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NON-ABELIAN COHOMOLOGY WITH COEFFICIENTS IN CROSSED BIMODULES

H. INASSARIDZE

ABSTRACT. When the coefficients are crossed bimodules, Guin's non-abelian cohomology [2], [3] is extended in dimensions 1 and 2, and a nine-term exact cohomology sequence is obtained.

We continue to study non-abelian cohomology of groups (see [1]) following Guin's approach to non-abelian cohomology [2], [3]. The pointed sets of cohomology $H^n(G,(A,\mu))$, $n=1,2$, will be defined when the group of coefficients (A,μ) is a crossed G-R-bimodule. The notion of a crossed bimodule has been introduced in [1]. $H^1(G,(A,\mu))$ is equipped with a partial product and coincides with Guin's cohomology group [3] when crossed G-modules are viewed as crossed G-G-bimodules. The pointed set of cohomology $H^2(G,(A,\mu))$ coincides with the second pointed set of cohomology defined in [1] when the coefficients are crossed modules. A coefficient short exact sequence of crossed G-R-bimodules gives rise to a nine-term exact cohomology sequence and we recover the exact cohomology sequence obtained in [1] when the coefficients are crossed modules. By analogy with the case $n=2$ the definition of a pointed set of cohomology $H^n(G,(A,\mu))$ of a group G with coefficients in a crossed G-R-bimodule (A,μ) is given for all $n\geq 1$.

The notation and diagrams of [1] will be used.

Recall the definitions of a crossed G-R-bimodule and the group $\mathrm{Der}(G,(A,\mu))$ of derivations from G to (A,μ).

Let G,R and A be groups. (A,μ) is a crossed G-R-bimodule if:

1) (A,μ) is a crossed R-module,
2) G acts on R and A,
3) the homomorphism $\mu: A\longrightarrow R$ is a homomorphism of G-groups,
4) ${}^{({}^g r)}a = {}^{grg^{-1}}a$ for $g\in G$, $r\in R$, $a\in A$.

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The group $\mathrm{Der}(G,(A,\mu))$ is defined as follows. It consists of pairs (α, r) where α is a crossed homomorphism from G to A and r is an element of R such that $\mu\alpha(x) = r\,{}^{x}r^{-1}$ for all $x \in G$. A product in $\mathrm{Der}(G,(A,\mu))$ is given by $(\alpha, r)(\beta, s) = (\alpha * \beta, rs)$ where $(\alpha * \beta)(x) = {}^{x}\beta(x)\alpha(x)$, $x \in G$. For any $a \in A$ and $(\alpha, r) \in \mathrm{Der}(G,(A,\mu))$ the following equality holds:

$$\alpha(x)\,{}^{xr}a = {}^{rx}a\,\alpha(x) \quad \text{for all} \quad x \in G\,.$$

Definition 1. *Let (A,μ) be a crossed G-R-bimodule. It will be said that a crossed homomorphism $\alpha : G \longrightarrow A$ satisfies condition* (j) (*resp. condition* (j′)) *if for $c \in H^0(G,R)$* (*resp. if for $c \in H^0(G,R)$ such that there is $b \in A$ with $\mu(b) = c$*) *there exists $a \in A$ such that ${}^{c}\alpha(x) = a^{-1}\,\alpha(x)\,{}^{x}a$ for $x \in G$ and $\mu(a) = 1$. It will be said that an element (α, r) of $\mathrm{Der}(G,(A,\mu))$ satisfies condition* (j) (*resp. condition* (j′)) *if α satisfies this condition.*

It is obvious that any element of the form $(\alpha, 1)$ satisfies condition (j′). If (A,μ) is a crossed G-R-bimodule induced by a surjective homomorphism $f : G \longrightarrow R$, then every element $(\alpha, r) \in \mathrm{Der}(G,(A,\mu))$ such that $\alpha(\ker f) = 1$ satisfies condition (j). In effect, for $c \in Z(R) = H^0(G,R)$ we have $z_x dx = xd$, $x \in G$, with $f(d) = c$ and $z_x \in \ker f$. Thus, $\alpha(z_x)\,{}^{z_x}\alpha(dx) = \alpha(xd)$, whence $\alpha(d)\,{}^{c}\alpha(x) = \alpha(x)\,{}^{x}\alpha(d)$ and $\mu\alpha(d) = r\,f(d)\,r^{-1}\,f(d)^{-1} = 1$.

Note that if $(\alpha, r) \sim (\alpha', r')$ (see below) and (α, r) satisfies condition (j) then (α', r') satisfies condition (j) too when $H^0(G,R) \subset Z(R)$. In effect, we have $\alpha'(x) = b^{-1}\alpha(x)\,{}^{x}b$, $r' = \mu(b)^{-1} r\, t$ and ${}^{c}\alpha(x) = a^{-1}\alpha(x)\,{}^{x}a$, $\mu(a) = 1$, where $c, t \in H^0(G,R) \subset Z(R)$. Thus

$$\begin{aligned} {}^{c}\alpha'(x) &= {}^{c}b^{-1}\,{}^{c}\alpha(x)\,{}^{cx}b = {}^{c}b^{-1}\,a^{-1}\alpha(x)\,{}^{x}a\,{}^{cx}b = \\ &= {}^{c}b^{-1}a^{-1}\alpha(x)\,{}^{x}(a\,{}^{c}b) = {}^{c}b^{-1}a^{-1}b\,\alpha'(x)\,{}^{x}b^{-1}\,{}^{x}(a\,{}^{c}b) = \\ &= {}^{c}b^{-1}ba^{-1}\alpha'(x)\,{}^{x}(ab^{-1}\,{}^{c}b) \end{aligned}$$

with $\mu(a\,b^{-1}\,{}^{c}b)^{-1} = (\mu(a)\,\mu(b^{-1})\,\mu({}^{c}b))^{-1} = \mu({}^{c}b^{-1})\,\mu(b) = c\,\mu(b)^{-1}c^{-1}$ $\mu(b) = 1$.

It is clear that if $f : (A,\mu) \longrightarrow (B,\lambda)$ is a homomorphism of crossed G-R-bimodules and $(\alpha, r) \in \mathrm{Der}(G,(A,\mu))$ satisfies condition (j), then $(f\alpha, r)$ satisfies condition (j).

Let (A,μ) be a crossed G-R-bimodule. In the group $\mathrm{Der}(G,(A,\mu))$ we introduce a relation $\sim$ defined as follows:

$$(\alpha, r) \sim (\beta, s) \Longleftrightarrow \begin{cases} \exists\, a \in A : \beta(x) = a^{-1}\alpha(x)\,{}^{x}a, \\ s = \mu(a)^{-1}r \mod H^0(G,R) \end{cases}.$$

Later we shall need the following assertion:

If (A,μ) is a precrossed G-R-bimodule the equality

$$ {}^{rx}a = {}^{xr}a \tag{1}$$

holds for any $x \in G$, $a \in A$, $r \in H^0(G,R)$.

In effect,we have

$$^{rx}a = {}^{xx^{-1}rx}a = {}^{x(^{x^{-1}}r)}a = {}^{xr}a.$$

Proposition 2. *The relation $\sim$ is an equivalence. Assume $H^0(G,R)$ is a normal subgroup of R; then the group* $\mathrm{Der}(G,(A,\mu))$ *induces on* $\mathrm{Der}(G,(A,\mu))/\sim$ *a partial product defined by*

$$[(\alpha,1)][(\beta,s)] = [(\alpha * \beta, s)]$$

if $[(\beta,s)]$ *contains an element satisfying condition* (j$'$)*, and by*

$$[(\alpha,r)][(\beta,s)] = [(\alpha * \beta, rs)]$$

if $[(\beta,s)]$ *contains an element satisfying condition* (j).

Proof. If $(\alpha,r) \sim (\alpha',r')$, i.e., $\alpha'(x) = a^{-1}\alpha(x)\,{}^{x}a$, $x \in G$, and $r' = \mu(a)^{-1}rz$, $z \in H^0(G,R)$, then $\alpha(x) = a\,\alpha(x)\,{}^{x}a^{-1}$ and $r = \mu(a)\,r'z^{-1}$, $z^{-1} \in H^0(G,R)$. Thus, $(\alpha',r') \sim (\alpha,r)$.

If $(\alpha,r) \sim (\alpha',r')$ and $(\alpha',r') \sim (\alpha'',r'')$ we have

$$\alpha'(x) = a^{-1}\alpha(x)\,{}^{x}a, \quad r' = \mu(a)^{-1}rz,$$
$$\alpha''(x) = b^{-1}\alpha'(x)\,{}^{x}b, \quad r'' = \mu(b)^{-1}r'z',$$

where $z, z' \in H^0(G,R)$. This implies $(\alpha,r) \sim (\alpha'',r'')$ and the relation $\sim$ is an equivalence.

It is clear that if $(\alpha,r) \in \mathrm{Der}(G,(A,\mu))$ and $c \in H^0(G,R)$ then $(\alpha,r) \sim (\alpha,rc)$.

We have yet to show the correctness of the partial product.

Let $(\alpha,1) \sim (\alpha',1)$, $(\beta,s) \sim (\beta',s')$ and (β,s) satisfy condition (j$'$). We will prove that $(\alpha,1)(\beta,s) \sim (\alpha',1)(\beta',s')$. One has

$$\alpha'(x) = a^{-1}\alpha(x)\,{}^{x}a, \quad x \in G,$$
$$1 = \mu(a)^{-1}z, \quad z \in H^0(G,R),$$

and

$$\beta'(x) = b^{-1}\beta(x)\,{}^{x}b, \quad x \in G,$$
$$s' = \mu(b)^{-1}sz', \quad z' \in H^0(G,R).$$

Then $\beta'(x)\alpha'(x) = b^{-1}\beta(x)\,{}^{x}ba^{-1}\alpha(x)\,{}^{x}a = b^{-1}\beta(x)a^{-1}\alpha(x)\,{}^{x}a\,{}^{x}b = b^{-1}a^{-1}\,{}^{\mu(a)}\beta(x)\alpha(x)\,{}^{x}a\,{}^{x}b = b^{-1}a^{-1}d^{-1}\beta(x)\alpha(x)\,{}^{x}(dab)$ and $s' = \mu(b)^{-1}\mu(a)^{-1}zsz' = \mu(b^{-1}a^{-1}d^{-1})\,sz''z'$ where $\beta(x) = d^{-1}\beta(x)\,{}^{x}d$, $\mu(d) = 1$ and $z'' \in H^0(G,R)$. Therefore $(\alpha,1)(\beta,s) \sim (\alpha',1)(\beta',s')$.

It is clear that the set of all elements of the form $[(\alpha,1)]$ forms an abelian group under this product.

Finally, we will prove that if $(\alpha, r) \sim (\alpha', r')$, $(\beta, s) \sim (\beta', s')$ and (β, s) satisfies condition (j) then $(\alpha, r)(\beta, s) \sim (\alpha', r')(\beta', s')$ and we check Guin's proof [3] in our case.

We first prove that

$$(\alpha, r)(\beta, s) \sim (\alpha, rc)(\beta, s)$$

for $c \in H^0(G, R)$.

Using condition (j) and equality (1) of [3] one gets

$$\begin{aligned} {}^{rc}\beta(x)\alpha(x) = {}^{r}(a^{-1}\beta(x)\,{}^{x}a)\,\alpha(x) = {}^{r}a^{-1}\,{}^{r}\beta(x)\,{}^{rx}a\,\alpha(x) = \\ = {}^{r}a^{-1}\,{}^{r}\beta(x)\alpha(x)\,{}^{rx}a. \end{aligned}$$

Since $\mu({}^{r}a)^{-1} = (r\,\mu(a)\,r^{-1})^{-1} = 1$, one has $rcs = \mu\,({}^{r}a)^{-1}rsc'$ with $c' \in H^0(G, R)$. Therefore, $(\alpha, r)(\beta, s) \sim (\alpha, rc)(\beta, s)$.

Further,we have

$$\alpha'(x) = b^{-1}\alpha(x)\,{}^{x}b, \quad r' = \mu(b)^{-1}rz,$$

and $\beta'(x) = d^{-1}\beta(x)\,{}^{x}d$, $s' = \mu(d)^{-1}st$ with $z, t \in H^0(G, R)$.

Put

$$(\alpha, rz)(\beta, s) = (\gamma, rzs),$$

where $\gamma(x) = {}^{rz}\beta(x)\alpha(x)$, $x \in G$, and $(\alpha', r')(\beta', s') = (\gamma', r's')$, where $\gamma'(x) = {}^{r'}\beta'(x)\alpha'(x)$, $x \in G$.

We will show that

$$(\alpha, rz)(\beta, s) \sim (\alpha', r')(\beta', s').$$

Using (1) and equality (1) of [1] one has

$$\begin{aligned} \gamma'(x) = {}^{r'}(d^{-1}\beta(x)\,{}^{x}d)\,b^{-1}\alpha(x)\,{}^{x}b = \\ = {}^{\mu(b)^{-1}r\cdot z}d^{-1}\,{}^{\mu(b)^{-1}rz}\beta(x)\,{}^{\mu(b)^{-1}rzx}d\,b^{-1}\alpha(x)\,{}^{x}b = \\ = b^{-1}\,{}^{r\cdot z}d^{-1}\,{}^{rz}\beta(x)\,{}^{r\cdot x}({}^{z}d)\,\alpha(x)\,{}^{x}b = b^{-1}\,{}^{r\cdot z}d^{-1}\,{}^{r\cdot z}\beta(x)\alpha(x)\,{}^{xrz}d\,{}^{x}b, \end{aligned}$$

and $\mu({}^{r\cdot z}d\,b)^{-1} = \mu(b)^{-1}rz\,\mu(d)^{-1}z^{-1}r^{-1} = r's't^{-1}s^{-1}z^{-1}r^{-1}$, $r's' = \mu({}^{r\cdot z}d\,b)^{-1}rzst$ with $t \in H^0(G, R)$.

Therefore $(\alpha, rz)(\beta, s) \sim (\alpha', r')(\beta', s')$, whence $(\alpha, r)(\beta, s) \sim (\alpha', r')$ (β', s'). $\square$

Definition 3. *Let (A, μ) be a crossed G-R-bimodule. One denotes by $H^1(G, (A, \mu))$ the quotient set $\mathrm{Der}(G, (A, \mu))/\sim$ equipped with the aforementioned partial product and it will be called the first set of cohomology of G with coefficients in the crossed G-R-bimodule (A, μ).*

If (A,μ) is a crossed G-module viewed as a crossed G-G-bimodule then $H^0(G,G)=Z(G)$ and for $(\alpha,g)\in \mathrm{Der}_G(G,A)=\mathrm{Der}(G,(A,\mu))$ and $c\in Z(G)$ the equality $\alpha(cx)=\alpha(xc)$, $x\in G$, implies

$$\alpha(c)\,{}^{c}\alpha(x)=\alpha(x)\,{}^{x}\alpha(c),$$

whence ${}^{c}\alpha(x)=\alpha(c)^{-1}\alpha(x)\,{}^{x}\alpha(c)$ and $\mu(\alpha(c))=gcg^{-1}c^{-1}=1$. Therefore every element of $\mathrm{Der}_G(G,A)$ satisfies condition (j). It follows that if (A,μ) is a crossed G-module we recover the group $H^1(G,A)$ defined by Guin [3].

It is clear that the map $H^1(G,A)\longrightarrow H^1(G,(A,1))$ given by $[\alpha]\longmapsto[(\alpha,1)]$ is an isomorphism where $(A,1)$ is a crossed G-R-bimodule.

Proposition 4. *Let (A,μ) be a crossed G-R-bimodule and assume $H^0(G,R)$ is a normal subgroup of R. If (α,r) and (β,s) satisfy condition* (j) *then $(\alpha,r)(\beta,s)$ and $(\alpha,r)^{-1}$ satisfy condition* (j).

Proof. Let $c\in H^0(G,R)$. Then ${}^{c}\alpha(x)=b^{-1}\alpha(x)\,{}^{x}b$ and $\mu(b)=1$. Since $H^0(G,R)$ is a normal subgroup of R, there is $c'\in H^0(G,R)$ such that $cr=rc'$. For c' we have ${}^{c'}\beta(x)=d^{-1}\beta(x)\,{}^{x}d$ and $\mu(d)=1$. Put $(\alpha,r)(\beta,s)=(\gamma,rs)$. Then

$$\begin{aligned}{}^{c}\gamma(x)&={}^{cr}\beta(x)\,{}^{c}\alpha(x)={}^{r}d^{-1}\,{}^{r}\beta(x)\,{}^{rx}d\,b^{-1}\alpha(x)\,{}^{x}b=\\&={}^{r}d^{-1}b^{-1}\,{}^{r}\beta(x)\alpha(x)\,{}^{x}(b\,{}^{r}d)\end{aligned}$$

with $\mu(b\,{}^{r}d)=\mu(b)\,r\,\mu(d)\,r^{-1}=1$. Thus, (γ,rs) satisfies condition (j). Put $(\alpha,r)^{-1}=(\overline{\alpha},r^{-1})$ where $\overline{\alpha}(x)={}^{r^{-1}}\alpha(x)^{-1}$, $x\in G$. If $c\in H^0(G,R)$ one has

$${}^{c}\overline{\alpha}(x)={}^{cr^{-1}}\alpha(x)^{-1}={}^{r^{-1}c'}\alpha(x)^{-1}={}^{r^{-1}}({}^{x}a^{-1}\alpha(x)^{-1}a),$$

where $cr^{-1}=r^{-1}c'$, $c'\in H^0(G,R)$ and ${}^{c'}\alpha(x)=a^{-1}\alpha(x)^{-1}\,{}^{x}a$, $\mu(a)=1$.

Hence

$$\begin{aligned}{}^{c}\overline{\alpha}(x)&={}^{r^{-1}x}a^{-1}\,{}^{r^{-1}}\alpha(x)^{-1}\,{}^{r^{-1}}a={}^{r^{-1}}\alpha(x)^{-1}\,{}^{xr^{-1}}a^{-1}\,{}^{r^{-1}}a=\\&={}^{r^{-1}}a\,{}^{r^{-1}}\alpha(x)^{-1}\,{}^{x}({}^{r^{-1}}a^{-1})\end{aligned}$$

with $\mu({}^{r^{-1}}a^{-1})=r^{-1}\mu(a^{-1})\,r=1$.

Therefore, $(\overline{\alpha},r^{-1})$ satisfies condition (j). □

Corollary 5. *The subset of $H^1(G,(A,\mu))$ of all equivalence classes containing an element with condition* (j) *forms a group if $H^0(G,R)$ is a normal subgroup of R.*

Proposition 6. *Let (A,μ) be a crossed G-R-bimodule such that $H^0(G,R)$ is a normal subgroup of R. If there is a map $\eta: H^0(G,R)\longrightarrow Z(G)$ such that $\mathrm{Im}\,\eta$ acts trivially on R and ${}^{\eta(r)}a={}^{r}a$, $a\in A$, then $H^1(G,(A,\mu))$ is a group.*

Proof. We have to show that every element $(\alpha, r) \in \mathrm{Der}(G, (A, \mu))$ satisfies condition (j). If $c \in H^0(G, R)$ take $\eta(c) = d \in Z(G)$. Then $\alpha(dx) = \alpha(xd)$ and $\alpha(d)\, {}^d\alpha(x) = \alpha(x)\, {}^x\alpha(d)$. Thus ${}^c\alpha(x) = \alpha(d)^{-1}\alpha(x)\, {}^x\alpha(d)$ and $\mu\alpha(d) = r\, {}^d r^{-1} = rr^{-1} = 1$. □

Corollary 7. *Let (A, μ) be either a crossed G-R-bimodule such that $H^0(G, R)$ is a normal subgroup of R trivially acting on A or induced by a surjective homomorphism $f : G \longrightarrow R$ such that $f(Z(G)) = Z(R)$. Then $H^1(G, (A, \mu))$ is a group.*

Proof. In the first case take η as the trivial map. In the second case take a map $\eta : Z(R) \longrightarrow Z(G)$ such that $f\eta = 1_{Z(R)}$. □

If $f : (A, \mu) \longrightarrow (B, \lambda)$ is a homomorphism of crossed G-R-bimodules then f induces a natural map

$$f^1 : H^1(G, (A, \mu)) \longrightarrow H^1(G, (B, \lambda))$$

which is a homomorphism in the following sense:

if xy is defined for $x, y \in H^1(G, (A, \mu))$ then $f^1(x)f^1(y)$ is defined and $f^1(xy) = f^1(x)f^1(y)$.

The above defined action of G on $\mathrm{Der}(G, (A, \mu))$ induces an action of G on $H^1(G, (A, \mu))$ given by

$${}^g[(\alpha, r)] = [{}^g(\alpha, r)], \quad g \in G.$$

We have to show that if $(\alpha, r) \sim (\alpha', r')$ then ${}^g(\alpha, r) \sim {}^g(\alpha', r')$. In effect, since

$$\alpha'(x) = a^{-1}\alpha(x)\, {}^x a, \quad x \in G,$$

this implies

$$\alpha'({}^{g^{-1}}x) = a^{-1}\alpha({}^{g^{-1}}x)\, {}^{g^{-1}xg}a, \quad x \in G.$$

Thus

$${}^g\alpha'({}^{g^{-1}}x) = {}^g a^{-1}\, {}^g\alpha({}^{g^{-1}}x)\, {}^{xg}a, \quad x \in G.$$

We also have $r' = \mu(a)^{-1}rz$, $z \in H^0(G, R)$, whence ${}^g r' = {}^g\mu(a^{-1})\, {}^g r\, {}^g z = \mu({}^g a)^{-1}\, {}^g r\, {}^g z$. Therefore ${}^g(\alpha, r) \sim {}^g(\alpha', r')$.

In what follows if f is a map from a group G to a group G' then $f^{-1} : G \longrightarrow G'$ denotes a map given by $f^{-1}(x) = f(x)^{-1}$.

Let (A, μ) be a crossed G-R-bimodule. The definition of $H^2(G, (A, \mu))$ is similar to the case of (A, μ) being a crossed G-module (see [1]).

Consider diagram (4) of [1] and the group $\mathrm{Der}(M, (A, \mu))$ where (A, μ) is viewed as a crossed M-R-bimodule induced by τl_0 and a crossed F-R-bimodule induced by τ. Let $\widetilde{Z^1}(M, (A, \mu))$ be the subset of $\mathrm{Der}(M, (A, \mu))$ consisting of elements of the form $(\alpha, 1)$.

Define, on $\widetilde{Z}^1(M, (A, \mu))$, relation

$$(\alpha', 1) \sim (\alpha, 1) \Longleftrightarrow (\beta, h) \in \mathrm{Der}(F, (A, \mu))$$

such that

$$(\alpha', 1) = (\beta l_0, h)(\alpha, 1)(\beta l_1, h)^{-1}$$

in the group $\mathrm{Der}(M, (A, \mu))$.

Definition 8. *Let (A, μ) be a crossed G-R-bimodule. The relation $\sim$ is an equivalence. Denote by $H^2(G,(A,\mu))$ the quotient set $\widetilde{Z}^1(M,(A,\mu))$ $/ \sim$. It will be called the second set of cohomology of G with coefficients in the crossed G-R-bimodule (A, μ).*

It can be proved (as for a crossed G-module (A, μ) (see Proposition 8 [1])) that $\widetilde{Z^1}(M, (A, \mu))\ / \sim$ is independent of diagram (4) of [1] and is unique up to bijection.

Let (A, μ) be a crossed G-R-bimodule. Then there is a canonical map

$$\vartheta' : H^2(G, \ker\ \mu) \longrightarrow H^2(G, (A, \mu))$$

defined by the composite

$$[E] \overset{\vartheta^{-1}}{\longmapsto} [\alpha] \longmapsto [(\alpha, 1)].$$

This map is surjective and was defined when (A, μ) is a crossed G-module [3].

Proposition 9. *Let (A, μ) be a crossed G-R-bimodule. There is an action of G on $H^2(G, (A, \mu))$ such that $Z(G)$ acts trivially. If R acts on G and satisfies the compatibility condition* (3) *of* [1] *then there is also an action of R on $H^2(G, (A, \mu))$.*

Proof. The action of G on $H^2(G, (A, \mu))$ is defined exactly in the same manner as for a crossed G-module (A, μ) (see Proposition 12 [1]). The action of R is defined similarly. Namely, we have an action of R on M_G given by

$$^r(|g_1|^\epsilon \cdots |g_n|^\epsilon, |g_1'|^\epsilon \cdots |g_m'|^\epsilon) = (|^r g_1|^\epsilon \cdots |^r g_n|^\epsilon, |^r g_1'|^\epsilon \cdots |^r g_m'|^\epsilon)$$

and one gets an action of R on $Der(M_G, (A, \mu))$ defined by

$$^r(\alpha, s) = (\widetilde{\alpha}, {}^r s),$$

where $\widetilde{\alpha}(m) = {}^r\alpha({}^{r^{-1}}m)$, $r \in R$, $m \in M_G$. Define ${}^r[(\alpha, 1)] = [{}^r(\alpha, 1)]$, $r \in R$. If $(\alpha, 1) \sim (\alpha', 1)$ it is easy to see that ${}^r(\alpha, 1) \sim {}^r(\alpha', 1)$. $\square$

Let (A, μ) be a crossed G-R-bimodule. Using (1) it can easily be shown that there is an action of $H^0(G, R)$ on $H^2(G, \ker\ \mu)$ given by

$$^r[\alpha] = [^r\alpha], \quad r \in H^0(G, R),$$

where $\alpha : M_G \longrightarrow \ker \mu$ is a crossed homomorphism under the action of G on A (see diagram (5) of [1]) such that $\alpha(\Delta) = 1$.

If this action of $H^0(G,R)$ is trivial and $\operatorname{Der}(F_G,(A,\mu))=\operatorname{IDer}(F_G,(A,\mu))$ then the map

$$\vartheta' : H^2(G,\ker\mu) \longrightarrow H^2(G,(A,\mu))$$

is a bijection.

Let

$$1 \longrightarrow (A,1) \xrightarrow{\varphi} (B,\mu) \xrightarrow{\psi} (C,\lambda) \longrightarrow 1 \tag{2}$$

be an exact sequence of crossed G-R-bimodules. If the action of $H^0(G,R)$ on $H^2(G,A)$ is trivial then there is an action of $H^1(G,(C,\lambda))$ on $H^2(G,A)$ given by

$$^{[(\alpha,r)]}[\gamma] = [^r\gamma].$$

We have to show that $^r\gamma$ is a crossed homomorphism and the correctness of the action.

Consider the diagram

$$\begin{array}{ccccc} M_G & \overset{l_0}{\underset{l_1}{\rightrightarrows}} & F_G & \xrightarrow{\tau_G} & G \\ & & & & \downarrow \alpha \\ A & \xrightarrow{\varphi} & B & \xrightarrow{\psi} & C \end{array} \tag{3}$$

There is a crossed homomorphism $\beta : F_G \longrightarrow B$ such that $\psi\beta = \alpha\tau_G$. Take the product

$$(\beta l_0, r)(\varphi\gamma, 1)(\beta l_0, r)^{-1} = (\widetilde{\gamma}, 1)$$

in the group $\operatorname{Der}(M_G\ ,(B,\mu))$. Then $\widetilde{\gamma}(x) = \beta(x)^{-1}\,{}^r\varphi\gamma(x)\,\beta(x) = {}^r\varphi\gamma(x)$, $x \in M$. Therefore $^r\gamma : M_G \longrightarrow A$ is a crossed homomorphism such that $^r\gamma(\Delta) = 1$.

If $(\alpha', r') \in [(\alpha, r)] \in H^1(G,(C,\lambda))$, i.e., $(\alpha,r) \sim (\alpha', r')$, then

$$\alpha'(x) = c^{-1}\alpha(x)\,{}^x c \quad \text{and} \quad r' = \lambda(c)^{-1} r t,$$

where $t \in H^0(G,R)$. It follows that

$$\begin{aligned} \varphi({}^{r'}\gamma(x)) &= {}^{r'}\varphi\gamma(x) = {}^{\lambda(c)^{-1}rt}\varphi\gamma(x) = {}^{\mu(b)^{-1}rt}\varphi\gamma(x) = \\ &= b^{-1}\,{}^{r\cdot t}\varphi\gamma(x)\,b = {}^{r\cdot t}\varphi\gamma(x) = \varphi({}^{r\cdot t}\gamma(x)), \quad x \in M_G, \end{aligned}$$

where $\psi(b) = c$.

Hence we have

$$[^{r'}\gamma] = [^{rt}\gamma] = [^r\gamma]$$

proving the correctness of the action.

Using diagram (3) for the exact sequence (2) one defines a connecting map

$$\delta^1 : H^1(G,(C,\lambda)) \longrightarrow H^2(G,A)$$

as follows.

For $[(\alpha, r)] \in H^1(G,(C,\lambda))$ take a crossed homomorphism $\beta : F_G \longrightarrow B$ such that $\psi\beta = \alpha\,\tau_G$. Thus there is a crossed homomorphism $\gamma : M_G \longrightarrow A$ such that

$$\varphi\gamma = (\beta l_1)^{-1}\beta l_0.$$

It is clear that $\gamma(\Delta) = 1$. Define

$$\delta^1([(\alpha, r)]) = [\gamma].$$

We must prove the correctness of δ^1. If $\beta' : F_G \longrightarrow B$ with $\psi\beta' = \alpha\tau$, then $\psi\beta' = \psi\beta$. Thus there is a crossed homomorphism $\sigma : F_G \longrightarrow A$ such that $\beta' = \beta\varphi\sigma$. Then we have

$$\begin{aligned}\varphi\gamma' &= (\beta' l_1)^{-1}\,\beta' l_0 = (\beta\varphi\sigma) {l_1}^{-1}\,(\beta\varphi\sigma) l_0 = \varphi\sigma l_1^{-1}\,\beta l_1^{-1}\,\beta l_0\,\varphi\sigma l_0 = \\ &= \beta l_1^{-1}\,\beta l_0 \varphi\sigma l_1^{-1}\varphi\sigma l_0 = \varphi(\gamma\sigma l_1^{-1}\sigma l_0).\end{aligned}$$

Hence $[\gamma'] = [\gamma]$.

If $(\alpha, r) \sim (\alpha', r')$ then

$$\alpha'(y) = c^{-1}\alpha(y)\,{}^{y}c, \quad c \in C, \quad y \in M,$$
$$r' = \lambda(c)^{-1} r t, \quad t \in H^0(G, R).$$

Take $\beta' : F_G \longrightarrow B$ such that

$$\beta'(x) = b^{-1}\beta(x)\,{}^{x}b$$

with $\psi(b) = c$ where $\psi\beta = \alpha\tau$. Then $(\beta' l_1^{-1}\beta' l_0)(y) = \beta'(x_2)^{-1}\beta'(x_1)$ where $y = (x_1, x_2) \in M_G$. Hence $\varphi\gamma'(y) = (\beta' l_1^{-1}\beta' l_0)(y) = (b^{-1}\beta(x_2)\,{}^{x_2}b)^{-1} b^{-1}\beta(x_1)\,{}^{x_1}b = {}^{x_2}b^{-1}\beta(x_2)^{-1}\beta(x_1)\,{}^{x_1}b = \beta(x_2)^{-1}\beta(x_1) = \varphi\gamma(y)$. Whence $\gamma' = \gamma$.

For any exact sequence (2) of crossed G-R-bimodules there is also an action of $\mathrm{Der}(F_0,(C,\lambda))$ on $H^3(G, A)$ defined as follows:

$$^{(\alpha,r)}[f] = [^{r}f],$$

where $f : F_2 \longrightarrow A$ is a crossed homomorphism with $\prod_{i=0}^{3}(f l_i^2 \tau_3)^{\epsilon} = 1$ where $\epsilon = (-1)^i$ (see diagram (7) of [1]) and $(\alpha, r) \in \mathrm{Der}(F_0,(C,\lambda))$. The correctness of this action is proved similarly to the case of a short exact sequence of crossed G-modules (see [1]).

If either the aforementioned action of $\mathrm{Der}(F_0,(C,\lambda))$ on $H^3(G, A)$ is trivial or $Der(F_0,(C,\lambda)) = \mathrm{IDer}(F_0,(C,\lambda))$ and $H^0(G, R)$ acts trivially on $H^2(G, \ker\lambda)$, then a connecting map

$$\delta^2 : H^2(G,(C,\lambda)) \longrightarrow H^3(G, A)$$

is defined by

$$\delta^2([(\alpha, 1)]) = [\gamma], \quad (\alpha, 1) \in \widetilde{\mathrm{Der}}(M_0,(C,\lambda)),$$

where $\varphi\gamma = \overline{\beta}\tau_2$ with $\overline{\beta} = \prod_{i=0}^{2}(\beta l_i^1)^\epsilon$, $\epsilon = (-1)^i$, and $\psi\beta = \alpha\tau_1$ (see diagram (7) of [1]). The correctness of δ^2 is proved similarly to the case of crossed G-modules [1], and if (2) is an exact sequence of crossed G-modules we recover the above-defined connecting map $\delta^2 : H^2(G, C) \longrightarrow H^3(G, A)$.

Theorem 10. *Let* (2) *be an exact sequence of crossed G-R-bimodules. Then there is an exact sequence*

$$1 \longrightarrow H^0(G,A) \xrightarrow{\varphi^0} H^0(G,B) \xrightarrow{\psi^0} H^0(G,C) \xrightarrow{\delta^0}$$
$$\xrightarrow{\delta^0} H^1(G,A) \xrightarrow{\varphi^1} H^1(G,(B,\mu)) \xrightarrow{\psi^1} H^1(G,(C,\lambda)) \xrightarrow{\delta^1} H^2(G,A) \xrightarrow{\varphi^2}$$
$$\xrightarrow{\varphi^2} H^2(G,(B,\mu)) \xrightarrow{\psi^2} H^2(G,(C,\lambda)),$$

where φ^0, ψ^0, δ^0, φ^1 are homomorphisms. If $H^0(G,R)$ is a normal subgroup of R, then ψ^1 and δ^1 are also homomorphisms. If in addition $H^0(G,R)$ acts trivially on $H^2(G,A)$, then δ^1 is a crossed homomorphism under the action of $H^1(G,(C,\lambda))$ on $H^2(G,A)$ induced by the action of R on A. Moreover, if either the action of $\mathrm{Der}(F_0,(C,\lambda))$ *on $H^3(G,A)$ is trivial (in particular if R acts trivially on A) or* $\mathrm{Der}(F_0,(C,\lambda)) = \mathrm{IDer}(F_0,(C,\lambda))$ *and $H^0(G,R)$ acts trivially on $H^2(G,\ker\lambda)$ then the sequence*

$$H^2(G,(B,\mu)) \xrightarrow{\psi^2} H^2(G,(C,\lambda)) \xrightarrow{\delta^2} H^3(G,A)$$

is exact.

Proof. The exactness of the sequence

$$1 \longrightarrow H^0(G,A) \xrightarrow{\varphi^0} H^0(G,B) \xrightarrow{\psi^0} H^0(G,C) \xrightarrow{\delta^0} H^1(G,A)$$

is known [4].

If $c \in H^0(G,C)$ then $\delta^0(c) = [\alpha]$ with $\alpha(x) = \varphi^{-1}(b^{-1}\,{}^x b)$, $x \in G$ and $\psi(b) = c$. It follows that $(\alpha_0, 1) \sim (\varphi\alpha, 1)$ where α_0 is the trivial map, since

$$\varphi\alpha(x) = b^{-1}\alpha_0\,{}^x b, \quad x \in G,$$

and $\mu(b) \in H^0(G,R)$ because $\mu(b) = \lambda\psi(b) = \lambda(c)$ and ${}^x\lambda(c) = \lambda({}^x c) = \lambda(c)$, $x \in G$. Therefore $\mathrm{Im}\,\delta^0 \subset \ker\varphi^1$.

Let $[\alpha] \in H^1(G,A)$ such that $(\alpha_0,1) \sim (\varphi\alpha,1)$. Then $\varphi\alpha(x) = b^{-1}\,{}^x b$, $x \in G$ and $\mu(b) \in H^0(G,R)$. We have $\psi(b^{-1}\,{}^x b) = \psi\varphi\alpha(x) = 1$. Thus $\psi(b) = \psi({}^x b) = {}^x\psi(b)$, whence $\psi(b) \in H^0(G,C)$. It is clear that $\delta^0(\psi(b)) = [\alpha]$. Therefore $\ker\varphi^1 \subset \mathrm{Im}\,\delta^0$.

Clearly, $\psi^1\varphi^1$ is the trivial map. Let $[(\alpha,r)] \in H^1(G,(B,\mu))$ such that $(\alpha_0,1) \sim (\varphi\alpha,1)$. Then $\psi\alpha(x) = c^{-1}\,{}^x c$, $c \in C$, and $r = \lambda(c)^{-1}t$, $t \in H^0(G,R)$. Let $\psi(b) = c$. Then $\mu(b) = \lambda(c)$ and $r = \mu(b)^{-1}t$. Take $\widetilde{\alpha}(x) =$

$b\,\alpha(x)\,{}^{x}b^{-1}$, $x \in G$. Since $\psi\widetilde{\alpha}(x) = 1$, $x \in G$, one has $\varphi^{-1}\widetilde{\alpha} : G \longrightarrow A$ and $(\alpha, r) \sim (\widetilde{\alpha}, 1)$. Therefore $\varphi^1([\varphi^{-1}\widetilde{\alpha}]) = [(\alpha, r)]$.

Let $[(\alpha, r)] \in H^1(G, (B, \mu))$. Then $\psi^1([(\alpha, r)]) = [(\psi\alpha, r)]$. Consider diagram (5) of [1] and take $\alpha\ \tau_G : F_G \longrightarrow B$. Then $\varphi\gamma = (\alpha\tau_G l_1)^{-1}\alpha\tau_G l_0$ and $\delta^1\psi^1([(\alpha, r)]) = [\gamma]$. But $\gamma = \alpha_0$ is the trivial map, since $\alpha\tau_G l_0 = \alpha\tau_G l_1$. Therefore Im $\psi^1 \subset \ker \delta^1$.

Let $[(\alpha, r)] \in H^1(G, (C, \lambda))$ such that $\delta^1([(\alpha, r)]) = 1$. If $\beta : F_G \longrightarrow B$ is a crossed homomorphism such that $\psi\beta = \alpha\tau_G$ then $\delta^1([(\alpha, r)]) = [\gamma]$, where $\varphi\gamma = (\beta l_1)^{-1}\beta l_0$. Thus there is a crossed homomorphism η: $F_G \longrightarrow A$ such that $\gamma = (\eta l_1)^{-1}\eta l_0$. Hence we have

$$(\beta l_1)^{-1}\beta l_0 = (\varphi\eta l_1)^{-1}\varphi\eta l_0, \quad (\varphi\eta^{-1}\,\beta)l_0 = (\varphi\eta^{-1}\,\beta)l_1.$$

Thus there is a crossed homomorphism $\overline{\alpha} : G \longrightarrow B$ such that $(\varphi\eta)^{-1}\beta = \overline{\alpha}\tau_G$. We have $\mu\beta(x) = \lambda\psi\beta(x) = \lambda\alpha\tau_G(x) = r^{\,\tau_G(x)}r^{-1}$, whence $(\beta, r) \in \mathrm{Der}(F_G, (B, \mu))$ and $(\overline{\alpha}, r) \in \mathrm{Der}(G, (B, \mu))$. Evidently, $\psi^1([(\overline{\alpha}, r)]) = [(\alpha, r)]$.

The rest of the proof repeats with minor modifications the proof of the exactness of the cohomology sequence for a coefficient short exact sequence of crossed G-modules (see Theorems 13 and 15 of [1]). □

It is clear that when (2) is an exact sequence of crossed G-modules, Theorem 10 implies Theorems 13 and 15 of [1].

By analogy with the case $n = 1$ we propose the following definition of the pointed set of cohomology $H^{n+1}(G, (A, \mu))$ of a group G with coefficients in a crossed G-R-bimodule (A, μ) (in particular, in crossed G-modules) for all $n \geq 1$.

Let (A, μ) be a crossed G-R-bimodule. Consider diagram (7) of [1] and the group $\mathrm{Der}(F_n, (A, \mu))$, $n \geq 1$, where (A, μ) is viewed as a crossed F_n-R-bimodule induced by $\tau_0\partial_0^1\partial_0^2 \cdots \partial_0^{n-1}\partial_0^n$ with $\partial_0^i = l_0^{i-1}\tau_i$, $i = 1, \ldots, n$. Denote by $\widetilde{Z^1}(F_n, (A, \mu))$ the subset of $\mathrm{Der}(F_n, (A, \mu))$ consisting of all elements of the form $(\alpha, 1)$ satisfying the condition

$$\prod_{j=0}^{n+1} (\alpha\partial_j^{n+1})^{\epsilon} = 1, \quad \epsilon = (-1)^i.$$

Note that since $\mu\alpha(x) = 1$, $x \in F_n$, we have $\alpha(F_n) \subset Z(A)$. In $\widetilde{Z^1}(F_n, (A, \mu))$ we introduce a relation $\sim$ as follows: $(\alpha', 1) \sim (\alpha, 1)$ if there is an element $(\beta, h) \in \mathrm{Der}(F_{n-1}, (A, \mu))$ such that

$$\alpha'(x) = {}^{h}\alpha(x) \prod_{i=0}^{n} (\beta\partial_i^n(x))^{\epsilon}, \quad x \in F_n, \tag{4}$$

where $\epsilon = (-1)^i$. Since the homomorphism $\tau_0 \partial^1_{i_n} \partial^2_{i_{n-1}} \cdots \partial^{n-1}_{i_2} \partial^n_{i_1}$ does not depend on the sequence $(i_1, i_2, \dots, i_{n-1}, i_n)$, we have

$$\beta\partial^n_j(x)(\beta\partial^n_l(x))^{-1} = (\beta\partial^n_l(x))^{-1}\beta\partial^n_j(x) \in \ker\mu, \quad x \in F_n,$$

for j even and l odd. It follows that the product $\prod_{i=0}^{n} (\beta\partial^n_i(x))^{\epsilon}$ in (4) does not depend on the order of the factors. Note that if n is even then $\beta(F_n) \subset \ker\mu \subset Z(A)$.

Similarly to the case $n = 1$ it can be shown that the relation $\sim$ is an equivalence, the quotient set $\widetilde{Z^1}(F_n, (A, \mu))/\sim$ is independent of diagram (7) of [1] (for instance, we can take the free cotriple resolution of the group G), and there is a surjective map

$$\vartheta'_n : H^{n+1}(G, \ker\mu) \longrightarrow \widetilde{Z^1}(F_n, (A, \mu))/\sim, \quad n \geq 1,$$

given by $[\alpha] \longmapsto [(\alpha, 1)]$ which is bijective if (A, μ) is a crossed G-G-bimodule and either μ is the trivial map or n is even.

Definition 11. *Let (A, μ) be a crossed G-R-bimodule. Define*

$$H^{n+1}(G, (A, \mu)) = \widetilde{Z^1}(F_n, (A, \mu))/\sim, \quad n \geq 1.$$

It is clear that for $n = 1$ we recover the second set of cohomology of G with coefficients in (A, μ).

Remark 1. Using the above-defined cohomology with coefficients in crossed bimodules it is possible to define a cohomology $H^n(G, A)$, $n \leq 2$, of a group G with coefficients in a G-group A.

Consider the quotient group $\overline{A} = A/Z(A)$ and define an action of $\overline{A}$ on A and an action of G on $\overline{A}$ as follows:

$$^{[a']}a = {}^{a'}a, \quad a, a' \in A,$$
$$^{g}[a] = [^{g}a], \quad g \in G, \quad a \in A.$$

Let $\mu_A : A \longrightarrow \overline{A}$ be the canonical homomorphism. Then (A, μ_A) is a crossed G-A-bimodule and we define

$$H^n(G, A) = H^n(G, (A, \mu_A)), \quad n \leq 2.$$

For $n = 1$ this cohomology differs from the pointed set of cohomology defined in [4]. If

$$1 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 1$$

is a central extension of G-groups then ψ induces an isomorphism $\vartheta : B/Z(B) \xrightarrow{\approx} C/\psi(Z(B))$ and one gets a short exact sequence of crossed G-$\overline{B}$-bimodules

$$1 \longrightarrow (A,1) \xrightarrow{\varphi} (B,\mu_B) \xrightarrow{\psi} (C,\overline{\mu}_C) \longrightarrow 1,$$

where $\overline{\mu}_C$ is the composite of the canonical map $\tau : C \longrightarrow C/\psi(Z(B))$ and the isomorphism ϑ^{-1}. Since $\overline{B}$ acts trivially on A, from Theorem 10 immediately follows the exact cohomology sequence

$$1 \longrightarrow H^0(G,A) \xrightarrow{\varphi^0} H^0(G,B) \xrightarrow{\psi^0} H^0(G,C) \xrightarrow{\delta^0} H^1(G,A) \xrightarrow{\varphi^1}$$
$$\xrightarrow{\varphi^1} H^1(G,B) \xrightarrow{\psi^1} H^1(G,(C,\overline{\mu}_C)) \xrightarrow{\delta^1} H^2(G,A) \xrightarrow{\varphi^2} H^2(G,B) \xrightarrow{\psi^2}$$
$$\xrightarrow{\psi^2} H^2(G,(C,\overline{\mu}_C)) \xrightarrow{\delta^2} H^3(G,A).$$

Remark 2. As for the case $n = 2$ (see Remark of [1]) it is possible to give an alternative more non-abelian definition of the third cohomology $\overline{H}^3(G,(A,\mu))$ of G with coefficients in a crossed G-R-bimodule (A,μ). To this end consider the commutative diagram

$$\begin{array}{ccccc} M_G^1 & \overset{\varphi_0}{\underset{\varphi_1}{\rightrightarrows}} & Q_G & \xrightarrow{\eta_G} & M_G \\ & & q_1 \downarrow\downarrow q_0 & & l_1 \downarrow\downarrow l_0 \\ & & F^2(G) & \xrightarrow{F(\tau_G)} & F(G) \\ & & \downarrow \tau_{F(G)} & & \downarrow \tau_G \\ & & F(G) & \xrightarrow{\tau_G} & G \end{array},$$

where $F(G) = F_G$, $F^2(G) = F(F(G))$, τ_G and $\tau_{F(G)}$ are canonical surjections, η_G is induced by $F(\tau_G)$, and (M_G, l_0, l_1), (Q_G, q_0, q_1), $(M_G^1, \varphi_0, \varphi_1)$ are the simplicial kernels of τ_G, $\tau_{F(G)}$ and η_G, respectively. It is clear that (A,μ) is a crossed Q_G-G-bimodule induced by $\tau_G l_0 \eta_G$. Let $\widetilde{Der}(Q_G,(A,\mu))$ be the subgroup of $\mathrm{Der}(Q_G,(A,\mu))$ consisting of elements (β,g) such that $\beta(\Delta_Q) = 1$, where $\Delta_Q = \{(x,x), x \in F^2(G)\}$. Consider the set $\widetilde{Z^1}(M_G^1,(A,\mu))$ of all crossed homomorphisms $\alpha : M_G^1 \longrightarrow A$ with $\alpha(\Delta) = 1$ where $\Delta = \{(y,y), y \in Q_G\}$ and M_G^1 acts on A via $\tau_G l_0 \eta_G \varphi_0$. Introduce, in $\widetilde{Z^1}(M_G^1,(A,\mu))$, a relation of equivalence as follows:

$$\alpha' \sim \alpha \quad \text{if} \quad \exists(\beta,g) \in \widetilde{\mathrm{Der}}(Q_G,(A,\mu))$$

such that $\alpha'(x) = \beta\varphi_1(x)^{-1}\alpha(x)\ \beta\varphi_0(x)$, $x \in M_G^1$. Define $\overline{H^3}(G,(A,\mu)) = \widetilde{Z^1}(M_G^1,(A,\mu))/\sim$. Then $\overline{H^3}(G,(A,\mu))$ is a covariant functor from the category of crossed G-R-bimodules to the category of pointed sets. It can be proved that $\overline{H^3}(G,(A,1))$ is isomorphic to the classical third cohomology group $H^3(G,A)$ if A is a G-module.

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Author's address:
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 380093
Georgia