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ON THE NUMBER OF REPRESENTATIONS OF POSITIVE INTEGERS BY A DIRECT SUM OF BINARY QUADRATIC FORMS WITH DISCRIMINANT −23

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ABSTRACT. Explicit exact formulas are obtained for the number of representations of positive integers by some direct sums of quadratic forms $F_1 = x_1^2 + x_1x_2 + 6x_2^2$ and $\Phi_1 = 2x_1^2 + x_1x_2 + 3x_2^2$.

Let

$$F_1 = x_1^2 + x_1x_2 + 6x_2^2 \quad \text{and} \quad \Phi_1 = 2x_1^2 + x_1x_2 + 3x_2^2.$$

These are primitive reduced binary quadratic forms with discriminant $\Delta = -23$. For each $k \geq 1$, let F_k and Φ_k denote the direct sums of k copies of F_1 and Φ_1, respectively.

In [1] exact formulas are derived for the number of representations of a positive integer by positive quadratic forms in six variables with integral coefficients, among which there are suitable formulas for the quadratic forms F_3, $F_2 \oplus \Phi_1$, $F_1 \oplus \Phi_2$ and Φ_3.

In [2] explicit exact formulas are derived for the arithmetical function $r(n;Q)$, the number of representations of a positive integer n by the quadratic forms Q, for $Q = F_2$, $F_1 \oplus \Phi_1$ and Φ_2.

In the present paper we obtain formulas for $r(n;Q)$ when $Q = F_4$, $F_3 \oplus \Phi_1$, $F_2 \oplus \Phi_2$, $F_1 \oplus \Phi_3$ and Φ_4.

It should be pointed out that the approach described here enables one to get formulas for $r(n;Q)$ for $Q = F_k$, Φ_k, $F_i \oplus \Phi_j$ $(i, j \geq 1,\ i + j = k)$, where $k > 4$. However the calculations will be very tedious.

In this paper the notation, definitions, and some results from [3] will be mostly used.

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§ 1. Some Known Results

Let

$$Q = Q(x_1, x_2, \dots, x_f) = \sum_{1 \le r \le s \le f} b_{rs} x_r x_s$$

be a positive quadratic form in f (f is even) variables with integral coefficients b_{rs}. Further let $\mathcal{D}$ be the determinant of the quadratic form

$$2Q = \sum_{r,s=1}^{f} a_{rs} x_r x_s \quad (a_{rr} = 2b_{rr}, \quad a_{rs} = a_{sr} = b_{rs}, \quad r < s);$$

A_{rs} the algebraic cofactors of elements a_{rs} in $\mathcal{D}$; Δ the discriminant of the form Q, i.e., $\Delta = (-1)^{f/2}\mathcal{D}$; $\delta = \text{g.c.d.}(\frac{A_{rr}}{2}, A_{rs})$ $(r, s = 1, \dots, f)$; $N = \frac{\mathcal{D}}{\delta}$ the level of the form Q; $\chi(d)$ the character of the form Q, i.e., $\chi(d) = 1$ if Δ is a perfect square, but if Δ is not a perfect square and $2 \nmid \Delta$, then $\chi(d) = (\frac{d}{|\Delta|})$ for $d > 0$ and $\chi(d) = (-1)^{f/2}\chi(-d)$ for $d < 0$ (here $(\frac{d}{|\Delta|})$ is the generalized Jacobi symbol). A positive quadratic form in f variables of level N and character $\chi(d)$ is called a quadratic form of type $(-f/2, N, \chi)$. Finally, let $\mathcal{P}_\nu = \mathcal{P}_\nu(x_1, x_2, \dots, x_f)$ be a spherical function of order ν (ν is a positive integer) with respect to the quadratic form Q.

In what follows q is an odd prime and $z = \exp(2\pi i\tau)$, $\operatorname{Im}\tau > 0$.

As is wellknown, to each positive quadratic form Q there corresponds the theta-series

$$\vartheta(\tau; Q) = 1 + \sum_{n=1}^{\infty} r(n; Q) z^n. \tag{1.1}$$

We shall formulate the well-known results in the form of the following lemmas.

Lemma 1 ([3], p. 874, 875, 817; see also [4], p. 15). *Any positive quadratic form Q of type $(-k, q, 1)$, $2|k$, $k > 2$, corresponds to one and the same Eisenstein series*

$$E(\tau; Q) = 1 + \sum_{n=1}^{\infty} (\alpha\sigma_{k-1}(n) z^n + \beta\sigma_{k-1}(n) z^{qn}),$$

where

$$\alpha = \frac{i^k}{\rho_k}\,\frac{q^{k/2} - i^k}{q^k - 1}, \quad \beta = \frac{1}{\rho_k}\,\frac{q^k - i^k q^{k/2}}{q^k - 1}, \tag{1.2}$$

$$\rho_k = (-1)^{k/2}\,\frac{(k-1)!}{(2\pi)^k}\,\zeta(k). \tag{1.3}$$

Lemma 2 ([3], pp.874, 875, 895). *If Q is a primitive quadratic form of type $(-k, q, 1)$, $2|k$, then the difference $\vartheta(\tau; Q) - E(\tau; Q)$ is a cusp form of type $(-k, q, 1)$.*

Lemma 3 ([3], p. 853, Theorem 33). *The homogeneous quadratic polynomials in f variables $\varphi_{rs} = x_r x_s - \frac{1}{f}\frac{A_{rs}}{\mathcal{D}} 2Q \quad (r, s = 1, 2, \ldots, f)$ are spherical functions of second order with respect to the positive quadratic form Q in f variables.*

Lemma 4 ([3], p. 855). *If Q is a quadratic form of type $(-f/2, N, \chi)$ and $\mathcal{P}_\nu$ is the spherical function of order ν with respect to Q, then the generalized multiple theta-series*

$$\vartheta(r; Q, \mathcal{P}_\nu) = \sum_{n=1}^{\infty} \Big(\sum_{Q=n} \mathcal{P}_\nu \Big) z^n$$

is a cusp form of type $(-(f/2 + \nu), N, \chi)$.

Lemma 5 ([3], p.846). *If the quadratic forms Q_1 and Q_2 have the same level N and characters $\chi_1(d)$ and $\chi_2(d)$, respectively, then the quadratic form $Q_1 \oplus Q_2$ will have the level N and the character $\chi_1(d)\chi_2(d)$.*

§ 2. Some Auxiliary Results

2.1. For the quadratic form F_1 we have $\mathcal{D} = 23$, $A_{11} = 12$, $A_{22} = 2$, i.e., $\delta = 1$, $N = 23$, $\Delta = -23$, $\chi = \chi(d) = (\frac{d}{23})$ if $d > 0$. For the quadratic form Φ_1 we have $\mathcal{D} = 23$, $A_{11} = 6$, $A_{22} = 4$, i.e., $\delta = 1$, $N = 23$, $\Delta = -23$, $\chi = \chi(d) = (\frac{d}{23})$ if $d > 0$. Hence F_1 and Φ_1 are quadratic forms of type $(-1, 23, \chi)$. Thus, by Lemma 5, F_2, Φ_2, and $F_1 \oplus \Phi_1$ are quadratic forms of type $(-2, 23, 1)$.

For the quadratic form F_2 we have $\mathcal{D} = 23^2$, $A_{11} = 12 \cdot 23$, $A_{22} = 2 \cdot 23$. Hence, if in Lemma 3 we put

$$f = 4, \quad Q = F_2, \quad r = s = 1,$$

then the polynomial

$$\varphi_{11} = x_1^2 - \frac{6}{23} F_2$$

will be a spherical function of second order with respect to F_2.

For the quadratic form Φ_2 we have $\mathcal{D} = 23^2$, $A_{11} = 6 \cdot 23$, $A_{22} = 4 \cdot 23$. Hence, if in Lemma 3 we put

$$f = 4, \quad Q = \Phi_2, \quad r = s = 1, \quad \text{and} \quad r = s = 2,$$

then the polynomials

$$\varphi_{11} = x_1^2 - \frac{3}{23} \Phi_2 \quad \text{and} \quad \varphi_{22} = x_2^2 - \frac{2}{23} \Phi_2$$

will be spherical functions of second order with respect to Φ_2.

For the quadratic form $F_1 \oplus \Phi_1$ we have $\mathcal{D} = 23^2$, $A_{11} = 12 \cdot 23$, $A_{22} = 2 \cdot 23$. Hence, if in Lemma 3 we put

$$f = 4, \quad Q = F_1 \oplus \Phi_1, \quad r = s = 1, \quad \text{and} \quad r = s = 2,$$

then the polynomials

$$\varphi_{11} = x_1^2 - \frac{6}{23}\,(F_1 \oplus \Phi_1) \quad \text{and} \quad \varphi_{22} = x_2^2 - \frac{1}{23}\,(F_1 \oplus \Phi_1)$$

will be spherical functions of second order with respect to $F_1 \oplus \Phi_1$.

2.2. It is easy to verify that the equation $F_1 = n$

(a) has two integral solutions for $n = 1$: $x_1 = \pm 1$, $x_2 = 0$;
(b) has no integral solutions for $n = 2, 3, 5$;
(c) has two integral solutions for $n = 4$: $x_1 = \pm 2$, $x_2 = 0$.

Also it is easy to verify that the equation $\Phi_1 = n$

(a) has no integral solutions for $n = 1, 5$;
(b) has two integral solutions for $n = 2$: $x_1 = \pm 1$, $x_2 = 0$;
(c) has two integral solutions for $n = 3$: $x_1 = 0$, $x_2 = \pm 1$.
(d) has two integral solutions for $n = 4$: $x_1 = \pm 1$, $x_2 = \mp 1$.

Hence, according to (1.1), we have

$$\vartheta(\tau; F_1) = 1 + 2z + 2z^4 + 0z^5 + \cdots, \tag{2.1}$$

$$\vartheta(\tau; \Phi_1) = 1 + 2z^2 + 2z^3 + 2z^4 + 0z^5 + \cdots. \tag{2.2}$$

From (2.1) it follows that

$$\vartheta(\tau; F_2) = \vartheta^2(\tau; F_1) = 1 + 4z + 4z^2 + 4z^4 + 8z^5 + \cdots, \tag{2.3}$$

whence

$$\vartheta(\tau; F_3) = \vartheta(\tau; F_2)\vartheta(\tau; F_1) = 1 + 6z + 12z^2 + 8z^3 + 6z^4 + 24z^5 + \cdots. \tag{2.4}$$

From (2.2) it follows that

$$\vartheta(\tau; \Phi_2) = \vartheta^2(\tau; \Phi_1) = 1 + 4z^2 + 4z^3 + 8z^4 + 8z^5 + \cdots, \tag{2.5}$$

whence

$$\vartheta(\tau; \Phi_3) = \vartheta(\tau; \Phi_2)\vartheta(\tau; \Phi_1) = 1 + 6z^2 + 6z^3 + 18z^4 + 24z^5 + \cdots. \tag{2.6}$$

From (2.1) and (2.2) it follows that

$$\begin{aligned} \vartheta(\tau; F_1 \oplus \Phi_1) &= \vartheta(\tau; F_1)\vartheta(\tau; \Phi_1) = \\ &= 1 + 2z + 2z^2 + 6z^3 + 8z^4 + 4z^5 + \cdots. \end{aligned} \tag{2.7}$$

§ 3. Formulas for $r(n;F_4)$, $r(n;\Phi_4)$, $r(n;F_3\oplus\Phi_1)$, $r(n;F_2\oplus\Phi_2)$, $r(n;F_1\oplus\Phi_3)$

Theorem 1. *The system of generalized fourfold theta-series*

$$\vartheta(\tau;F_2,\varphi_{11})=\frac{1}{23}\sum_{n=1}^{\infty}\Big(\sum_{F_2=n}23x_1^2-6n\Big)z^n, \tag{3.1}$$

$$\vartheta(\tau;F_1\oplus\Phi_1,\varphi_{11})=\frac{1}{23}\sum_{n=1}^{\infty}\Big(\sum_{F_1\oplus\Phi_1=n}23x_1^2-6n\Big)z^n, \tag{3.2}$$

$$\vartheta(\tau;F_1\oplus\Phi_1,\varphi_{22})=\frac{1}{23}\sum_{n=1}^{\infty}\Big(\sum_{F_1\oplus\Phi_1=n}23x_2^2-n\Big)z^n, \tag{3.3}$$

$$\vartheta(\tau;\Phi_2,\varphi_{11})=\frac{1}{23}\sum_{n=1}^{\infty}\Big(\sum_{\Phi_2=n}23x_1^2-3n\Big)z^n, \tag{3.4}$$

$$\vartheta(\tau;\Phi_2,\varphi_{22})=\frac{1}{23}\sum_{n=1}^{\infty}\Big(\sum_{\Phi_2=n}23x_2^2-2n\Big)z^n \tag{3.5}$$

is a basis of the space $S_4(23,1)$ (*the space of cusp forms of type* $(-4,23,1)$).

Proof. I. As said above, F_2 is a quadratic form of type $(-2,23,1)$ and

$$\varphi_{11}=x_1^2-\frac{6}{23}F_2$$

is a spherical function of second order with respect to F_2. Hence, by Lemma 4, the theta-series (3.1) is a cusp form of type $(-4,23,1)$.

Taking into account (2.3), it is not difficult to verify that the equation $F_2=n$

(a) has four integral solutions for $n=1$: $x_1=\pm1$, $x_2=x_3=x_4=0$; $x_3=\pm1$, $x_1=x_2=x_4=0$;
(b) has four integral solutions for $n=3$: $x_1=\pm1$, $x_3=1$, $x_2=x_4=0$; $x_1=\pm1$, $x_3=-1$, $x_2=x_4=0$ for $n=2$;
(c) has no integral solutions for $n=3$;
(d) has four integral solutions for $n=4$: $x_1=\pm2$, $x_2=x_3=x_4=0$; $x_3=\pm2$, $x_1=x_2=x_4=0$;
(e) has eight integral solutions for $n=5$: $x_1=\pm2$, $x_3=1$, $x_2=x_4=0$; $x_1=\pm2$, $x_3=-1$, $x_2=x_4=0$; $x_1=\pm1$, $x_3=2$, $x_2=x_4=0$; $x_1=\pm1$, $x_3=-2$, $x_2=x_4=0$.

Hence

$$\vartheta(\tau;F_2,\varphi_{11})=\frac{1}{23}\{(23\cdot2-6\cdot4)z+(23-6\cdot2)4z^2+$$
$$+(23\cdot4\cdot2-6\cdot4\cdot4)z^4+(23\cdot4\cdot4+23\cdot4-6\cdot5\cdot8)z^5+\cdots\}=$$

$$= \frac{22}{23}\,z + \frac{44}{23}\,z^2 + \frac{88}{23}\,z^4 + \frac{220}{23}\,z^5 + \cdots . \tag{3.6}$$

II. As said above $F_1 \oplus \Phi_1$ is a quadratic form of type $(-2, 23, 1)$ and

$$\varphi_{11} = x_1^2 - \frac{6}{23}(F_1 \oplus \Phi_1) \quad \text{and} \quad \varphi_{22} = x_2^2 - \frac{1}{23}(F_1 \oplus \Phi_1)$$

are spherical functions of second order with respect to $F_1 \oplus \Phi_1$.

Hence, by Lemma 4, the theta-series (3.2) and (3.3) are cusp forms of type $(-4, 23, 1)$.

Taking into account (2.7), it is not difficult to verify that the equation $F_1 \oplus \Phi_1 = n$

(a) has two integral solutions for $n = 1$: $x_1 = \pm 1$, $x_2 = x_3 = x_4 = 0$;
(b) has two integral solutions for $n = 2$: $x_3 = \pm 1$, $x_1 = x_2 = x_4 = 0$;
(c) has six integral solutions for $n = 3$: $x_1 = \pm 1$, $x_3 = 1$, $x_2 = x_4 = 0$; $x_1 = \pm 1$, $x_3 = -1$, $x_2 = x_4 = 0$; $x_1 = x_2 = x_3 = 0$, $x_4 = \pm 1$;
(d) has eight integral solutions for $n = 4$: $x_1 = \pm 1$, $x_4 = 1$, $x_2 = x_3 = 0$; $x_1 = \pm 1$, $x_4 = -1$, $x_2 = x_3 = 0$; $x_1 = \pm 2$, $x_2 = x_3 = x_4 = 0$; $x_1 = x_2 = 0$, $x_3 = 1$, $x_4 = -1$; $x_1 = x_2 = 0$, $x_3 = -1$, $x_4 = 1$;
(e) has four integral solutions for $n = 5$: $x_1 = \pm 1$, $x_3 = 1$, $x_4 = -1$, $x_2 = 0$; $x_1 = \pm 1$, $x_2 = 0$, $x_3 = -1$, $x_4 = 1$.

Hence,

$$\vartheta(\tau; F_1 \oplus \Phi_1, \varphi_{11}) = \frac{1}{23}\big\{(23 - 6)2z + (-6 \cdot 2 \cdot 2)z^2 +$$
$$+ (23{\cdot}4 - 6{\cdot}3{\cdot}6)z^3 + (23{\cdot}4{\cdot}2 + 23{\cdot}4 - 6{\cdot}4{\cdot}8)z^4 + (23 - 6{\cdot}5{\cdot}4)z^5 + \cdots\big\} =$$
$$= \frac{34}{23}\,z - \frac{24}{23}\,z^2 - \frac{16}{23}\,z^3 + \frac{84}{23}\,z^4 - \frac{28}{23}\,z^5 + \cdots , \tag{3.7}$$

$$\vartheta(\tau; F_1 \oplus \Phi_1, \varphi_{22}) =$$
$$= \frac{1}{23}\big\{ -2z - 2 \cdot 2z^2 - 3 \cdot 6z^3 - 4 \cdot 8z^4 - 5 \cdot 4z^5 + \cdots \big\} =$$
$$= -\frac{2}{23}\,z - \frac{4}{23}\,z^2 - \frac{18}{23}\,z^3 - \frac{32}{23}\,z^4 - \frac{20}{23}\,z^5 + \cdots . \tag{3.8}$$

III. As said above Φ_2 is a quadratic form of type $(-2, 23, 1)$ and

$$\varphi_{11} = x_1^2 - \frac{3}{23}\,\Phi_2 \quad \text{and} \quad \varphi_{22} = x_2^2 - \frac{2}{23}\,\Phi_2$$

are spherical functions of second order with respect to Φ_2. Hence, by Lemma 4, theta-series (3.4) and (3.5) are cusp forms of type $(-4, 23, 1)$.

Taking into account (2.5) it is easy to verify that the equation $\Phi_2 = n$

(a) has no integral solutions for $n = 1$;
(b) has four integral solutions for $n = 2$: $x_1 = \pm 1$, $x_2 = x_3 = x_4 = 0$; $x_3 = \pm 1$, $x_1 = x_2 = x_4 = 0$;

(c) has four integral solutions for $n = 3$: $x_2 = \pm 1$, $x_1 = x_3 = x_4 = 0$; $x_1 = x_2 = x_3 = 0$, $x_4 = \pm 1$;

(d) has eight integral solutions for $n = 4$: $x_1 = \pm 1$, $x_3 = 1$, $x_2 = x_4 = 0$; $x_1 = \pm 1$, $x_3 = -1$, $x_2 = x_4 = 0$; $x_1 = 1$, $x_2 = -1$, $x_3 = x_4 = 0$; $x_1 = -1$, $x_2 = 1$, $x_3 = x_4 = 0$; $x_1 = x_2 = 0$, $x_3 = 1$, $x_4 = -1$; $x_1 = x_2 = 0$, $x_3 = -1$, $x_4 = 1$;

(e) has eight integral solutions for $n = 5$: $x_1 = \pm 1$, $x_2 = x_3 = 0$, $x_4 = 1$; $x_1 = \pm 1$, $x_2 = x_3 = 0$, $x_4 = -1$; $x_1 = x_4 = 0$, $x_2 = \pm 1$, $x_3 = 1$; $x_1 = x_4 = 0$, $x_2 = \pm 1$, $x_3 = -1$.

Hence

$$\begin{aligned}\vartheta(\tau;\Phi_2,\varphi_{11}) &= \frac{1}{23}\big\{(23\cdot 1\cdot 2 - 3\cdot 2\cdot 4)z^2 - 3\cdot 3\cdot 4z^3 + \\ &+(23\cdot 1\cdot 6 - 3\cdot 4\cdot 8)z^4 + (23\cdot 1\cdot 4 - 3\cdot 5\cdot 8)z^5 + \cdots\big\} = \\ &= \frac{22}{23}z^2 - \frac{36}{23}z^3 + \frac{42}{23}z^4 - \frac{28}{23}z^5 + \cdots, \end{aligned} \tag{3.9}$$

$$\begin{aligned}\vartheta(\tau;\Phi_2,\varphi_{22}) &= \frac{1}{23}\big\{-2\cdot 2\cdot 4z^2 + (23\cdot 1\cdot 2 - 2\cdot 3\cdot 4)z^3 + \\ &+(23\cdot 1\cdot 2 - 2\cdot 4\cdot 8)z^4 + (23\cdot 1\cdot 4 - 2\cdot 5\cdot 8)z^5 + \cdots\big\} = \\ &= -\frac{16}{23}z^2 + \frac{22}{23}z^3 - \frac{18}{23}z^4 + \frac{12}{23}z^5 + \cdots. \end{aligned} \tag{3.10}$$

The system of theta-series (3.1)–(3.5) is linearly independent, since the determinant of fifth order whose elements are coefficients in the expansions of these theta-series is different from zero. Thus the theorem is proved, since $\dim S_4(23,1) = 5$ [3, p. 900]. □

Theorem 2.

$$\begin{aligned} r(n;F_4) &= \frac{24}{53}\sigma_3^*(n) - \frac{240}{11\cdot 53}\Big(\sum_{F_2=n} 23x_1^2 - 6n\Big) + \\ &+\frac{440}{23\cdot 53}\Big(\sum_{F_1\oplus\Phi_1=n} 23x_1^2 - 6n\Big) - \frac{2640}{23\cdot 53}\Big(\sum_{F_1\oplus\Phi_1=n} 23x_2^2 - n\Big) - \\ &-\frac{8144}{23\cdot 53}\Big(\sum_{\Phi_2=n} 23x_1^2 - 3n\Big) - \frac{14096}{23\cdot 53}\Big(\sum_{\Phi_2=n} 23x_2^2 - 2n\Big), \end{aligned} \tag{I}$$

$$\begin{aligned} r(n;\Phi_4) &= \frac{24}{53}\sigma_3^*(n) - \frac{28}{11\cdot 53}\Big(\sum_{F_2=n} 23x_1^2 - 6n\Big) + \\ &+\frac{16}{23\cdot 53}\Big(\sum_{F_1\oplus\Phi_1=n} 23x_1^2 - 6n\Big) - \frac{96}{23\cdot 53}\Big(\sum_{F_1\oplus\Phi_1=n} 23x_2^2 - n\Big) - \end{aligned}$$

$$-\frac{512}{23\cdot 53}\Big(\sum_{\Phi_2=n} 23x_1^2-3n\Big)-\frac{1164}{23\cdot 53}\Big(\sum_{\Phi_2=n} 23x_2^2-2n\Big), \qquad \text{(II)}$$

$$r(n;F_3\oplus\Phi_1)=\frac{24}{53}\,\sigma_3^*(n)-\frac{28}{11\cdot 53}\Big(\sum_{F_2=n} 23x_1^2-6n\Big)+$$
$$+\frac{175}{23\cdot 53}\Big(\sum_{F_1\oplus\Phi_1=n} 23x_1^2-6n\Big)-\frac{1050}{23\cdot 53}\Big(\sum_{F_1\oplus\Phi_1=n} 23x_2^2-n\Big)-$$
$$-\frac{2685}{23\cdot 53}\Big(\sum_{\Phi_2=n} 23x_1^2-3n\Big)-\frac{4609}{23\cdot 53}\Big(\sum_{\Phi_2=n} 23x_2^2-2n\Big), \qquad \text{(III)}$$

$$r(n;F_2\oplus\Phi_2)=\frac{24}{53}\,\sigma_3^*(n)-\frac{28}{11\cdot 53}\Big(\sum_{F_2=n} 23x_1^2-6n\Big)+$$
$$+\frac{122}{23\cdot 53}\Big(\sum_{F_1\oplus\Phi_1=n} 23x_1^2-6n\Big)-\frac{732}{23\cdot 53}\Big(\sum_{F_1\oplus\Phi_1=n} 23x_2^2-n\Big)-$$
$$-\frac{1360}{23\cdot 53}\Big(\sum_{\Phi_2=n} 23x_1^2-3n\Big)-\frac{2330}{23\cdot 53}\Big(\sum_{\Phi_2=n} 23x_2^2-2n\Big), \qquad \text{(IV)}$$

$$r(n;F_1\oplus\Phi_3)=\frac{24}{53}\,\sigma_3^*(n)-\frac{28}{11\cdot 53}\Big(\sum_{F_2=n} 23x_1^2-6n\Big)+$$
$$+\frac{3}{53}\Big(\sum_{F_1\oplus\Phi_1=n} 23x_1^2-6n\Big)-\frac{18}{53}\Big(\sum_{F_1\oplus\Phi_1=n} 23x_2^2-n\Big)-$$
$$-\frac{1201}{23\cdot 53}\Big(\sum_{\Phi_2=n} 23x_1^2-3n\Big)-\frac{1959}{23\cdot 53}\Big(\sum_{\Phi_2=n} 23x_2^2-2n\Big), \qquad \text{(V)}$$

where

$$\begin{aligned}\sigma_3^*(n)&=\sigma_3(n) && \text{if } 23\nmid n,\\ &=\sigma_3(n)+23^2\sigma_3\Big(\frac{n}{23}\Big) && \text{if } 23|n.\end{aligned}$$

Proof. From (2.3), (2.5) follow the relations

$$\vartheta(\tau;F_4)=\vartheta^2(\tau;F_2)=1+8z+24z^2+32z^3+24z^4+48z^5+\cdots, \qquad (3.11)$$
$$\vartheta(\tau;\Phi_4)=\vartheta^2(\tau;\Phi_2)=1+8z^2+8z^3+32z^4+48z^5+\cdots, \qquad (3.12)$$

respectively. The couples of relations (2.2) and (2.4), (2.3) and (2.4), (2.1) and (2.6) lead to

$$\vartheta(\tau;F_3\oplus\Phi_1)=\vartheta(\tau;F_3)\vartheta(\tau;\Phi_1)=$$
$$=1+6z+14z^2+22z^3+44z^4+76z^5+\cdots, \qquad (3.13)$$

$$\vartheta(\tau; F_2 \oplus \Phi_2) = \vartheta(\tau; F_2)\vartheta(\tau; \Phi_2) = \\ = 1 + 4z + 8z^2 + 20z^3 + 44z^4 + 64z^5 + \cdots, \tag{3.14}$$

$$\vartheta(\tau; F_1 \oplus \Phi_3) = \vartheta(\tau; F_1)\vartheta(\tau; \Phi_3) = \\ = 1 + 2z + 6z^2 + 18z^3 + 32z^4 + 60z^5 + \cdots, \tag{3.15}$$

respectively. By Lemma 5, F_4, Φ_4, $F_3 \oplus \Phi_1$, $F_2 \oplus \Phi_2$ and $F_1 \oplus \Phi_3$ are quadratic forms of type $(-4, 23, 1)$, to which by Lemma 1 there corresponds one and the same Eisenstein series. For $k = 4$, from (1.2) we have

$$\alpha = \frac{1}{\rho_4}\frac{23^2-1}{23^4-1} = \frac{1}{\rho_4}\frac{1}{23^2+1}, \quad \beta = \frac{1}{\rho_4}\frac{23^4-23^2}{23^4-1} = \frac{1}{\rho_4}\frac{23^2}{23^2+1}$$

where $\rho_4 = \frac{1}{240}$ [3, p. 823] . Thus for all these forms we have

$$E(\tau; F_4) = E(\tau; \Phi_4) = E(\tau; F_3 \oplus \Phi_1) = E(\tau; F_2 \oplus \Phi_2) = E(\tau; F_1 \oplus \Phi_3) = \\ = 1 + \frac{24}{53}\sum_{n=1}^{\infty}(\sigma_3(n)z^n + 23^2\sigma_3(n)z^{23n}) = \tag{3.16}$$

$$= 1 + \frac{24}{53}z + \frac{24\cdot 9}{53}z^2 + \frac{24\cdot 28}{53}z^3 + \frac{24\cdot 73}{53}z^4 + \frac{24\cdot 126}{53}z^5 + \cdots. \tag{3.16$_1$}$$

(I). By Lemma 2, the difference $\vartheta(\tau; F_4) - E(\tau; F_4)$ is a cusp form of type $(-4, 23, 1)$. Hence, by Theorem 1, there exist numbers $c_1, \ldots, c_5$ such that

$$\vartheta(\tau; F_4) - E(\tau; F_4) = c_1\vartheta(\tau; F_2, \varphi_{11}) + c_2\vartheta(\tau; F_1 \oplus \Phi_1, \varphi_{11}) + \\ + c_3\vartheta(\tau; F_1 \oplus \Phi_1, \varphi_{22}) + c_4\vartheta(\tau; \Phi_2, \varphi_{11}) + c_5\vartheta(\tau; \Phi_2, \varphi_{22}).$$

Equating the coefficients of $z, z^2, \ldots, z^5$ on both sides of this equality and taking into account (3.11), (3.16$_1$) and (3.6)–(3.10), we can find these numbers and obtain

$$\vartheta(\tau, F_4) = E(\tau; F_4) - \frac{240\cdot 23}{11\cdot 53}\vartheta(\tau; F_2, \varphi_{11}) + \\ + \frac{440}{53}\vartheta(\tau; F_1 \oplus \Phi_1, \varphi_{11}) - \frac{2640}{53}\vartheta(\tau; F_1 \oplus \Phi_1, \varphi_{22}) - \\ - \frac{8144}{53}\vartheta(\tau; \Phi_2, \varphi_{11}) - \frac{14096}{53}\vartheta(\tau; \Phi_2, \varphi_{22}).$$

Equating now the coefficients of z^n on both sides of this equation, by (1.1), (3.16), and (3.1)–(3.5), we get the desired formula (I).

(II)-(V). Applying the same arguments as above except that (3.11) is replaced by (3.12), (3.13), (3.14), and (3.15) for the quadratic forms $\Phi_4, \Phi_3 \oplus \Phi_1, F_2 \oplus \Phi_2$, and $\Phi_1 \oplus \Phi_3$, respectively, we obtain

$$\vartheta(\tau, \Phi_4) = E(\tau; \Phi_4) - \frac{644}{11\cdot 53}\vartheta(\tau; F_2, \varphi_{11}) +$$

$$
\begin{aligned}
&+\frac{16}{53}\,\vartheta(\tau;F_1\oplus\Phi_1,\varphi_{11})-\frac{96}{53}\,\vartheta(\tau;F_1\oplus\Phi_1,\varphi_{22})-\\
&-\frac{512}{53}\,\vartheta(\tau;\Phi_2,\varphi_{11})-\frac{1164}{53}\,\vartheta(\tau;\Phi_2,\varphi_{22}),\\
\vartheta(\tau,F_3\oplus\Phi_1)=\;&E(\tau;F_3\oplus\Phi_1)-\frac{644}{11\cdot 53}\,\vartheta(\tau;F_2,\varphi_{11})+\\
&+\frac{175}{53}\,\vartheta(\tau;F_1\oplus\Phi_1,\varphi_{11})-\frac{1050}{53}\,\vartheta(\tau;F_1\oplus\Phi_1,\varphi_{22})-\\
&-\frac{2685}{53}\,\vartheta(\tau;\Phi_2,\varphi_{11})-\frac{4609}{53}\,\vartheta(\tau;\Phi_2,\varphi_{22}),\\
\vartheta(\tau,F_2\oplus\Phi_2)=\;&E(\tau;F_2\oplus\Phi_2)-\frac{644}{11\cdot 53}\,\vartheta(\tau;F_2,\varphi_{11})+\\
&+\frac{122}{53}\,\vartheta(\tau;F_1\oplus\Phi_1,\varphi_{11})-\frac{732}{53}\,\vartheta(\tau;F_1\oplus\Phi_1,\varphi_{22})-\\
&-\frac{1360}{53}\,\vartheta(\tau;\Phi_2,\varphi_{11})-\frac{2330}{53}\,\vartheta(\tau;\Phi_2,\varphi_{22}),\\
\vartheta(\tau,F_1\oplus\Phi_3)=\;&E(\tau;F_1\oplus\Phi_3)-\frac{644}{11\cdot 53}\,\vartheta(\tau;F_2,\varphi_{11})+\\
&+\frac{69}{53}\,\vartheta(\tau;F_1\oplus\Phi_1,\varphi_{11})-\frac{414}{53}\,\vartheta(\tau;F_1\oplus\Phi_1,\varphi_{22})-\\
&-\frac{1201}{53}\,\vartheta(\tau;\Phi_2,\varphi_{11})-\frac{1959}{53}\,\vartheta(\tau;\Phi_2,\varphi_{22}).
\end{aligned}
$$

From these identities, as above, we get the formulas (II)–(V). $\square$

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