces a functor $S(f):S(E')\to S(E)$ as follows: for $(P',\phi')\in S(\mathbf{E}')$, $S(f)$ is a pair (P,ϕ) such that the squares

$$\begin{tikzcd} P_0 \arrow[r] \arrow[d, "\phi_0"'] & P_0' \arrow[d, "\phi_0'"] \\ \mathbf{E} \arrow[r, "f"'] & \mathbf{E}' \end{tikzcd} , \qquad \begin{tikzcd} P_1 \arrow[r] \arrow[d, "\phi_1'"'] & P_1' \arrow[d, "\phi_1'"] \\ \mathbf{E}\underset{\mathbf{Set}}{\times}\mathbf{E} \arrow[r, "f\times f"'] & \mathbf{E}'\underset{\mathbf{Set}}{\times}\mathbf{E}' \end{tikzcd}$$

are pullbacks. By [3, Corollary 4.35] it is clear that since in the above commutative diagrams the geometric morphisms ϕ_0' and ϕ_1' are the local homeomorphisms, ϕ_0 and ϕ_1 will be the local homeomorphisms too.

Theorem 8. *Let* $\mathbf{E}$ *be a topos over* **Sets**. *Then there is an equivalence of categories*

$$\mathbf{Cat}\,\Delta(E) \underset{\Psi_E}{\overset{\Phi_E}{\leftrightarrows}} S(\mathbf{E})$$

and this equivalence is natural in $\mathbf{E}$.

Proof. First let us construct the functor Φ_E. Suppose $(X_0,X_1,\partial_1,\partial_0,s,m)\in \mathrm{Cat}\,\Delta(E)$. Then $X_0\in\mathbf{E}$ and $X_1\in\mathbf{E}\underset{\mathbf{Set}}{\times}\mathbf{E}$. Denote the comma categories $\mathbf{E}/X_0$, $\mathbf{E}\underset{\mathbf{Set}}{\times}\mathbf{E}/X_1$ by $\mathbf{E}_0$ and $\mathbf{E}_1$, respectively.

For $i=0,1$ the morphisms $\partial_i: X_1\to\delta^i(X_0)$ in $\mathbf{E}\underset{\mathbf{Set}}{\times}\mathbf{E}$ induce by [3, corollary 4.35] the geometric morphisms $\bar\partial_i:\mathbf{E}_1\to\mathbf{E}_0$ such that the diagrams

$$\begin{tikzcd} \mathbf{E}_1 \arrow[r, "\bar\partial_1"] \arrow[d] & \mathbf{E}_0 \arrow[d] \\ \mathbf{E} \arrow[r, "\partial_1"] & \mathbf{E} \end{tikzcd} , \qquad \begin{tikzcd} \mathbf{E}_1 \arrow[r, "\bar\partial_0"] \arrow[d] & \mathbf{E}_0 \arrow[d] \\ \mathbf{E} \arrow[r, "\partial_0"] & \mathbf{E} \end{tikzcd}$$

are commutative.

Similarly, the morphisms $s: X_0\to\sigma(X_1)$, $m:\varprojlim(\rho_2^{0,1}(X_1)\xrightarrow{\rho_2^{0,1}}\rho_2^1(X_0)\xleftarrow{\rho_2^{1,2}}\rho_2^{1,2}(X_1))\to\rho_2^{0,2}(X_1)$ induce the geometric morphisms $\bar s:\mathbf{E}_0\to\mathbf{E}_1$ and $\bar m:\varprojlim(\mathbf{E}_1\to\mathbf{E}_0\leftarrow\mathbf{E}_1)\to\mathbf{E}_1$ for which the diagrams

$$\begin{tikzcd} \mathbf{E}_0 \arrow[r, "\bar s"] \arrow[d] & \mathbf{E}_1 \arrow[d] \\ \mathbf{E} \arrow[r, "(1,1)"] & \mathbf{E}\underset{\mathbf{Set}}{\times}\mathbf{E} \end{tikzcd} , \qquad \begin{tikzcd} \varprojlim(\mathbf{E}_1\to\mathbf{E}_0\leftarrow\mathbf{E}_1) \arrow[r, "\bar m"] \arrow[d] & \mathbf{E}_1 \arrow[d] \\ \mathbf{E}\underset{\mathbf{Set}}{\times}\mathbf{E}\underset{\mathbf{Set}}{\times}\mathbf{E} \arrow[r, "(pr_0,pr_1)"] & \mathbf{E}\underset{\mathbf{Set}}{\times}\mathbf{E} \end{tikzcd}$$

are commutative.

Using $(1')$–$(6')$ for $(X_0, X_1, \partial_1, \partial_0, s, m)$ one can easily prove that the sixtuple $(\mathbf{E}_0, \mathbf{E}_1, \bar{\partial}_1, \bar{\partial}_0, \bar{s}, \bar{m})$ satisfies conditions (1)-(6) for an internal category. Hence $\Phi_{\mathbf{E}}((X_0, X_1, \partial_1, \partial_0, s, m))$ is determined as a pair $((\mathbf{E}_0, \mathbf{E}_1, \bar{\partial}_1, \bar{\partial}_0, \bar{s}, \bar{m}), (\alpha_0, \alpha_1))$ where α_0 and α_1 are natural geometric morphisms from the comma category into the underlying topos.

Now let us construct the functor $\Psi_{\mathbf{E}}$. Let $((\mathbf{E}_0, \mathbf{E}_1, \bar{\partial}_1, \bar{\partial}_0, \bar{s}, \bar{m}), (\alpha_0, \alpha_1)) \in S(\mathbf{E})$. Then $((\mathbf{E}_0, \mathbf{E}_1, \bar{\partial}_1, \bar{\partial}_0, \bar{s}, \bar{m})$ is an internal category in **Top** and (α_0, α_1) is an internal functor from $((\mathbf{E}_0, \mathbf{E}_1, \bar{\partial}_1, \bar{\partial}_0, \bar{s}, \bar{m})$ to $ad(\mathbf{E})$ such that $\alpha_0 : \mathbf{E}_0 \to \mathbf{E}$ and $\alpha_1 : \mathbf{E}_1 \to \mathbf{E} \underset{\mathbf{Set}}{\times} \mathbf{E}$ are local homeomorphisms. Suppose the local homeomorphisms α_0 and α_1 are induced by $X_0 \in \mathbf{E}$ and $X_1 \in \mathbf{E} \underset{\mathbf{Set}}{\times} \mathbf{E}$ respectively. By [3, Corollary 4.35] the morphisms $\bar{\partial}_1$, $\bar{\partial}_0$, $\bar{s}$, $\bar{m}$ induce the morphisms $\partial_1 : X_1 \to \delta^1(X_0)$,$\partial_0 : X_1 \to \delta^0(X_0)$, $s : X_0 \to \sigma(X_1)$, $m : \varprojlim(\rho_2^{0,1}(X_1) \xrightarrow{\rho_2^{0,1}(\partial_0)} \rho_2^1(X_0) \xleftarrow{\rho_2^{1,2}(\partial_0)} \rho_2^{1,2}(X_1)) \to \rho_2^{0,2}(X_1)$ (here the functors δ, σ and ρ are from the cosimplicial category $\boldsymbol{\Delta}(E)$).

The fact that the sixtuple $(X_0, X_1, \partial_1, \partial_0, s, m)$ satisfies conditions $(1')$–$(6')$ follows from the fact that $(\mathbf{E}_0, \mathbf{E}_1, \bar{\partial}_1, \bar{\partial}_0, \bar{s}, \bar{m})$ satisfies conditions (1)–(6) for an internal category. Therefore $\Psi_{\mathbf{E}}(((\mathbf{E}_0, \mathbf{E}_1 \bar{\partial}_1, \bar{\partial}_0, \bar{s}, \bar{m}), (\alpha_0, \alpha_1)))$ is determined to be the sixtuple $(X_0, X_1, \partial_1, \partial_0, s, m)$.

It is easily seen that the functors $\Phi_{\mathbf{E}}$, $\Psi_{\mathbf{E}}$ are inverse to each other. $\square$

For any $\mathbf{E} \in \mathbf{Top}$ we define the forgetful functor $T_{\mathbf{E}} : S(\mathbf{E}) \to \mathbf{E}$ as follows: for any $((\mathbf{E}_0, \mathbf{E}_1, \bar{\partial}_1, \bar{\partial}_0, \bar{s}, \bar{m}), (\alpha_0, \alpha_1)) \in S(\mathbf{E})$ let $T_{\mathbf{E}}(((\mathbf{E}_0, \mathbf{E}_1, \bar{\partial}_1, \bar{\partial}_0, \bar{s}, \bar{m}), (\alpha_0, \alpha_1)))$ be $X \in \mathbf{E}$ such that $\mathbf{E}_0 \cong \mathbf{E}/X$.

If a geometric morphism $f : \mathbf{E} \to \mathbf{E}'$ is given, then its inverse image functor f^* is interchangeable with the functor T_*, i.e., the diagram

$$\begin{tikzcd} S(\mathbf{E}) \arrow[d, "T_{\mathbf{E}}"'] & S(\mathbf{E}') \arrow[l, "S(f)"'] \arrow[d, "T_{\mathbf{E}'}"'] \\ \mathbf{E} & \mathbf{E}' \arrow[l, "f^*"'] \end{tikzcd}$$

is commutative.

Definition 9. Let $P = ((\mathbf{E}_0, \mathbf{E}_1, \partial_1, \partial_0, s, m), (\alpha_0, \alpha_1)) \in S(\mathbf{E})$.

(1) We define the opposite internal category P^{op} of P to be that obtained by interchanging ∂_1 and ∂_0, and "twisting" the definition of m, i.e., in the internal category P^{op} we must replace $(\mathbf{E}_0, \mathbf{E}_1, \partial_1, \partial_0, s, m)$ by its opposite.

(2) We call P an internal groupoid if the underlying internal category $(\mathbf{E}_0, \mathbf{E}_1, \partial_1, \partial_0, s, m)$ is an internal groupoid in **Top**.

Remark 10. For any $Q = (X_0, X_1, \partial_1, \partial_0, s, m) \in \mathbf{Cat}\,\Delta(E)$, the opposite category of Q is $Q^{op} = (X_0, X_1^*, \partial_1^*, \partial_0^*, s, m^*)$, where X_1 is $(pr_1, pr_0)^*(X_1)$,

$\partial_i^* = (pr_1, pr_0)^*(\partial_i)$ for $i = 0, 1$ and $m^* = (pr_2, pr_1, pr_0)^*(m)$; here

$(pr_1, pr_0) : \mathbf{E} \underset{\mathbf{Sets}}{\times} \mathbf{E} \to \mathbf{E} \underset{\mathbf{Sets}}{\times} \mathbf{E}$ is the "swapping" geometric morphism.

Note that internal groupoids in $\mathbf{\Delta}(E)$ form a full subcategory in $S(\mathbf{E})$. Denote this subcategory by $\mathbf{Gpd}\,\Delta(E)$ or simply by $\mathbf{Gpd}(\mathbf{E})$.

It is easy to see that $\mathbf{Gpd}(-)$ is a functor (as $S(-)$); for any $f : \mathbf{E} \to \mathbf{E}'$ the geometric morphism in **Top**, $S(f) : S(\mathbf{E}') \to S(\mathbf{E})$ assigns the internal groupoid in $\mathbf{Gpd}(\mathbf{E})$ to each internal groupoid in $\mathbf{Gpd}(\mathbf{E}')$.

We will consider two examples of our constructions.

Example 1. Consider the case with $\mathbf{E}$ as a topos of sheaves over a locally compact Hausdorff topological space A. Denote this topos by $\mathbf{shv}(A)$. In this case, for each $n \geq 0$ $\underbrace{\mathbf{shv}(A) \underset{\mathbf{Sets}}{\times} \cdots \underset{\mathbf{Sets}}{\times} \mathbf{shv}(A)}_{n+1} = \mathbf{shv}(A^{n+1})$ [3], any geometric morphism from $\mathbf{shv}(A^{n+1})$ to $\mathbf{shv}(A^{m+1})$ for $n, m \geq 0$ is induced by a continuous map from A^{n+1} to A^{m+1} [4], and this geometric morphism is a local homeomorphism if and only if the continuous map between the underlying topological spaces is a local homeomorphism. Therefore the category $S(\mathbf{shv}(a))$ can be represented as a category of pairs (C, α), where $C = (C_0, C_1, \partial_1, \partial_0, s, m)$ is a continuous category, $\alpha = (\alpha_0, \alpha_1) : C \to ad(A)$ is a continuous functor, such that both maps $\alpha_0 : C_0 \to A$ and $\alpha_1 : C_1 \to A \times A$ are local homeomorphisms. Let us denote this new representation of the category $S(\mathbf{shv}(a))$, i.e., the representation by continuous categories and functors, simply by $S(A)$.

Now we will derive some technical properties of some elements of $S(A)$ to be used in Section 3.

Proposition 11. *Let A be a topological space as above and let $(C, \alpha) \in S(A)$. Then the continuous map $C_1 \xrightarrow{(\partial_1, \partial_0)} C_0 \times C_0$ satisfies the unique lifting property of paths* [5]. *This means that for any two paths $(\omega, \omega') : I \to C_1$ in C_1, for which $(\partial_1, \partial_0) \circ \omega = (\partial_1, \partial_0) \circ \omega'$ and $\omega(0) = \omega'(0)$, we have $\omega = \omega'$ (a path is a continuous function $I \to C_1$).*

Proof. Suppose there exist two paths $\omega, \omega' : I \to C_1$ with $(\partial_1, \partial_0) \circ \omega = (\partial_1, \partial_0) \circ \omega'$, $\omega(0) = \omega'(0)$ and $\omega \neq \omega'$. The proposition will be proved in three steps. In the 1st step it will be shown that $(\partial_1, \partial_0) : C_1 \to C_0 \times C_0$ is a local homeomorphism; in the 2nd step a sequence p_n will be found, which has two distinct limits in C_1, and in the 3rd step a new sequence $\widetilde{p}_n$ will be constructed using the sequence p_n, which leads to a contradiction.

1*st step.* Consider the commutative diagram

$$\begin{tikzcd} C_1 \arrow[rr, "(\partial_1,\partial_0)"] \arrow[dr, "\alpha_1"'] & & C_0 \times C_0 \arrow[dl, "\alpha_0 \times \alpha_0"] \\ & A \times A & \end{tikzcd} .$$

Since in this diagram α_1 and $\alpha_0 \times \alpha_0$ are local homeomorphisms, (∂_1, ∂_0) is a local homeomorphism too [5].

2nd step. Denote by κ_i for $i = 0, 1$ the path $\partial_i \circ \omega' = \partial_i \circ \omega$ in C_0. Consider the path $\kappa = (\kappa_1, \kappa_0)$ in $C_0 \times C_0$ and the object $(P \xrightarrow{\partial} I) = \kappa^*((\partial_1, \partial_0) : C_1 \to C_0 \times C_0) \in \mathbf{shv}(I)$. Two paths ω and ω' in C_1 induce two global sections of the local homeomorphism $\partial : P \to I$ or, equivalently, two global elements $\bar{\omega}, \bar{\omega}' : 1 \to (\partial : P \to I)$ in $\mathbf{shv}(I)$. Let $\iota : U \rightarrowtail 1$ be an equalizer of the parallel arrows $\bar{\omega}$ and $\bar{\omega}'$. We can represent the subobject $\iota : U \rightarrowtail I$ as an open subset U of I. This is a subspace of those $t \in I$ for which $\omega(t) = \omega'(t)$. Therefore $0 \in U$. Consider a maximal open connected subspace in I which contains $0 \in I$. Since $\omega \neq \omega'$, this component of the connectivity of U has the shape $[0, t_0)$. It is clear that $0 < t_0 < 1$ and $\omega(t_0) \neq \omega'(t_0)$. Denote $\omega(t_0)$ by p and $\omega'(t_0)$ by p'. Choose a sequence t_n in $[0, t_0)$ whose limit is t_0. Then for $n \geq 0$, $\omega(t_n) = \omega'(t_n)$; denote this latter by p_n. Thus we have constructed a sequence p_n in C_1 which has two distinct limits p and p'.

3rd step. Consider $\kappa_1(t_0) \in C_0$ (resp., $\kappa_0(t_0) \in C_0$) and $s(\kappa_1(t_0)) \in C_1$ (resp., $s(\kappa_0(t_0)) \in C_1$). Since (∂_1, ∂_0) is a local homeomorphism, there exists an open neighborhood W_1 (resp., W_0) of $s(\kappa_1(t_0))$ (resp., $s(\kappa_0(t_0))$) which is mapped homeomorphically onto the open neighborhood $U_1 \times U_1$ (resp., $U_0 \times U_0$) of $(\kappa_1(t_0), \kappa_1(t_0))$ (resp., of $(\kappa_0(t_0), \kappa_0(t_0))$). Denote the inverse of the homeomorphism $(\partial_1, \partial_0)|_{W_1} : W_1 \to U_1 \times U_1$ (resp., $(\partial_1, \partial_0)|_{W_0} : W_0 \to U_0 \times U_0$) by f_1 (resp., by f_0).

Consider an open neighborhood $U_1 \times U_0$ of $(\kappa_1(t_0), \kappa(t_0))$ in $C_0 \times C_0$. Since $\lim\limits_{n\to\infty} (d_1, d_0)(p_n) = (\kappa_1(t_0), \kappa_0(t_0))$, there exists $n_0 > 0$ such that $n > n_0$ implies $(\partial_1, \partial_0)(p_n) \in U_1 \times U_0$. Denote by q_n^1 (resp., q_n^0) the point $(\kappa_1(t_0), \partial_1(p_n)) \in C_0 \times C_0$ (resp., $(\partial_0(p_n), \kappa_0(t_0)) \in C_0 \times C_0$). Obviously, $\lim\limits_{n\to\infty} q_n^1 = (\kappa_1(t_0), \kappa_1(t_0))$ (resp., $\lim\limits_{n\to\infty} q_n^0 = (\kappa_0(t_0), \kappa_0(t_0))$ and $\lim\limits_{n\to\infty} f_1(q_n^1) = s\kappa_1(t_0)$ (resp., $\lim\limits_{n\to\infty} f_0(q_n^0) = s\kappa_0(t_0)$).

Now consider the following sequence in C_1:

$$\widetilde{p}_n = f_0(q_n) \circ p_n \circ f_1(q_n^1), \qquad n > n_0;$$

here $\circ$ means the internal composition in the continuous category C.

We have

$$\begin{aligned} \lim_{n\to\infty} \widetilde{p}_n = \lim_{n\to\infty} f_0(q_n^0) \circ \lim_{n\to\infty} p_n \circ \lim_{n\to\infty} f_1(q_n^1) = \\ = s(\kappa_0(1)) \circ \lim_{n\to\infty} p_n \circ s(\kappa_1(t_0)) = \lim_{n\to\infty} p_n. \end{aligned}$$

But p and p' are limits of p_n so that the limits of $\widetilde{p}_n$ will be p and p'.

For each $n > n_0$, $\widetilde{p}_n \in (\partial_1, \partial_0)^{-1}(\kappa_1(t_0), \kappa_0(t_0))$ and the subspace $(\partial_1, \partial_0)^{-1}(\kappa_1(t_0), \kappa_0(t_0)) \subset C_1$ is discrete, since (∂_1, ∂_0) is a local homeomorphism. Therefore in the discrete topological space $(\partial_1, \partial_0)^{-1}(\kappa_1(t_0), \kappa_0(t_0))$, there is a sequence $\widetilde{p}_n$ having two distinct limits p and p', which is a contradiction. □

Proposition 12. *Let A be a locally path connected topological space, and $(C, \alpha) \in S(A)$. Then the continuous function $(\partial_1, \partial_0) : C_1 \to C_0 \times C_0$ is a covering map* [5]. *This means that each $p \in C_0 \times C_0$ has an open neighborhood U with $p \in U \subset C_0 \times C_0$ for which $(\partial_1, \partial_0)^{-1}(U)$ is a disjoint union of subspaces $U_i \subset C_1$, each of which is mapped homeomorphically onto U by (∂_1, ∂_0).*

Proof. We will prove this proposition in three steps. In the 1st step, for each $p \in C_0$ we will find an open neighborhood U_p of p in C_0 and an open neighborhood W_p of $s(p)$ in C_1 for which W_p is mapped homeomorphically onto $U_p \times U_p$ by (∂_1, ∂_0). In the 2nd step, for each $(p, p') \in C_0 \times C_0$ and $q \in (\partial_1, \partial_0)^{-1}(U_p \times U_p)$ we will find an open neighborhood V_q of q in C_1, which is mapped homeomorphically onto $U_p \times U'_p$ by (∂_1, ∂_0). And in the 3rd step we will prove that for each $(p, p') \in C_0 \times C_0$ and each $q, q' \in (\partial_1, \partial_0)^{-1}(U_p \times U'_p)$ either $V_q = V_{q'}$ or $V_q \cap V_{q'} = \varnothing$.

1st step. Since (∂_1, ∂_0) is a local homeomorphism and C_0 is locally path connected as well as A, it is possible to choose, for each $p \in C_0$, an open neighborhood W_p of $s(p) \in C_1$ which is mapped homeomorphically onto $U_p \times U_p$, where U_p is an open connected neighborhood of p in C_0. Let us prove that $s(p) \in W_p$ for any $p' \in U_p$. Denote by h_p the inverse homeomorphism of $(\partial_1, \partial_0)|_{W_p} : W_p \to U_p \times U_p$. Consider a path ω in C_0 from p to p'. Then we have two paths $s \circ \omega$ and $h_p(\omega, \omega)$ in C_1 with the common origin. By Proposition 11 $s \circ \omega = h_p(\omega, \omega)$. Therefore $s(p') = s\omega(1) = h_p(\omega(1), \omega(1)) \in W_p$.

2nd step. Suppose $q \in (\partial_1, \partial_0)^{-1}(p, p')$. Consider the continuous map

$$\chi_q : U_p \times U_{p'} \to C_1, \quad \chi_q(\widetilde{p}, \widetilde{p}') = h_{p'}(p', \widetilde{p}') \circ q \circ (\widetilde{p}, p);$$

here as above we mean by $\circ$ the internal composition in the internal category C.

Denote $\chi_q(U_p \times U_{p'})$ by V_q. By Proposition 11 it is clear that V_q is an open neighborhood of q which is mapped homeomorphically onto $U_p \times U_{p'}$ by (∂_1, ∂_0).

3rd step. V_q and $V_{q'}$ are connected open neighborhoods; so this step follows immediately from Proposition 11. □

Corollary 13. *Let A be a locally path connected topological space and $(C, \alpha) \in S(A)$. Then the continuous function $(\partial_1, \partial_0) : C_1 \to C_0 \times C_0$ is a fibration in the sense of Hurevich.*

Proof. This follows from Proposition 12 and [5, Theorem 2.33]. $\square$

Example 2. Consider the case with $\mathbf{E}$ as a topos of presheaves on a small category C. Denote this topos as $\mathbf{Sets}^C$. Then for each $n \geq 0$, $\underbrace{\mathbf{Sets}^C \underset{\mathbf{Sets}}{\times} \cdots \underset{\mathbf{Sets}}{\times} \mathbf{Sets}^C}_{n+1} = \mathbf{Sets}^{A^{n+1}}$ [3], any face and degeneracy geometric morphism from $\mathbf{Sets}^{C^{n+1}}$ to $\mathbf{Sets}^{C^{m+1}}$ is induced by the corresponding internal face and degeneracy functors in the natural simplicial internal category over $\mathbf{Sets}^C$, and any geometric morphism $\mathbf{Sets}^{C'} \to \mathbf{Sets}^{C''}$ is a local homeomorphism if and only if it is induced by a discrete fibration $C' \to C''$. Therefore the category $S(\mathbf{Sets}^C)$ can be represented as the category of pairs (P, ϕ), where P is an internal category in $\mathbf{Cat}$ and ϕ is an internal functor from P to the antidiscrete internal category $ad(C)$ such that $\phi_0 : P_0 \to ad(C)_0 = C$ and $\phi_1 : P_1 \to ad(C)_1 = C \times C$ are discrete fibrations. Let us denote this new representation of the category $S(\mathbf{Sets}^C)$, i.e., the representation by internal categories and functors in $\mathbf{Sets}^C$, simply by $S(C)$.

Internal categories in $\mathbf{Cat}$ are called double categories [1]. Any double category D can be represented as a structure with objects, vertical morphisms, horizontal morphisms, and double morphisms (or cells). The vertical morphisms form a category whose composition is denoted by $*$ and identities by id. The horizontal structure also forms a category with composition denoted by $\circ$ and identities by 1. In fact, the whole structure can be described by the pasting of double morphisms. A double morphism has a horizontal domain and codomain, and a vertical domain and codomain. It can be pictured as:

$$\begin{tikzcd} A \arrow[r, "h'"] & B \\ C \arrow[u, "v'"] \arrow[r, "h"] & D \arrow[u, "v"'] \end{tikzcd}$$

Definition 14. For any small category C, let $S'(C)$ be the category whose objects are quadruples (F, p, Z, g), where $p : F \to C$ is a discrete fibration, Z is a small category, and $g : F \to Z$ is a functor, such that $g_0 : F_0 \to Z_0$ is a bijection and $g_1 : F_1 \to Z_1$ assigns an isomorphism in Z to each $\alpha \in F_1$.

A morphism between (F, p, Z, g) and (F', p', Z', g') is a pair of functors $(\beta : F \to F'; \gamma : Z \to Z')$ such that $p' \circ \beta = p$ and the diagram

$$\begin{array}{ccc} F & \xrightarrow{\beta} & F' \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle g'} \\ Z & \xrightarrow{\gamma} & Z' \end{array}$$

is commutative.

Proposition 15. *Let C be a small category. Then there is an equivalence of the categories*

$$S(C) \underset{\Phi_C}{\overset{\Psi_C}{\leftrightarrows}} S'(C)$$

and this equivalence is natural in C.

Proof. First let us construct the functor Φ_C.

Suppose $(P, \phi) \in S(C)$. Represent the double category P as a diagram:

$$\begin{array}{ccc} P_0'' & \leftleftarrows & P_1'' \\ \downdownarrows & & \downdownarrows \\ P_0' & \leftleftarrows & P_1' \end{array},$$

where $P_0'' \rightrightarrows P_0'$ is the category P_0 and $P_1'' \rightrightarrows P_1'$ is the category P_1, P_0' is the set of objects in P, P_0'' the set of vertical morphisms, P_1' the set of horizontal morphisms, and P_1'' the set of double morphisms or cells. The functors

$$\phi_0 = (\phi_0', \phi_0'') : (P_0'' \leftleftarrows P_0') \to (C_1 \leftleftarrows C_0),$$

$$\phi_1 = (\phi_1', \phi_1'') : (P_1'' \leftleftarrows P_1') \to (C_1 \times C_1 \underset{\partial_0 \times \partial_0}{\overset{\partial_1 \times \partial_1}{\leftleftarrows}} C_0 \times C_0)$$

are discrete fibrations.

Construct the functor $g = (g_0, g_1) : (P_0'' \leftleftarrows P_0') \to (P_0' \leftleftarrows P_1')$ as follows:

Let $g_0 : P_0' \to P_0'$ be the identity map $1_{P_0'}$.

For a vertical morphism $\alpha : x \to y$ of P let $g_1(\alpha)$ be a horizontal morphism λ such that there is a double morphism

$$\begin{array}{ccc} x & \xrightarrow{\lambda} & y \\ {\scriptstyle id_x}\big\uparrow & & \big\uparrow \\ x & \xrightarrow{1_x} & x \end{array}.$$

Since ϕ_0 and ϕ_1 are discrete fibrations, such λ exists and is unique.

Let us prove that $g = (g_0, g_1)$ is a functor.

For each $u \in P_0'$ and $x \xrightarrow{\alpha} y \xrightarrow{\beta} z$ $\alpha, \beta \in P_0''$, consider the diagrams

$$\begin{tikzcd} u \arrow[r, "1_u"] & u \\ u \arrow[u, "id_u"] \arrow[r, "1_u"] & u \arrow[u, "id_u"'] \end{tikzcd}, \qquad \begin{tikzcd} x \arrow[r, "g(\alpha)"] & y \arrow[r, "g(\beta)"] & z \\ x \arrow[u, "id_x"] \arrow[r, "g(\alpha)"] & y \arrow[u, "id_y"] \arrow[r, "1_y"] & y \arrow[u, "\beta"] \\ x \arrow[u, "id_x"] \arrow[r, "1_x"] & x \arrow[u, "\alpha"] \arrow[r, "1_x"] & x \arrow[u, "\alpha"] \end{tikzcd}.$$

Since in these diagrams all small squares determine cells in P, we have $g(id_u) = 1_u$ and $g(\beta \circ \alpha) = g(\beta) \circ g(\alpha)$. So g is a functor.

Now let us prove that for each $(\alpha : x \to y) \in P_0''$ $g(\alpha)$ is invertible. Consider the diagrams

$$\begin{tikzcd} x \arrow[r, "g(\alpha)"] & y \arrow[r, "\lambda'"] & x \\ x \arrow[u, "id_x"] \arrow[r, "1_x"] & x \arrow[u, "\alpha"'] \arrow[r, "1_x"] & x \arrow[u, "id_x"'] \end{tikzcd}, \qquad \begin{tikzcd} x \arrow[r, "\lambda'"] & y \arrow[r, "g(\alpha)"] & x \\ x \arrow[u, "\alpha"] \arrow[r, "1_x"] & x \arrow[u, "id_x"'] \arrow[r, "1_x"] & x \arrow[u, "\alpha"'] \end{tikzcd}$$

in which all squares represent cells in P. Since ϕ_1 is a discrete fibration, λ' is the inverse of $g(\alpha)$

Determine $\Phi_C(P, \phi)$ to be the quadruple $((P_0'' \rightrightarrows P_0'), \phi_0, (P_1' \rightrightarrows P_0'), g)$.

Now let construct a functor $\Psi_C : S'(C) \to S(C)$.

Suppose $(F, p, Z, g) \in S'(C)$. We can identify objects of F with objects of Z via the bijection $p_0 : F_0 \to Z_0$. Determine a double category P as follows:

Let the set of objects of P be $F_0 \overset{p_0}{\cong} Z_0$, the set of vertical morphisms F_1, the set of horizontal morphisms Z_1, and let the set of cells be the set of squares

$$\begin{tikzcd} u \arrow[r, "\gamma"] & z \\ x \arrow[u, "\epsilon"'] \arrow[r, "\alpha"] & y \arrow[u, "\beta"'] \end{tikzcd},$$

where $\epsilon, \beta \in F_1,\ \alpha, \gamma \in Z_1$, and $g(\beta) \circ \alpha = \gamma \circ g(\epsilon)$.

There is a naturally determined double functor $\phi : P \to ad(C)$ which assigns $p_0(x)$ to each $x \in F_0 \cong Z_0$, $p_1(\beta)$ to each $(\beta : x \to y) \in F_1$, $(p_0(x), p_0(y)) \in C_0 \times C_0$ to each $(\alpha : x \to y) \in Z_1$, and $(p_1(\epsilon), p_1(\beta)) \in$

$C_1 \times C_1$ to each cell

$$\begin{tikzcd} u \arrow[r, "\gamma"] & z \\ x \arrow[u, "\epsilon"] \arrow[r, "\alpha"] & y \arrow[u, "\beta"'] \end{tikzcd} .$$

We determine $\Psi_C(F, p, Z, g))$ to be the pair (P, ϕ).

One can easily check that the functors Φ_C and Ψ_C are inverse to each other.

The equivalences Φ_C, Ψ_C are clearly natural in C. $\square$

Remark 16. It is not difficult to see that the full subcategory of $S'(C)$, which under the equivalence (Φ_C, Ψ_C) corresponds to the full subcategory of internal groupoids in $S(C)$, consists of quadruples (F, p, Z, g) such that Z is a groupoid.

Remark 17. Define the forgetful functor $T'_C : S(C)' \to \mathbf{Sets}^C$ by

$$(F, p, Z, g) \mapsto (F, p).$$

Then the diagram

$$\begin{tikzcd} S(C) \arrow[rr, shift left, "\Phi_C"] \arrow[dr, "T_C"'] & & S'_C(C) \arrow[ll, shift left, "\Psi_C"] \arrow[dl, "T'_C"] \\ & \mathbf{Sets}^C & \end{tikzcd}$$

is commutative.

Corollary 18. *The category* $\mathbf{Cat}\,\Delta(Sets)$ *is equivalent to the category of small categories and the category* $\mathbf{Grp}\,\Delta(Sets)$ *is equivalent to the category of small groupoids.*

Proof. The corollary follows from Example 2 when A is a one-point space, and also from Proposition 15 when C is a category with one object and one morphism. $\square$

3. A Fundamental Group

Definition 19. Let $\mathbf{E} \in \mathbf{Top}$. We will say that $\mathbf{E}$ admits the notion of a discrete category if the forgetful functor $T_{\mathbf{E}} : S(\mathbf{E}) \to \mathbf{E}$ has a left adjoint $F_{\mathbf{E}} : \mathbf{E} \to S(\mathbf{E})$ such that $T_{\mathbf{E}} \circ F_{\mathbf{E}} \cong 1_{\mathbf{E}}$. We will call this left adjoint functor $F_{\mathbf{E}}$ the discrete category functor for $\mathbf{E}$, and internal categories in its range discrete internal categories.

Proposition 20. *Let a topos* $\mathbf{E}$ *admit the notion of a discrete category; then the discrete category functor* $F_{\mathbf{E}} : \mathbf{E} \to S(\mathbf{E})$ *sends each* $X \in \mathbf{E}$ *to an internal groupoid in* $\boldsymbol{\Delta(E)}$*, i.e., any "discrete category" in* $\boldsymbol{Cat\Delta(E)} \cong S(\mathbf{E})$ *is an internal groupoid.*

Proof. We will prove this proposition in two steps. In the first step we will construct for each $X \in \mathbf{E}$ an internal equivalence η_X from $F_{\mathbf{E}}(X)$ to $F^{op}_{\mathbf{E}}(X)$, which is the identity on objects, and in the second step we will prove that the internal functor η_X "sends each morphism to its inverse."

1*st step.* Since $F_{\mathbf{E}}$ is a left adjoint of $T_{\mathbf{E}}$, there exists a function ϕ which assigns, to each pair of objects $X \in \mathbf{E}$ and $\Lambda \in S(\mathbf{E})$, the bijection

$$\phi = \phi_{x,\Lambda} : \mathcal{M}or_{S(\mathbf{E})}(F(X); \Lambda) \xrightarrow{\approx} \mathcal{M}or_{\mathbf{E}}(X; T(\Lambda)) \tag{7}$$

(we write $T_{\mathbf{E}}$ and $F_{\mathbf{E}}$ without the subscript $\mathbf{E}$), and which is natural in X and Λ. Denote by η_X the internal functor $(\phi^{op}_{X,F(X)})^{-1}(1_X) : F(X) \to F^{op}$ (note that $TF(X) = X$).

Consider an internal functor $\eta^{op}_X : F(X)^{op} \to F(X)$. Since η_X is the identity on objects, η^{op} will also be the identity on objects. So $\phi_{X,F(X)}(\eta^{op}_X \circ \eta_X) = 1_X$, and by (7) we have $\eta^{op}_X \circ \eta_X = 1_{F_{\mathbf{E}}(X)}$. Similarly, we prove $\eta_X \circ \eta^{op}_X = 1_{F_{\mathbf{E}}(X)^{op}}$.

2*nd step.* Let $X \in \mathbf{E}$. Suppose $F_{\mathbf{E}}(X) = (X_0, X_1, \partial_1, \partial_0, s, m)$, where $F_{\mathbf{E}}(X)$ is represented as an object of $\mathbf{Cat}\,\boldsymbol{\Delta(E)}$. Consider the subobject Y of X_1 in $\mathbf{E} \underset{\mathbf{Set}}{\times} \mathbf{E}$, which is the "subobject of those morphisms $\xi : \partial_1(\xi) \to \partial_0(\xi)$ in X_1 for which ξ^{op} is the left inverse of ξ." Such a subobject will be the limit of the diagram

$$\begin{tikzcd}
& \varprojlim \left(\begin{array}{ccc} X_1 & & X_1^* \\ \searrow^{\partial_0} & & {}^{\partial_1^*}\swarrow \\ & (1_E, 1_E)^*(X_0) & \end{array} \right) \arrow[dr, "{(pr_0, pr_1, pr_0)^*(m)}"] & \\
X_1 \arrow[ur, "{\left(\begin{smallmatrix} 1 & & \eta \\ & \partial_0 & \end{smallmatrix}\right)}"] \arrow[dr, "\partial_1"'] & & (pr_0, pr_0)^*(X_1) \\
& (1_E, 1_E)^*(X) \arrow[ur, "{pr_0^*(s)}"'] &
\end{tikzcd}$$

Here ∂_i^* and X_1^* are the same as $(pr_1, pr_0)^*(\partial_i)$ and $(pr_1, pr_0)^*(X_1)$.

Now let us construct a subcategory $\lambda = (X, Y, \bar{\partial}_1, \bar{\partial}_0, \bar{s}, \bar{m})$ of the internal category $F_{\mathbf{E}}(X) = (X, X_1, \partial_1, \partial_0, s, m)$ with a natural embedding $i = (1_X, i_1) : \lambda \rightarrowtail F_{\mathbf{E}}(X)$. We must determine the morphisms $\bar{\partial}_1$, $\bar{\partial}_0$, $\bar{s}$, $\bar{m}$

in such a way that they satisfy conditions (1′)–(6′) for an internal category and the following four conditions for an internal functor:

$$\begin{tikzcd} Y \arrow[rr, "i_1"] \arrow[dr, "\bar{\partial}_1"'] & & X_1 \arrow[dl, "\partial_1"] \\ & \delta^1(X) & \end{tikzcd} \quad , \tag{8}$$

$$\begin{tikzcd} Y \arrow[rr, "i_1"] \arrow[dr, "\bar{\partial}_0"'] & & X_1 \arrow[dl, "\partial_0"] \\ & \delta^0(X) & \end{tikzcd} \quad , \tag{9}$$

$$\begin{tikzcd} & \sigma(Y) \arrow[dd, "\sigma(i_1)"] \\ X \arrow[ur, "\bar{s}"] \arrow[dr, "s"'] & \\ & \sigma(X_1) \end{tikzcd} \quad , \tag{10}$$

$$\begin{tikzcd} \varprojlim \left(\begin{smallmatrix} \rho_2^{0,1}(Y) & & \rho_2^{1,2}(Y) \\ & \searrow \quad \swarrow & \\ & \rho_2^1(X) & \end{smallmatrix} \right) \arrow[r, "\bar{m}"] \arrow[d] & \rho_2^{0,2}(Y) \arrow[d, "\rho_2^{0,2}(i_1)"] \\ \varprojlim \left(\begin{smallmatrix} \rho_2^{0,1}(X_1) & & \rho_2^{1,2}(X_1) \\ & \searrow \quad \swarrow & \\ & \rho_2^1(X) & \end{smallmatrix} \right) \arrow[r, "m"] & \rho_2^{0,2}(X_1). \end{tikzcd} \tag{11}$$

The commutativity of diagrams (8), (9) determines $\bar{\partial}_1$ and $\bar{\partial}_0$, respectively. By the definition of Y it is clear that the existence of a morphism $\bar{s}$ (resp., $\bar{m}$) such that diagram (10) (resp., (11)) is commutative is equivalent to the commutativity of diagram (12) (resp., (13)):

$$\begin{tikzcd} & \sigma(X_1) \arrow[dr, "\sigma(\alpha)"] & \\ X_1 \arrow[ur, "s"] \arrow[dr, "s"'] & & \sigma\delta^1\sigma(X_1) \\ & \sigma(X_1) \arrow[ur, "\sigma(\beta)"'] & \end{tikzcd} \quad , \tag{12}$$

$$\begin{tikzcd}[ampersand replacement=\&, column sep=small]
\& \varprojlim\left(\begin{smallmatrix} \rho_2^{0,1}(X_1) & & \rho_2^{1,2}(X_1) \\ & \searrow\ \swarrow & \\ & \rho_2^{1}(X) & \end{smallmatrix}\right) \arrow[r, "m"] \& \rho_2^{0,2}(X_1) \arrow[dr, "\rho_2^{0,2}(\alpha)"] \& \\
\varprojlim\left(\begin{smallmatrix} \rho_2^{0,1}(Y) & & \rho_2^{1,2}(Y) \\ & \searrow\ \swarrow & \\ & \rho_2^{1}(X) & \end{smallmatrix}\right) \arrow[ur] \arrow[dr] \& \& \& \rho_2^{0,2}\delta^1\sigma(X_1) \\
\& \varprojlim\left(\begin{smallmatrix} \rho_2^{0,1}(X_1) & & \rho_2^{1,2}(X_1) \\ & \searrow\ \swarrow & \\ & \rho_2^{1}(X) & \end{smallmatrix}\right) \arrow[r, "m"] \& \rho_2^{0,2}(X_1)\,. \arrow[ur, "\rho_2^{0,2}(\beta)"'] \&
\end{tikzcd} \tag{13}$$

But the commutativity of diagrams (12), (13) as well as that of diagrams $(1')$–$(6')$ for $\bar{\partial}_1$, $\bar{\partial}_0$, $\bar{s}$, $\bar{m}$ easily follow from conditions $(1')$–$(6')$ for an internal category $F(X) = (X, X_1, \partial_1, \partial_0, s, m)$.

So we have constructed an internal category λ and an internal inclusion $i : \lambda \to F_{\mathbf{E}}$ such that $i_0 : \lambda_0 = X \to X = F_{\mathbf{E}}(X)_0$ is the identity. But by the bijection $\phi_{X,\lambda}$ from (7) there exists an internal retraction $j = \phi_{X,\lambda} : F(X) \to \lambda$ which is also the identity on objects. Consider the composition $i \circ j : F(X) \to F(X)$ (note that $i_0 \circ j_0 = 1_X$). Then by (7) we have $i \circ j = 1$. Therefore $i_1 \circ j_1 = 1_{X_1}$. So j_1 is an epimorphism, but it is a monomorphism too by definition. As is wellknown [3], in any topos a morphism which is both mono- and epi- is an isomorphism. Therefore i_1 is an isomorphism and hence λ is isomorphic to $F(X)$ via i. So "every morphism in $F_{\mathbf{E}}(X)$ has a left inverse." Similarly we prove that "every morphism in $F_{\mathbf{E}}(X)$ has the right inverse." Hence $F_{\mathbf{E}}(X)$ is an internal groupoid. $\square$

Now we will define the fundamental group of those toposes which admit the notion of a discrete category. Suppose $\mathbf{E}$ is such a topos and θ is a point in it, i.e., θ is a geometric morphism $\theta : \mathbf{Sets} \to \mathbf{E}$. Consider the internal groupoid $F_{\mathbf{E}}(1)$ in $\boldsymbol{\Delta(E)}$, where 1 is the terminal object in $\mathbf{E}$. Then $F_{\mathbf{E}}(1)_0 = 1$ and therefore $\theta^*(F_{\mathbf{E}}(1))$ is an internal groupoid in $\mathbf{Sets}$ (i.e., a small groupoid) which has only one object. We can consider the small groupoid $\theta^*(F_{\mathbf{E}}(1))$ as a group. Elements of this group are morphisms in $\theta(F_{\mathbf{E}}(1))$ and the product is composition. Denote this group by $\pi(\mathbf{E}, \theta)$. We will call the group $\pi(\mathbf{E}, \theta)$ the fundamental group of the topos $\mathbf{E}$ at the point θ.

Let us prove that construction of the fundamental group from a topos which admit it is functorial. Suppose $\mathbf{E}_1$ and $\mathbf{E}_2$ are toposes which admit the notion of a discrete category, θ_1 and θ_2 are the points in them, respectively, and f is a geometric morphism $f : \mathbf{E}_1 \to \mathbf{E}_2$ such that $f \circ \theta_1 = \theta_2$.

Consider two internal groupoids $F_{\mathbf{E}_1}(1_{\mathbf{E}_1})$ and $f^*(F_{\mathbf{E}_2}(1_{\mathbf{E}_2}))$ in $\boldsymbol{\Delta(E_1)}$. Both have the terminal object in $\mathbf{E}_1$ as the object of objects. Therefore by bijection (7) there exists a unique internal functor $\widetilde{f}$ from $F_{\mathbf{E}_1}(1_{\mathbf{E}_1})$ to

$f^*(F_{\mathbf{E}_2}(1_{\mathbf{E}_2}))$. Consider the functor

$$\theta_1^*(\widetilde{f}) : \theta_1^*(F_{\mathbf{E}_1}(1_{\mathbf{E}_1})) \to \theta_1^* f^*(F_{bfE_2}(1_{\mathbf{E}_2})) = \theta_2^*(F_{\mathbf{E}_2}(1_{\mathbf{E}_2})).$$

This functor determines the group homomorphism

$$\pi(\mathbf{E}_1, \theta_1) \to \pi(\mathbf{E}_2, \theta_2)$$

which we denote by $\pi(f)$.

One can easily check that $\pi(1_{\mathbf{E}}) = 1_{\pi(\mathbf{E})}$, and if the geometric morphisms f and g are composable, then $\pi(f \circ g) = \pi(f) \circ \pi(g)$. So fundamental groups of toposes with points determine a functor from the category of pointed toposes admitting the notion of a discrete category to the category of groups.

Now we will consider two examples which continue the examples from Section 2.

Example 1. $\mathbf{E}$ is a topos of sheaves on a topological space.

Proposition 21. *Let A be a locally compact, locally path connected, and locally simply connected topological space. Then the topos of sheaves over A admits the notion of a discrete category.*

Proof. We will prove this proposition in two steps. In the first step we will construct a functor $F_A : \mathbf{shv}(A) \to S(A)$ such that $T_A \circ F_A = 1_{\mathbf{shv}(A)}$ and in the second step we will prove that F_A is left adjoint to T_A.

$1st$ *step.* Suppose a local homeomorphism $p : E \to A$ is any object of $\mathbf{shv}(A)$. We must assign to $(p : E \to A) \in \mathbf{shv}(A)$ a pair (P^A, ϕ^A), where P^A is a continuous category and $\phi^A : P^A \to ad(A)$ is a continuous functor such that $\phi_1^A : P_1^A \to A \times A$ and $\phi_0^A : P_0^A \to A$ are local homeomorphisms.

Let P^A without a topology (or with a discrete topology as a small category) be the fundamental groupoid $\pi(E)$ of the space E as in [5]. Any object of $\pi(E)$ is a point of E, and an arrow $x \to x'$ of $\pi(E)$ is a homotopy class of paths from x to x'. (Such a path f is a continuous function $I \to E$ with $f(0) = x$, $f(1) = x'$, while two paths f, g with the same end points x and x' are homotopic, when there is a continuous function $F : I \times I \to E$ with $F(t, 0) = f(t)$, $F(t, 1) = g(t)$ and $F(0, s) = x$, $F(1, s) = x'$ for all s and t in I.) The composite of paths $g : x' \to x''$ and $f : x \to x'$ is the path h which is "f followed by g." Composition applies also to homotopy classes and makes $\pi(E)$ a category and a groupoid. (The inverse of any path is the same path traced in the opposite direction.)

Now determine topologies on P_1^A and P_0^A. Such a construction of the topological fundamental groupoid is considered in [6]. Since P_0^A as a set is the same as E, let a topology in P_0^A be the same as in E. (i.e., $P_0^A = E$ as a topological space).

For any open subsets U and V in E, and any path ω in E also with $\omega(0) \in U,\ \omega(1) \in V$, consider the following subset of P_1^A:

$$< U, \omega, V >= \{\omega_2 * \omega * \omega_1 | \omega_1 \text{ is a path in } U \text{ and } \omega_2 \text{ is a path in } V\}$$

(here by $< \kappa >$ is denoted the homotopy class of a path κ and $*$ denotes the composition in the groupoid P_A).

The subsets $< U, \omega, V >$ determine an open base in P_1^A. Using this open base we will generate a topology in P_1^A. Equipped with these topologies, P^A becomes a continuous groupoid. (It is easy to check that the maps $\partial_1, \partial_0 : P_1^A \to P_0^A = E,\ s : E \to P_1^A$, and $m : \varprojlim(P_1^A \xrightarrow{\partial_0} E \xleftarrow{\partial_1} O_1^A) \to P_1^A$ are continuous.) There is a naturally determined continuous functor: $\phi^A : P^A \to ad(a)$ $\phi_1^A = (p \times p) \circ (\partial_1, \partial_0)$ $\phi_0^A = p$. It remains to prove that ϕ_1^A is a local homeomorphism ($\phi_0^A = p$ is a local homeomorphism because $(p : E \to A) \in \mathbf{shv}(A)$). For this it is sufficient to prove that (∂_1, ∂_0) is a local homeomorphism because $p \times p$ is. First we prove that (∂_1, ∂_0) is an open map and then prove that (∂_1, ∂_0) is a local homeomorphism.

Any open $W \subset P_1^A$ can be represented as a union $W = \underset{i}{\cup} < U_i, \omega_i, V_i >$, where U_i and V_i are linear connected open subsets. (Such a representation is possible because the space E is locally path connected as well as A.) Then $(\partial_1, \partial_0)(W) = \underset{i}{\cup} U_i \times V_i$ and therefore is open, i.e., (∂_1, ∂_0) is an open map.

For any $< \omega >\in P_1^A$, choose an open neighborhood $< U, \omega, V >$ of $< \omega >$, where U and V are path connected and simply connected open sets. (This choice is also possible because E is a locally simply connected space as well as A.) It is easy to check that $(\partial_1, \partial_0)|_{< U, \omega, V >} :< U, \omega, V >\to U \times V$ is a bijection and also a continuous open map. Therefore (∂_1, ∂_0) is a local homeomorphism.

So we have constructed the object $(P^A, \phi^A) \in S(A)$. Let $F_A(p : E \to A)$ be (P^A, ϕ^A). This construction clearly implies that $T_A \circ F_A = 1_{\mathbf{shv}(A)}$.

2nd step. Suppose $(p : E \to A) \in \mathbf{shv}(A)$ and $(\Lambda, \psi) \in S(A)$. Consider the function

$$\phi : \mathcal{M}or_{S(A)}(F_A(p : E \to A); (\Lambda, \psi)) \to \mathcal{M}or_{\mathbf{shv}(A)}((p : E \to A); \Lambda_0),$$
$$(f : F_A(p : E \to A) \to \Lambda) \mapsto (f_0 : E \to \Lambda_0).$$

We must prove that ϕ is a bijection. For this let us construct its inverse ϕ^{-1}. Suppose we are given $f_0 : E \to \Lambda_0$ with $\psi_0 \circ f_0 = p$ and an element $< \omega >$ in $F_A(p : E \to A)_1$ (here ω is a path in E). Consider the following path κ in $F_A(p : E \to A)_1$:

$$\kappa : I \to F_A(p : E \to A)_1, \quad t \mapsto \kappa(t) =< \underset{s \mapsto \omega(st)}{I \to E} > .$$

The source of this path is the homotopy class $s(\omega(0)) \in F_A(p : E \to A)_1$, and the target is the homotopy class of ω. The image of the path κ under the map $(\partial_1, \partial_0) : F_A(p : E \to A)_1 \to E \times E$ is the pair of paths $(\omega(0), \omega)$. The path κ is the lifting of the path $(\omega(0), \omega)$ with the source $s(<\omega(0)>) \in F_A(p : E \to A)_1$. Therefore consider the path $(f_0 \circ \omega(0), f_0 \circ \omega)$ in $\Lambda_0 \times \Lambda_0$ and the lifting of this path to Λ_1 via (∂_1, ∂_0) with a source $s(f_0 \circ \omega(0)) \in \Lambda_1$. (Such a lifting is possible and is unique by Proposition 13.) We determine $\phi^{-1}(f_0)(<\omega>)$ as the target of the lifted path. It is easy to check that $\phi^{-1}(f_0)$ so defined is a morphism in $S(A)$ and ϕ^{-1} is the inverse of ϕ. $\square$

Let us determine what $\pi(\mathbf{shv}(A), \theta)$ will be for a given point θ in $\mathbf{shv}(A)$ (A is a topological space with properties from Proposition 21). As is well-known (see [4]), if A is sufficiently separated (for instance, if A is Hausdorff), then θ induced by a continuous map $\widetilde{\theta} : \{*\} \to A$ ($\{*\}$ is a one point space with the unique point $*$. In this case $\pi(\mathbf{shv}(A), \theta)$ is a group of automorphisms of the object $\widetilde{\theta}(*) \in A$ in the fundamental groupoid $\pi(A)$, i.e., $\pi(\mathbf{shv}(A), \theta) = \pi(A, \widetilde{\theta}(*))$ (here $\pi(A, \widetilde{\theta}(*))$ is the ordinary fundamental group of the space A at the point $\widetilde{\theta}(*)$).

Example 2. E is a topos of presheaves.

Proposition 22. *Let* $\mathbf{C}$ *be a small category; then the topos* $\mathbf{Sets}^{C}$ *admits the notion of a discrete category.*

Proof. It is easily seen from [2] that the left adjoint functor of the forgetful functor $T_{\mathbf{C}} : S_1(\mathbf{C}) \to \mathbf{Sets}^{\mathbf{C}}$ exists and has the shape

$$F_{\mathbf{C}} : \mathbf{Sets}^{\mathbf{C}} \to S_1(\mathbf{C}),$$
$$F_{\mathbf{C}} : (p : \mathbf{Q} \to \mathbf{C}) \mapsto (\mathbf{Q}, p, \mathbf{Z}, g),$$

where $\mathbf{Z}$ is the category of fractions or the universal groupoid of $\mathbf{Q}$ and g is the natural functor from $\mathbf{Q}$ to $\mathbf{Z}$. (Note that $\mathbf{Q}$ and $\mathbf{Z}$ have the same objects and g is identical on objects.)

The condition $T_{\mathbf{C}} \circ F_{\mathbf{C}} = 1_{\mathbf{Sets}^{\mathbf{C}}}$ is satisfied trivially. $\square$

Now let us determine what $\pi(\mathbf{Sets}^{\mathbf{C}}, \theta)$ will be for a given point θ in $\mathbf{Sets}^{\mathbf{C}}$. Suppose $\mathbf{G}$ is the category of fractions of $\mathbf{C}$ and $i : \mathbf{C} \to \mathbf{G}$ is a natural embedding. From the construction of the functor $F_{\mathbf{C}}$ it follows that $i^*(F_{\mathbf{G}}(1) = i^*(\mathbf{G}, 1_{\mathbf{G}}, \mathbf{G}, 1_{\mathbf{G}}) = (\mathbf{C}, 1_{\mathbf{C}}, \mathbf{G}, i) = F_{\mathbf{C}}(1)$. Therefore $\pi(\mathbf{Sets}^{\mathbf{C}}, \theta) = \pi(\mathbf{Sets}^{\mathbf{G}}, \widetilde{i} \circ \theta)$; here by $\widetilde{i}$ is denoted the geometric morphism from $\mathbf{Sets}^{\mathbf{C}}$ to $\mathbf{Sets}^{\mathbf{G}}$ induced by i.

By Diaconescu's theorem [3] the point $\theta : \mathbf{Sets} \to \mathbf{Sets}^{\mathbf{C}}$ determines the flat presheaf $\bar{\theta} : \mathbf{C}^{op} \to \mathbf{Sets}$. From the definition of flat presheaves it follows that the value of the functor $\bar{\theta}$ is a nonempty set only on the objects of exactly one connected component of $\mathbf{C}^{op}$. Denote this connected component by $\mathbf{C}_0^{op}$. Then the group $\pi(\mathbf{Sets}^{\mathbf{C}}, \theta) = \pi(\mathbf{Sets}^{\mathbf{G}}, \widetilde{i} \circ \theta)$ will be isomorphic to the group of automorphisms of some object of $\mathbf{C}_0$ in the groupoid $\mathbf{G}$.

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