

**MODULAR PROPERTIES OF THETA-FUNCTIONS AND
REPRESENTATION OF NUMBERS BY POSITIVE
QUADRATIC FORMS**

T. VEPKHAVADZE

ABSTRACT. By means of the theory of modular forms the formulas for a number of representations of positive integers by two positive quaternary quadratic forms of steps 36 and 60 and by all positive diagonal quadratic forms with seven variables of step 8 are obtain.

Let $r(n; f)$ denote a number of representations of a positive integer n by a positive definite quadratic form f with a number of variables s . It is well known that, for the case $s > 4$, $r(n; f)$ can be represented as

$$r(n; f) = \rho(n; f) + \nu(n; f),$$

where $\rho(n; f)$ is a "singular series" and $\nu(n; f)$ is a Fourier coefficient of cusp form. This can be expressed in terms of the theory of modular forms by stating that

$$\vartheta(\tau; f) = E(\tau; f) + X(\tau),$$

where $E(\tau; f)$ is the Eisenstein series and $X(\tau)$ is a cusp form.

In his work [1] Malyshev formulated the following problem: to define the Eisenstein series and to develop a full theory of singular series for arbitrary $s \geq 2$. For $s \geq 3$, its solution follows from Ramanathan's results [2].

In the present paper we work out a full solution of this problem. Moreover, convenient formulas are obtained for calculating values of the function $\rho(n; f)$.

Thus, if the genus of the quadratic form f contains one class, then according to Siegel's theorem ([2], [3], [4]), $\vartheta(\tau; f) = E(\tau; f)$ and in that case the problem for obtaining "exact" formulas for $r(n; f)$ is solved completely. If the genus contains more than one class, then it is necessary to find a cusp form $X(\tau)$.

1991 *Mathematics Subject Classification*. 11E20.

Key words and phrases. Quadratic form, number of representations, modular form, generalized theta-function.

A large number of papers is devoted to finding such formulas. Cusp forms in these works are constructed in the form of linear combinations of products of simple theta-functions with characteristics or their derivatives (see, e.g., [5]), products of Jacobi theta-functions or their derivatives (see, e.g., [6]), and theta-functions with spherical polynomials (see, e.g., [7]). All these functions are certain particular cases of linear combinations of the so-called generalized theta-functions with characteristics defined below by the formula (1). In the present paper, using modular properties of these functions, we have obtained by the unique method the exact formulas for a number of representations of numbers by positive quadratic forms both with an even (forms of such a kind were considered earlier) and an odd number of variables. Moreover, it is shown that using [8], one can reduce cumbersome calculations for obtaining formulas for a number of representations of numbers by the quadratic forms considered earlier and to obtain new formulas.

§ 1. Let $f = \frac{1}{2}x'Ax$ be a positive definite quadratic form, let A be an integral matrix with even diagonal elements, and the vector column $x \in \mathbb{Z}^s$, $s \in \mathbb{N}$, $s \geq 2$. We call $u \in \mathbb{Z}^s$ a special vector with respect to the form f if $Au \equiv 0 \pmod{N}$, where N is a step of the form f . Moreover, let $P_\nu = P_\nu(x)$ be a spherical function of the ν th order (ν is a positive integer) corresponding to the form f (see [9], p. 454). Then the generalized theta-function with characteristics we define as follows:

$$\vartheta_{gh}(\tau; p_\nu, f) = \sum_{x \equiv g \pmod{N}} (-1)^{\frac{h'A(x-g)}{N^2}} p_\nu(x) e^{\frac{\pi i \tau x'Ax}{N^2}}; \quad (1)$$

here and below, g and h are the special vectors with respect to the form f .

In the sequel, use will be made of the following lemmas (see Lemmas 1 and 4 in [8]).

Lemma 1. *Let k be an arbitrary integral vector and l be a special vector with respect to the form f . Then the following equalities hold:*

$$\begin{aligned} \vartheta_{g+Nk,h}(\tau; p_\nu, f) &= (-1)^{\frac{h'Ak}{N}} \vartheta_{gh}(\tau; p_\nu, f), \\ \vartheta_{g,h+2l}(\tau; p_\nu, f) &= \vartheta_{gh}(\tau; p_\nu, f). \end{aligned}$$

Lemma 2. *Let $F(\tau)$ be an entire modular form of the type $(-r, N, v(L))$ and let there exist an integer l for which $(v(L))^l = 1$. Then the function $F(\tau)$ is identically equal to zero if in its expansion in powers of $Q = e^{2\pi i \tau}$ the coefficients c_n equal 0 for $n \leq (r/12)N \prod_{p|N} (1 + p^{-1})$.*

In the main theorem below we formulate modular properties of linear combinations of functions (1).

Theorem 1. Let $f_1 = f_1(x) = \frac{1}{2}x'A_1x, \dots, f_j = f_j(x) = \frac{1}{2}x'A_jx$ be positive definite quadratic forms with a number of variables s , let $P_\nu^{(k)} = P_\nu^{(k)}(x)$ be the corresponding spherical functions of order ν , Δ_k be the determinant of the matrix A_k , N_k the step of the form f_k ($k = 1, \dots, j$), Δ the determinant of the matrix A of some positive definite quadratic form $\frac{1}{2}x'Ax$ with a number of variables $s + 2\nu$ and the step N .

Next, let $g^{(k)}$ and $h^{(k)}$ be special vectors with respect to f_k ; moreover, given $2 \nmid \frac{N}{N_k}$, let h_k be the vector with even components ($k = 1, \dots, j$);

$$L = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(N);$$

$$\begin{aligned} v(L) &= (i^{\frac{1}{2}}\eta(\gamma)(\text{sgn } \delta - 1))^{s+2\nu} (i^{\frac{|\delta|-1}{2}})^{s+2\nu} \left(\frac{2(\text{sgn } \delta)\beta\Delta}{|\delta|} \right) \text{ for } 2 \nmid s, \\ &= (\text{sgn } \delta)^{\frac{s}{2}+\nu} \left(\frac{(-1)^{\frac{s}{2}+\nu}\Delta}{|\delta|} \right) \text{ for } 2 \mid s; \end{aligned} \tag{2}$$

$$\begin{aligned} \eta(\gamma) &= 1 \text{ for } \gamma \geq 0, \\ &= -1 \text{ for } \gamma < 0; \end{aligned}$$

$\left(\frac{(-1)^{\frac{s}{2}+\nu}\Delta}{|\Delta|} \right)$ is the Kronecker symbol, $\left(\frac{2(\text{sgn } \delta)\beta\Delta}{|\delta|} \right)$ is the Jacobi symbol. Then the function

$$X(\tau) = \sum_{k=1}^j B_k \vartheta_{g^{(k)}h^{(k)}}(\tau; P_\nu^{(k)}, f_k) \tag{3}$$

for arbitrary complex numbers B_k is an entire modular form of the type $(-\frac{s}{2} + \nu, N, v(L))$ if and only if the conditions

$$N_k \mid N, \quad N_k^2 \mid f_k(g^{(k)}), \quad 4N_k \mid \frac{N}{N_k} f_k(h^{(k)})$$

are fulfilled, and for all α and δ such that $\alpha\delta \equiv 1 \pmod{N}$ we have

$$\begin{aligned} \sum_{k=1}^j B_k \vartheta_{\alpha g^{(k)}, h^{(k)}}(\tau; p_\nu^{(k)}, f_k) (\text{sgn } \delta)^\nu \left(\frac{(-1)^{[\frac{1}{2}]\Delta_k}}{|\delta|} \right) &= \\ = \left(\frac{(-1)^{[\frac{s+2\nu}{2}]\Delta}}{|\delta|} \right) \sum_{k=1}^j B_k \vartheta_{g^{(k)}h^{(k)}}(\tau; p_\nu^{(k)}, f_k). \end{aligned}$$

This theorem has been proved in [8] for even h_k . It can easily be adjusted to the case $2 \mid \frac{N}{N_k} h_k$. From this theorem we obtain the following two theorems which are analogues of Theorems 4 and 2 from [8].

Theorem 2. *If all the conditions of Theorem 1 are fulfilled and either $\nu > 0$, or $\nu = 0$ and all the $g^{(k)}$ vectors are nonzero, then the function (3) is a cusp form of the type $(-\frac{1}{2} + \nu, N, \nu(L))$.*

Theorem 3. *Let f be an integral positive quadratic form with a number of variables s and let Δ be a determinant of the form f . Then the function $\vartheta(\tau; f)$ defined by the formula*

$$\vartheta(\tau; f) = 1 + \sum_{n=1}^{\infty} r(n; f) e^{2\pi i \tau f} \quad (\text{Im } \tau > 0) \quad (4)$$

is the entire modular form of the type $(-\frac{s}{2}, N, \nu(L))$, where $\nu(L)$ are defined by the formulas (2) for $\nu = 0$.

From the results of [2], [3], [4] and [10] we obtain

Theorem 4. *Let f be a positive quadratic form with a number of variables s and let Δ be its determinant. Then the function $E(\tau, z; f)$, determined for $\text{Re } z \geq 2 - \frac{s}{2}$ and $\text{Im } \tau > 0$ by the formula*

$$E(\tau, z; f) = 1 + \frac{e^{\frac{\pi i s}{4}}}{2^{\frac{s}{2}} \Delta^{\frac{s}{2}}} \sum_{q=1}^{\infty} \sum_{\substack{H=-\infty \\ (H,q)=1}}^{\infty} \frac{S(fh, q)}{q^{\frac{s}{2}} (q\tau - H)^{\frac{s}{2}} |q\tau - H|^z},$$

where $S(fh, q)$ is the Gaussian sum, can be continued analytically into the neighborhood of the point $z = 0$. Further, having defined the Eisenstein series $E(\tau; f)$ by the formulas

$$\begin{aligned} E(\tau; f) &= \frac{1}{2} E(\tau, z; f) \Big|_{z=0} \quad \text{for } s = 2 \\ &= E(\tau, z; f) \Big|_{z=0} \quad \text{for } s > 2, \end{aligned}$$

we have

$$\begin{aligned} E(\tau; f) &= 1 + \frac{1}{2} \sum_{n=1}^{\infty} \rho(n; f) e^{2\pi i \tau n} \quad \text{for } s = 2, \\ &= 1 + \sum_{n=1}^{\infty} \rho(n; f) e^{2\pi i \tau n} \quad \text{for } s > 2; \end{aligned} \quad (5)$$

here $\rho(n; f)$ is a singular series which is calculated as follows:

(1) If $2 \mid s$, $v = \prod_{\substack{p \mid n \\ p \nmid 2\Delta}} p^w$, $\Delta = r^2\omega$ (ω is a square-free number), then

$$\begin{aligned} \rho(n; f) &= \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})\Delta^{1/2}} n^{\frac{s}{2}-1} \chi_2 \prod_{\substack{p \mid \Delta \\ p > 2}} \chi_p \prod_{\substack{p \mid r \\ p > 2}} \left(1 - \left(\frac{(-1)^{\frac{s}{2}}\omega}{p}\right) p^{-\frac{s}{2}}\right)^{-1} \times \\ &\times \mathcal{L}^{-1}\left(\frac{s}{2}; (-1)^{\frac{1}{2}}\omega\right) \sum_{k \mid v} \left(\frac{(-1)^{\frac{s}{2}}\Delta}{k}\right) k^{1-\frac{s}{2}}. \end{aligned}$$

(2) If $2 \nmid s$, $\Delta n = 2^{\alpha+\gamma}v_1v_2 = r^2\omega$, $2^\alpha \parallel n$, $2^\gamma \parallel \Delta$, $p^l \parallel \Delta$, $p^w \parallel n$ ($p > 2$), $v_1 = \prod_{\substack{p \mid n \\ p \nmid 2\Delta}} p^w = r_1^2\omega_1$, $v_2 = \prod_{\substack{p \mid \Delta n \\ p \nmid \Delta, p > 2}} p^{w+l} = r_2^2\omega_2$ ($\omega, \omega_1, \omega_2$ are square-free numbers), then

$$\begin{aligned} \rho(n; f) &= \frac{(s-1)! r_1^{2-s} n^{\frac{s}{2}-1}}{\Gamma(\frac{s}{2})2^{s-2}\pi^{\frac{s}{2}-1}|B_{s-1}|\Delta^{\frac{1}{2}}} \chi_2 \prod_{\substack{p \mid \Delta \\ p > 2}} \chi_p \times \\ &\times \prod_{p \mid 2\Delta} (1-p^{1-s})^{-1} \mathcal{L}\left(\frac{s-1}{2}; (-1)^{\frac{s-1}{2}}\omega\right) \prod_{\substack{p \mid r_2 \\ p > 2}} \left(1 - \left(\frac{(-1)^{\frac{s-1}{2}}\omega}{p}\right) p^{\frac{1-s}{2}}\right) \times \\ &\times \sum_{k \mid r_1} k^{s-2} \prod_{p \mid k} \left(1 - \left(\frac{(-1)^{\frac{s-1}{2}}\omega}{p}\right) p^{\frac{1-s}{2}}\right). \end{aligned}$$

The values of χ_2 and χ_p are given in [11] (formulas (9)–(13), p. 66);

$$\mathcal{L}(k; (-1)^k\omega) = \sum_{\substack{l=1 \\ 2 \nmid l}}^{\infty} \left(\frac{(-1)^k\omega}{l}\right) \frac{1}{l^k} = \prod_{\substack{p \\ p > 2}} \left(1 - \left(\frac{(-1)^k\omega}{p}\right) p^{-k}\right)^{-1};$$

B_{s-1} are Bernoulli's numbers.

§ 2. In this section we will obtain exact formulas for a number of representations of numbers by quaternary quadratic forms:

$$x_1^2 + x_2^2 + x_3^2 + 15x_4^2 \quad \text{and} \quad 2x_1^2 + 2x_1x_2 + 5x_2^2 + 2x_3^2 + 2x_3x_4 + 5x_4^2.$$

The first form has been considered by Lomadze in [5] who, for the construction of cusp form $X(\tau)$, used products of simple theta-functions with characteristics and of their derivatives (some particular cases of the function (1)) and therefore he had to use modular forms of step 240 instead of 60.

Theorem 5. *Let*

$$\begin{aligned} f &= x_1^2 + x_2^2 + x_3^2 + 15x_4^2, & f_1 &= 3x_1^2 + 15x_2^2, \\ f_2 &= 4x_1^2 + 2x_1x_2 + 4x_2^2, & g^{(1)} &= \begin{pmatrix} 20 \\ 20 \end{pmatrix}, & h^{(1)} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ g^{(2)} &= \begin{pmatrix} 15 \\ 0 \end{pmatrix}, & h^{(2)} &= \begin{pmatrix} 0 \\ 15 \end{pmatrix}, & p_1^{(1)} &= x_2, & p_1^{(2)} &= x_1 + x_2. \end{aligned}$$

Then the equality

$$\vartheta(\tau; f) = E(\tau; f) + \frac{4}{15}\vartheta_{g^{(1)}h^{(1)}}(\tau; p_1^{(1)}, f_1) + \frac{1}{10}\vartheta_{g^{(2)}h^{(2)}}(\tau; p_1^{(2)}, f_2) \quad (6)$$

holds, where the functions

$$\vartheta(\tau; f), \quad \vartheta_{g^{(1)}h^{(1)}}(\tau; p_1^{(1)}, f_1), \quad \vartheta_{g^{(2)}h^{(2)}}(\tau; p_1^{(2)}, f_2)$$

are defined by formulas (4) and (1), while the function $E(\tau; f)$ by formula (5).

Proof. By Theorem 3, the function $\vartheta(\tau; f)$ belongs to the space of entire modular forms of the type $(-2, 60, v(L))$, where $v(L)$ is the corresponding multiplier system. Then, according to Siegel's theorem (see [2]), $E(\tau; f)$ also belongs to this space. Using Lemma 1, we can check that the function

$$X(\tau) = \frac{4}{15}\vartheta_{g^{(1)}h^{(1)}}(\tau; p_1^{(1)}, f_1) + \frac{1}{10}\vartheta_{g^{(2)}h^{(2)}}(\tau; p_1^{(2)}, f_2)$$

satisfies all the conditions of Theorem 1.

Indeed, f_1 is the binary form of step 60 and f_2 is the binary form of step 30 ($N_1 = 60$, $N_2 = 30$), $g^{(1)} = \begin{pmatrix} 20 \\ 20 \end{pmatrix}$, and $h^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ are special vectors with respect to the form $f_1 = 3x_1^2 + 15x_2^2$, and $g^{(2)} = \begin{pmatrix} 15 \\ 0 \end{pmatrix}$ and $h^{(2)} = \begin{pmatrix} 0 \\ 15 \end{pmatrix}$ are special vectors with respect to the form $f_2 = 4x_1^2 + 2x_1x_2 + 4x_2^2$. $2 \mid h^{(1)}$, $2 \mid \frac{N}{N_2}h^{(2)}$, since $N_1 = N = 60$, $N_2 = 30$; but $60 \mid N$, $30 \mid N$, $60^2 \mid f_1(g^{(1)})$, $30^2 \mid f_2(g^{(2)})$, $240 \mid f_1(h^{(1)})$, $120 \mid 2f_2(h^{(2)})$.

If $\alpha\delta \equiv 1 \pmod{60}$, then either

$$\alpha \equiv \delta \equiv 1 \pmod{3} \quad \text{or} \quad \alpha \equiv \delta \equiv -1 \pmod{3}.$$

Because of Lemma 1,

$$\vartheta_{\alpha g^{(1)}, h^{(1)}}(\tau; p_1^{(1)}, f_1) = \begin{cases} \vartheta_{g^{(1)}h^{(1)}}(\tau; p_1^{(1)}, f_1) & \text{for } \alpha \equiv \delta \equiv 1 \pmod{3}, \\ \vartheta_{-g^{(1)}h^{(1)}}(\tau; p_1^{(1)}, f_1) & \text{for } \alpha \equiv \delta \equiv -1 \pmod{3}. \end{cases}$$

Due to (1) we have

$$\begin{aligned} \vartheta_{g^{(1)h^{(1)}}}(\tau; p_1, f_1) &= \sum_{x \equiv g^{(1)} \pmod{60}} x_2 e^{2\pi i \tau \frac{3x_1^2 + 15x_2^2}{60^2}} \\ &= - \sum_{x \equiv -g^{(1)} \pmod{60}} x_2 e^{2\pi i \tau \frac{3x_1^2 + 15x_2^2}{60^2}} = -\vartheta_{-g^{(1)h^{(1)}}}(\tau; p_1^{(1)}, f_1). \end{aligned}$$

Thus

$$\vartheta_{\alpha g^{(1)h^{(1)}}}(\tau; p_1^{(1)}, f_1) = \begin{cases} \vartheta_{g^{(1)h^{(1)}}}(\tau; p_1^{(1)}, f_1) & \text{for } \alpha \equiv \delta \equiv 1 \pmod{3}, \\ \vartheta_{-g^{(1)h^{(1)}}}(\tau; p_1^{(1)}, f_1) & \text{for } \alpha \equiv \delta \equiv -1 \pmod{3}. \end{cases} \quad (7)$$

We have

$$\operatorname{sgn} \delta \left(\frac{-\Delta_1}{|\delta|} \right) = \operatorname{sgn} \delta \left(\frac{-1}{|\delta|} \right) \left(\frac{5}{|\delta|} \right), \quad \left(\frac{(-1)^2 \Delta}{|\delta|} \right) = \left(\frac{-1}{|\delta|} \right) \left(\frac{|\delta|}{3} \right) \left(\frac{5}{|\delta|} \right). \quad (8)$$

Furthermore, we have

$$\left(\frac{|\delta|}{3} \right) = \begin{cases} \operatorname{sgn} \delta & \text{for } \delta \equiv 1 \pmod{3}, \\ -\operatorname{sgn} \delta & \text{for } \delta \equiv -1 \pmod{3}. \end{cases}$$

We can easily verify that formulas (7) and (8) imply

$$\vartheta_{\alpha g^{(1)h^{(1)}}}(\tau; p_1^{(1)}, f_1) \operatorname{sgn} \delta \left(\frac{-\Delta_1}{|\delta|} \right) = \left(\frac{\Delta}{|\delta|} \right) \vartheta_{g^{(1)h^{(1)}}}(\tau; p_1^{(1)}, f_1^{(1)}). \quad (9)$$

Analogously, we get

$$\vartheta_{\alpha g^{(2)h^{(2)}}}(\tau; p_1^{(2)}, f_2) \operatorname{sgn} \delta \left(\frac{-\Delta_2}{|\delta|} \right) = \left(\frac{\Delta}{|\delta|} \right) \vartheta_{\alpha g^{(2)h^{(2)}}}(\tau; p_1^{(2)}, f_2). \quad (10)$$

Consequently, according to (9) and (10), the function

$$X(\tau) = \frac{4}{15} \vartheta_{g^{(1)h^{(1)}}}(\tau; p_1^{(1)}, f_1) + \frac{1}{10} \vartheta_{g^{(2)h^{(2)}}}(\tau; p_1^{(2)}, f_2) \quad (11)$$

satisfies the conditions of Theorem 1 and, due to Theorem 2, belongs to the space of cusp forms of the type $(-2, 60, v(L))$.

Thus, owing to Lemma 2, the function

$$\Psi(\tau) = \vartheta(\tau; f) - E(\tau; f) - X(\tau),$$

where $X(\tau)$ is defined by (11), will be identically zero if all coefficients for Q^n ($n \leq 24$) are zero in its expansion in powers of $Q = e^{2\pi i \tau}$.

Next, let $n = 2^\alpha 3^{\beta_1} 5^{\beta_2} u$, $(u, 30) = 1$. Then by Theorem 4,

$$E(\tau; f) = 1 + \sum_{n=1}^{\infty} \rho(n; f) Q^n \quad (Q = e^{2\pi i \tau}), \quad (12)$$

where

$$\begin{aligned} \rho(n; f) &= \frac{1}{12} \left(2^{\alpha+1} + (-1)^{\beta_1} \left(\frac{-1}{u} \right) \right) \left(3^{\beta_1+1} - (-1)^{\alpha+\beta_2} \left(\frac{u}{3} \right) \right) \times \\ &\times \left(5^{\beta_2+1} + (-1)^{\alpha+\beta_1+\beta_2} \left(\frac{u}{5} \right) \right) \sum_{d_1 d_2 = u} \left(\frac{15}{d_1} \right) d_2. \end{aligned} \tag{13}$$

Having calculated the values $\rho(n; f)$ for all $n \leq 24$ by formula (13), we obtain because of (12):

$$\begin{aligned} E(\tau; f) &= 1 + 3Q + \frac{20}{3}Q^2 + \frac{8}{3}Q^3 + 9Q^4 + 24Q^5 + 15Q^6 + \\ &+ \frac{16}{3}Q^7 + \frac{68}{3}Q^8 + 39Q^9 + \frac{65}{3}Q^{10} + 24Q^{11} + \frac{56}{3}Q^{12} + 24Q^{13} + \\ &+ 48Q^{14} + \frac{65}{3}Q^{15} + 33Q^{16} + 72Q^{17} + \frac{140}{3}Q^{18} + 18Q^{19} + \\ &+ 72Q^{20} + 96Q^{21} + 24Q^{22} + \frac{88}{3}Q^{23} + 75Q^{24} + \dots \end{aligned} \tag{14}$$

Formulas (4) and (1) yield

$$\begin{aligned} \vartheta(\tau; f) &= 1 + 6Q + 12Q^2 + 8Q^3 + 6Q^4 + 24Q^5 + 24Q^6 + \\ &+ 12Q^8 + 30Q^9 + 24Q^{10} + 24Q^{11} + 8Q^{12} + 24Q^{13} + \\ &+ 480Q^{14} + 2Q^{15} + 18Q^{16} + 72Q^{17} + 52Q^{18} + 36Q^{19} + \\ &+ 72Q^{20} + 96Q^{21} + 24Q^{22} + 24Q^{23} + 84Q^{24} + \dots \end{aligned} \tag{15}$$

$$\begin{aligned} \frac{4}{15} \vartheta_{g^{(1)}h^{(1)}}(\tau; p_1^{(1)}, f_1) &= \frac{16}{3}(Q^2 + Q^3 - Q^7 - 2Q^8 + Q^{10} - \\ &- 2Q^{12} - 2Q^{15} + Q^{18} - Q^{23} + 4Q^{27} + \dots), \end{aligned} \tag{16}$$

$$\begin{aligned} \frac{1}{10} \vartheta_{g^{(2)}h^{(2)}}(\tau; p_1^{(2)}, f_2) &= \frac{3}{2}(2Q - 2Q^4 - 6Q^6 - 6Q^9 - \\ &- 2Q^{10} - 6Q^{15} - 10Q^{16} + 12Q^{19} + 6Q^{24} + \dots). \end{aligned} \tag{17}$$

Taking into account (14)–(17), we can easily verify that all coefficients for $Q^n (n \leq 24)$ in the expansion of the function $\psi(\tau)$ in powers of Q are zero. Thus identity (6) is proved. \square

Theorem 6. Let $f = x_1^2 + x_2^2 + x_3^2 + 15x_4^2$, $n = 2^\alpha 3^{\beta_1} 5^{\beta_2} u$, $(u, 30) = 1$. Then

$$\begin{aligned} r(n; f) &= \frac{1}{6} \left(3^{\beta_1+1} - (-1)^{\beta_2} \left(\frac{u}{3} \right) \right) \times \\ &\times \left(5^{\beta_2+1} + (-1)^{\beta_1+\beta_2} \left(\frac{u}{5} \right) \right) \sum_{d_1 d_2 = u} \left(\frac{15}{d_1} \right) d_2 + \end{aligned}$$

$$\begin{aligned}
 & + \frac{3}{2} \sum_{\substack{n=x_1^2+x_1x_2+4x_2^2 \\ 2|x_1}} (-1)^{\frac{x_1-1}{2}} (x_1 + 2x_2) \text{ for } n \equiv 1 \pmod{4}, \\
 & = \frac{1}{12} \left(2^{\alpha+1} + (-1)^{\beta_1} \left(\frac{-1}{u} \right) \right) \left(3^{\beta_1+1} - (-1)^{\alpha+\beta_2} \left(\frac{u}{3} \right) \right) \times \\
 & \quad \times \left(5^{\beta_2+1} + (-1)^{\alpha+\beta_1+\beta_2} \left(\frac{u}{5} \right) \right) \sum_{d_1 d_2 = u} \left(\frac{15}{d_1} \right) d_2 + \\
 & + \frac{3}{2} \sum_{\substack{n=x_1^2+x_1x_2+4x_2^2 \\ 2|x_1}} (-1)^{\frac{x_1-1}{2}} (x_1 + 2x_2) + \\
 & + \frac{16}{3} \sum_{\substack{3n=x_1^2+5x_2^2 \\ x_1 \equiv x_2 \equiv 1 \pmod{3}}} x_2 \text{ otherwise.}
 \end{aligned}$$

Proof. Equating coefficients of the same powers Q in both parts of identity (6), we get

$$r(n; f) = \rho(n; f) + \frac{16}{3} \nu_1(n) + \frac{3}{2} \nu_2(n), \tag{18}$$

where $\nu_1(n), \nu_2(n)$ denote respectively the coefficients for Q in the expansions of the functions

$$\frac{1}{20} \vartheta_{g^{(1)}h^{(1)}}(\tau; p_1^{(1)}, f_1), \quad \frac{1}{15} \vartheta_{g^{(2)}h^{(2)}}(\tau; p_1^{(2)}, f_2)$$

in powers of Q .

From (1) we have

$$\frac{1}{20} \vartheta_{g^{(1)}h^{(1)}}(\tau; p_1^{(1)}, f_1) = \sum_{x_1, x_2 = -\infty} (3x_2 + 1) e^{\frac{2\pi i \tau ((3x_1+1)^2 + 5(3x_2+1)^2)}{3}},$$

i.e.,

$$\nu_1(n) = \sum_{\substack{3n=x_1^2+5x_2^2 \\ x_1 \equiv x_2 \equiv 1 \pmod{3}}} x_2. \tag{19}$$

It follows from (4) that

$$\begin{aligned}
 & \frac{1}{15} \vartheta_{g^{(2)}h^{(2)}}(\tau; p_1^{(2)}, f_2) = \\
 & = \sum_{x_1, x_2 = -\infty} (-1)^{x_1} (2x_1 + 1 + 2x_2) e^{\frac{2\pi i \tau [(2x_1+1)^2 + (2x_1+1)x_2 + 4x_2^2]}{1}},
 \end{aligned}$$

i.e.,

$$\nu_2(n) = \sum_{\substack{n=x_1^2+x_1x_2+4x_2^2 \\ 2|x_1}} (-1)^{\frac{x_1-1}{2}}(x_1 + 2x_2). \tag{20}$$

From formulas (18), (13), (19), (20) we obtain the desired expression for $r(n; f)$. \square

Theorem 7. *Let $f = 2x_1^2 + 2x_1x_2 + 5x_2^2 + 2x_3^2 + 2x_3x_4 + 5x_4^2$, $f_1 = 3x_1^2 + 9x_2^2$, $g = \begin{pmatrix} 18 \\ 6 \end{pmatrix}$, $h = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $p_1 = x_2$. Then*

$$\vartheta(\tau; f) = E(\tau; f) - \frac{1}{9}\vartheta_{gh}(\tau; p_1, f_1),$$

where the functions $\vartheta(\tau; f)$, $E(\tau; f)$ and $\vartheta_{gh}(\tau; p_1, f_1)$ are defined respectively by the formulas (4), (5) and (1).

Proof. Let $n = 2^\alpha 3^\beta u$, $(u, 6) = 1$. Then by Theorem 4, $E(\tau; f) = 1 + \sum_{n=1}^\infty \rho(n; f)Q^n$ ($Q = e^{2\pi i\tau}$), where

$$\begin{aligned} \rho(n; f) &= 12(3^{\beta-1} - 1) \sum_{\mu|u} \mu \text{ for } \alpha > 0, \beta > 0, \\ &= 4(3^{\beta-1} - 1) \sum_{\mu|u} \mu \text{ for } \alpha = 0, \beta > 0, \\ &= 4 \sum_{\mu|u} \mu \text{ for } \alpha > 0, \beta > 0, \\ &= \frac{4}{3} \sum_{\mu|u} \mu \text{ for } (n, 6) = 1. \end{aligned} \tag{21}$$

Formulas (5) and (21) imply

$$\begin{aligned} E(\tau; f) &= 1 + \frac{4}{3}Q + \dots, \quad \vartheta(\tau; f) = 1 + 4Q^2 + \dots, \\ -\frac{1}{9}\vartheta_{9h}(\tau; p_1, f_1) &= -\frac{4}{3}Q + \dots \end{aligned} \tag{22}$$

By Theorem 3, the function $\vartheta(r; f)$ belongs to the space of entire modular forms of the type $(-2, 36, 1)$. Then by Siegel's theorem (see [2]), $E(\tau; f)$ also belongs to this space. Using Lemma 1, we can easily verify that the function $\vartheta_{gh}(\tau; p_1, f_1)$ satisfies all the conditions of Theorem 1. Therefore by Theorem 2, it belongs to the space of cusp forms of the type $(-2, 36, 1)$. It is well known that this space is one-dimensional (see [12]). Therefore from (22) we obtain the above assertion. \square

From Theorem 7 immediately follows

Theorem 8. Let $n = 2^\alpha 3^\beta u$, $(u, 6) = 1$, $f = 2x_1^2 + 2x_1x_2 + 5x_2^2 + 2x_3^2 + 2x_3x_4 + 5x_4^2$. Then

$$\begin{aligned} r(n; f) &= 12(3^{\beta-1} - 1) \sum_{\mu|u} \mu \text{ for } \alpha > 0, \beta > 0, \\ &= 4(3^{\beta-1} - 1) \sum_{\mu|u} \mu \text{ for } \alpha = 0, \beta > 0, \\ &= 4 \sum_{\mu|u} \mu \text{ for } \alpha > 0, \beta = 0, \\ &= \frac{4}{3} \sum_{\mu|u} \mu - \frac{2}{3} \sum_{\substack{4n=3x_1^2+x_2^2 \\ x_1 \equiv 1 \pmod{2} \\ x_2 \equiv 1 \pmod{6}}} x_2 \text{ for } (n, 6) = 1. \end{aligned}$$

Remark to Theorem 8. Let

$$\nu(n) = \frac{1}{2} \sum_{\substack{4n=3x_1^2+x_2^2 \\ x_1 \equiv 1 \pmod{2} \\ x_2 \equiv 1 \pmod{6}}} x_2.$$

It can be easily shown that

$$\nu(n) = \frac{1}{2} \sum_{\substack{4n=3x_1^2+x_2^2 \\ x_1 \equiv 1 \pmod{2} \\ x_2 \equiv 1 \pmod{6}}} (x_1 + x_2).$$

Further, arguing as in [12] (p. 233), we can easily show that

- (1) $\nu(n_1 n_2) = \nu(n_1) \nu(n_2)$ if $(n_1, n_2) = 1$;
- (2) $\nu(p^\beta) = \sum_{0 \leq k < \frac{\beta}{2}} p^k \text{Tr}(\pi^{\beta-2k}(p)) + \delta\left(\frac{\beta}{2}\right) p^{\frac{\beta}{2}}$,

where $\pi(p)$ is the Frobenius endomorphism of a curve $y^2 = x^3 + 1$ reduced in modulo p , $\delta(r)$ is equal to one or to zero according to whether the number r is an integer or not. In particular, if $n = p$ is a prime number, then

$$\nu(p) = - \sum_{x=0}^{p-1} \left(\frac{x^3 + 1}{p} \right),$$

where $\left(\frac{x^3 + 1}{p} \right)$ is the Legendre symbol.

§ 3. In this section we obtain formulas for a number of representations of numbers by quadratic forms with seven variables

$$f = 2 \sum_{j=1}^s x_j^2 + \sum_{j=s+1}^7 x_j^2 \quad (1 \leq s \leq 6). \tag{23}$$

The cases $s = 0$ and $s = 1$ are considered earlier (see, e.g., [13], Vol. II, pp. 305, 309, 335 and Vol. III, p. 237). In these cases the corresponding forms belong to one-class genera. The case $s = 3$ was considered in [6].

Theorem 9. *Let f be of the kind (23), $f_1 = 2x_1^2 + 2x_2^2 + x_3^2$, $f_2 = x_1^2 + x_2^2 + 2x_3^2$, $g^{(1)} = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix}$, $h^{(1)} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$, $p_2 = x_1x_2$, $g^{(2)} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}$, $h^{(2)} = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix}$. Then the following equality holds,*

$$\vartheta(\tau; f) = E(\tau; f) + X(\tau; f), \tag{24}$$

where

$$\begin{aligned} X(\tau; f) &= \frac{1}{32} \vartheta_{g^{(1)}h^{(1)}}(\tau; p_2, f_1) \quad \text{for } s = 2, 4, \\ &= \frac{1}{32(s-1)} \vartheta_{g^{(2)}h^{(2)}}(\tau; p_2, f_2) \quad \text{for } s = 3, 5, \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{25}$$

Proof. By Theorem 3, the functions $\vartheta(\tau; f)$ belong to two different spaces of modular forms $(-\frac{7}{2}, 8, v(L))$, where $v(L)$ is a system of multipliers corresponding to the form f . This system is the same for all s with the same evenness. Then, according to Siegel's theorem (see [1]), the functions $E(\tau; f)$ also belong to appropriate spaces of modular forms. It can be easily verified that functions (25) satisfy all the conditions of Theorem 1 and, by Theorem 2, they belong to two different spaces of cusp forms depending on s .

Let $n = 2^\alpha u$ ($2 \nmid u$, $\alpha \geq 0$), $2^s n = r_s^2 \omega_s$, $u = r^2 \omega$ ($s = 1, 2, \dots, 6$) and let ω and ω_s be square-free numbers. Then by Theorem 4 we have

$$E(\tau; f) = 1 + \sum_{n=1}^{\infty} \rho(n; f) Q^n \quad (Q = e^{2\pi i \tau}), \tag{26}$$

where

$$\rho(n; f) = 2^{\frac{5\alpha}{2} + 9 - \frac{s}{2}} \pi^{-3} \omega^{\frac{5}{2}} \mathcal{L}(3; -\omega_s) \chi_2 \sum_{\mu|r} \mu^5 \prod_{p|\mu} \left(1 - \left(\frac{-\omega_s}{p}\right) p^{-3}\right).$$

By Lemma 27 from [14] we have

$$\begin{aligned}
 \mathcal{L}(3; -1) &= \frac{\pi^3}{32}, & \mathcal{L}(3; -2) &= \frac{3\pi^3}{64\sqrt{2}}; & (27) \\
 \mathcal{L}(3; -\omega) &= \frac{\pi^3}{16\omega^{\frac{5}{2}}} \left\{ \sum_{1 \leq h \leq \frac{\omega}{4}} (\omega^2 - 16h^2) \left(\frac{h}{\omega}\right) + 3\omega^2 \sum_{\frac{\omega}{4} < h < \frac{\omega}{2}} \left(\frac{h}{\omega}\right) + \right. \\
 &\quad \left. + 16 \sum_{\frac{\omega}{4} < h \leq \frac{\omega}{2}} h(h - \omega) \left(\frac{h}{\omega}\right) \right\}, \text{ if } \omega \equiv 1 \pmod{4}, \omega > 1, \\
 &= \frac{\pi^3}{2\omega^{\frac{5}{2}}} \sum_{1 \leq h \leq \frac{\omega}{2}} h(\omega - 2h) \left(\frac{h}{\omega}\right), \text{ if } \omega \equiv 3 \pmod{4}, \\
 &= \frac{\pi^3}{32\omega^{\frac{5}{2}}} \left\{ \sum_{1 \leq h \leq \frac{\omega}{16}} (3\omega^2 - 256h^2) \left(\frac{h}{\frac{1}{2}\omega}\right) + \right. \\
 &\quad \left. + 4\omega \sum_{\frac{\omega}{16} < h < \frac{3\omega}{16}} (\omega - 8h) \left(\frac{h}{\frac{1}{2}\omega}\right) + 13\omega^2 \sum_{\frac{3\omega}{16} < h \leq \frac{\omega}{4}} \left(\frac{h}{\frac{1}{2}\omega}\right) - \right. \\
 &\quad \left. - 128 \sum_{\frac{3\omega}{16} < h \leq \frac{\omega}{4}} h(\omega - 2h) \left(\frac{h}{\frac{1}{2}\omega}\right) \right\}, \text{ if } \omega \equiv 2 \pmod{8}, \omega > 2, \\
 &= \frac{\pi^3}{32\omega^{\frac{5}{2}}} \left\{ 32\omega \sum_{1 \leq h \leq \frac{\omega}{16}} h \left(\frac{h}{\frac{1}{2}\omega}\right) - \omega^2 \sum_{\frac{\omega}{16} < h < \frac{3\omega}{16}} \left(\frac{h}{\frac{1}{2}\omega}\right) + \right. \\
 &\quad \left. + 64 \sum_{\frac{\omega}{16} < h \leq \frac{3\omega}{16}} h(\omega - 4h) \left(\frac{h}{\frac{1}{2}\omega}\right) + 8\omega \sum_{\frac{3\omega}{16} < h \leq \frac{\omega}{4}} (\omega - 4h) \left(\frac{h}{\frac{1}{2}\omega}\right) \right\}, \\
 &\quad \text{if } \omega \equiv 6 \pmod{8}. & (28)
 \end{aligned}$$

Using formulas (33) of [9], after calculation of values χ_2 , we obtain

$$\begin{aligned}
 \chi_2 &= 1 \text{ for } 2 \nmid s, \alpha = 0, \text{ or for } 2|s, \alpha = 0, u \equiv 1 \pmod{4}, \\
 &\quad \text{or } 2|s, \alpha = 1, \\
 &= 1 + (-1)^{\frac{u^2-1}{6}} 2^{\frac{s}{2}-5}, \text{ for } 2|s, \alpha = 0, u \equiv 3 \pmod{4}, \\
 &= 1 + \frac{2^{\frac{s}{2}-3}(1 - 2^{-\frac{5\alpha}{2}} \cdot 63)}{31} \text{ for } 2|s, 2|\alpha, u \equiv 1 \pmod{4}, \\
 &= 1 + \frac{2^{\frac{s}{2}-3}(1 - 2^{-\frac{5\alpha}{2}} + (-1)^{\frac{u^2-1}{8}} 2^{-\frac{5\alpha}{2}-2} \cdot 31)}{31} \text{ for } 2|s, \\
 &\quad 2|\alpha, u \equiv 3 \pmod{4} \\
 &= 1 + \frac{2^{\frac{s}{2}-3}(1 - 2^{-\frac{5\alpha}{2} + \frac{5}{2}} \cdot 63)}{31} \text{ for } 2|s, 2 \nmid \alpha, \alpha > 1,
 \end{aligned}$$

$$\begin{aligned}
 &= 1 + \frac{2^{\frac{s}{2}-\frac{1}{2}}(1 - 2^{-\frac{5\alpha}{2}} \cdot 63)}{31} \text{ for } 2 \nmid s, 2|\alpha, \alpha > 0, \\
 &= 1 + \frac{2^{\frac{s}{2}-\frac{1}{2}}(1 - 2^{-\frac{5\alpha}{2}-\frac{5}{2}} \cdot 63)}{31} \text{ for } 2 \nmid s, 2 \nmid \alpha, u \equiv 1 \pmod{4}, \\
 &= 1 + \frac{2^{\frac{s}{2}-\frac{1}{2}}(1 - 2^{-\frac{5\alpha}{2}-\frac{5}{2}} + (-1)^{\frac{u^2-1}{8}} 2^{-\frac{5\alpha}{2}-\frac{9}{2}} \cdot 31)}{31} \\
 &\quad \text{for } 2 \nmid s, 2 \nmid \alpha, u \equiv 3 \pmod{4}. \tag{29}
 \end{aligned}$$

By (1) we have

$$\begin{aligned}
 \vartheta_{g^{(1)h^{(1)}}}(\tau; p_2, f_1) &= 16 \sum_{x_1, x_2, x_3 = -\infty}^{\infty} (-1)^{x_1+x_2+x_3} (2x_1+1)(2x_2+1) \times \\
 &\quad \times e^{\frac{2\pi i \tau [(2x_1+1)^2 + (2x_2+1)^2 + 2x_3^2]}{2}}, \tag{30}
 \end{aligned}$$

$$\begin{aligned}
 \vartheta_{g^{(2)h^{(2)}}}(\tau; p_2, f_2) &= 16 \sum_{x_1, x_2, x_3 = -\infty}^{\infty} (-1)^{x_1+x_2} (2x_1+1)(2x_2+1) \times \\
 &\quad \times e^{\frac{2\pi i \tau [(2x_1+1)^2 + (2x_2+1)^2 + 2(2x_3+1)^2]}{4}}. \tag{31}
 \end{aligned}$$

Taking then into account (26)–(31) and arguing as in the proof Theorem 5, we obtain the above assertion. \square

From Theorem 9 we have

Theorem 10. *Let $n = 2^\alpha u$ ($2 \nmid u, \alpha \geq 0$), $2^s n = r_1^2 \omega_s$, $u = r^2 \omega$, and let ω and ω_s be square-free numbers, $s = 1, 2, \dots, 6$,*

$$f = 2 \sum_{j=1}^s x_j^2 + \sum_{j=s+1}^7 x_j^2.$$

Then

$$\begin{aligned}
 r(n; f) &= 2^{\frac{5\alpha}{2}-\frac{s}{2}+9} \omega^{\frac{5}{2}} \pi^{-3} \mathcal{L}(3; -\omega_s) \chi_2 \times \\
 &\quad \times \sum_{\mu|r} \mu^5 \prod_{p|\mu} \left(1 - \left(\frac{-\omega_s}{p}\right) p^{-3}\right) + \nu(n; f),
 \end{aligned}$$

where

$$\begin{aligned} \nu(n; f) &= 0 \quad \text{for } s = 1, 6, \\ &= \frac{1}{2} \sum_{\substack{2n=x_1^2+x_2^2+2x_3^2 \\ 2|x_1, 2|x_2}} (-1)^{\frac{x_1x_2-1}{2}+x_3} x_1x_2 \quad \text{for } s = 2, 4, \\ &= \frac{1}{2s-2} \sum_{\substack{4n=x_1^2+x_2^2+2x_3^2 \\ 2|x_1, 2|x_2, 2|x_3}} (-1)^{\frac{x_1x_2-1}{2}} x_1x_2 \quad \text{for } s = 3, 5. \end{aligned}$$

The values $\mathcal{L}(3; -\omega_3)$ and χ_2 can be calculated by formulas (27)–(29).

REFERENCES

1. A. B. Malyshev, Formulas for a number of representations of numbers by positive quadratic forms. (Russian) *Actual problems of analytic number theory (Russian)* 119–137, Minsk, 1974.
2. K. G. Ramanathan, On the analytic theory of quadratic forms. *Acta Arithmetica* **21**(1972), 423–436.
3. T. V. Vepkhvadze, To one Siegel's formula. (Russian) *Acta Arithmetica* **40**(1982), 125–142.
4. T. V. Vepkhvadze, To the analytic theory of quadratic forms. (Russian) *Trudy Tbilis. Mat. Inst. Razmadze* **72**(1983), 12–31.
5. G. A. Lomadze, On a number of representations of numbers by quadratic forms with four variables. (Russian) *Trudy Tbilis. Mat. Inst. Razmadze* **40**(1971), 106–139.
6. H. Peterson, *Modulfunktionen und quadratische Formen*, Springer-Verlag, Berlin, 1982.
7. G. A. Lomadze, On cusp forms of a prime step and of basic type II. (Russian) *Trudy Tbilis. Mat. Inst. Razmadze* **57**(1977), 63–81.
8. T. V. Vepkhvadze, Generalized theta-functions with characteristics and representation of numbers by quadratic forms. (Russian) *Acta Arithmetica* **53**(1990), 433–451.
9. C. Shimura, On modular forms of half-integral weight. *Ann. of Math.* **97**(1973), 440–481.
10. R. I. Beridze, On summation of a singular Hardy-Littlewood series. (Russian) *Soobshch. Acad. Nauk Gruz. SSR* **38**(1965), No. 3, 529–534.
11. A. V. Malyshev, On the representation of integers by positive quadratic forms. (Russian) *Trudy Mat. Inst. Steklov.* **65**(1962), 1–211.

12. A. N. Andrianov, Representation of numbers by some quadratic forms in connection with the theory of elliptic curves. (Russian) *Izv. Acad. Nauk SSSR, Ser. Mat.* **29**(1965), No. 1, 227–238.

13. L. E. Dickson, History of the theory of numbers, Vols. II and III. *New York*, 1934.

14. H. Steefkerk, Over het aantal oplossingen der diophantische vergelijking $U = \sum_{i=1}^s Ax_i^2 + Bx_i + C_i$. *Dissertation, Amsterdam*, 1943.

(Received 22.01.1995)

Author's address:

Faculty of Physics

I. Javakhishvili Tbilisi State University

1, I. Chavchavadze Ave., Tbilisi 380028

Georgia