

**ON THE GLOBAL SOLVABILITY OF THE CAUCHY  
PROBLEM FOR SINGULAR FUNCTIONAL  
DIFFERENTIAL EQUATIONS**

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ABSTRACT. Sufficient conditions are found for the global solvability of the weighted Cauchy problem

$$\frac{dx(t)}{dt} = f(x)(t), \quad \lim_{t \rightarrow a} \frac{\|x(t) - c_0\|}{h(t)} = 0,$$

where  $f : C([a, b]; R^n) \rightarrow L_{loc}([a, b]; R^n)$  is a singular Volterra operator,  $c_0 \in R^n$ ,  $h : [a, b] \rightarrow [0, +\infty[$  is a function continuous and positive on  $]a, b]$ , and  $\|\cdot\|$  is the norm in  $R^n$ .

Throughout the paper the following notation will be used:

$R$  is the set of real numbers,  $R_+ = [0, +\infty[$ ; if  $u \in R$ , then  $[u]_+ = \frac{1}{2}(|u| + u)$ ;

$R^n$  is the space of  $n$ -dimensional column vectors  $x = (x_i)_{i=1}^n$  with elements  $x_i \in R$  ( $i = 1, \dots, n$ ) and the norm  $\|x\| = \sum_{i=1}^n |x_i|$ ;

$R_\rho^n = \{x \in R^n : \|x\| \leq \rho\}$ ;

if  $x = (x_i)_{i=1}^n$ , then  $\text{sgn}(x) = (\text{sgn } x_i)_{i=1}^n$ ;

$x \cdot y$  is the scalar product of the vectors  $x$  and  $y \in R^n$ ;

$C([a, b]; R^n)$  is the space of continuous vector functions  $x : [a, b] \rightarrow R^n$  with the norm  $\|x\|_C = \max\{\|x(t)\| : a \leq t \leq b\}$ ;

$$C_\rho([a, b]; R^n) = \{x \in C([a, b]; R^n) : \|x\|_C \leq \rho\};$$

$$C([a, b]; R_+) = \{x \in C([a, b]; R) : x(t) \geq 0 \text{ for } a \leq t \leq b\};$$

if  $x \in C([a, b]; R^n)$  and  $a \leq s \leq t \leq b$ , then

$$\nu(x)(s, t) = \max\{\|x(\xi)\| : s \leq \xi \leq t\};$$

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$L_{loc}(]a, b[; R^n)$  is the space of vector functions  $x : ]a, b[ \rightarrow R^n$  which are summable on each segment of  $]a, b[$  with the topology of convergence in the mean on each segment from  $]a, b[$ ;

$$L_{loc}(]a, b[; R_+) = \{x \in L_{loc}(]a, b[; R) : x(t) \geq 0 \text{ for almost all } t \in ]a, b[\}.$$

**Definition 1.** An operator  $f : C([a, b]; R^n) \rightarrow L_{loc}(]a, b[; R^n)$  is called a *Volterra* one if the equality  $f(x)(t) = f(y)(t)$  holds almost everywhere on  $]a, t_0[$  for any  $t_0 \in ]a, b[$  and any vector-functions  $x$  and  $y \in C([a, b]; R^n)$  satisfying the condition  $x(t) = y(t)$  when  $a < t \leq t_0$ .

**Definition 2.** An operator  $f : C([a, b]; R^n) \rightarrow L_{loc}(]a, b[; R^n)$  will be said to satisfy the *local Carathéodory conditions* if it is continuous and there exists a function  $\gamma : ]a, b[ \times R_+ \rightarrow R_+$  nondecreasing with respect to the second argument such that  $\gamma(\cdot, \rho) \in L_{loc}(]a, b[; R)$  for  $\rho \in R_+$ , and the inequality

$$\|f(x)(t)\| \leq \gamma(t, \|x\|_C)$$

is fulfilled for any  $x \in C([a, b]; R^n)$  almost everywhere on  $]a, b[$ .

If

$$\int_a^b \gamma(t, \rho) dt < +\infty \quad \text{for } \rho \in R_+,$$

then the operator  $f$  is called *regular*, and, otherwise, *singular*.

Here we will consider the vector functional differential equation

$$\frac{dx(t)}{dt} = f(x)(t) \tag{1}$$

with the weighted initial condition

$$\lim_{t \rightarrow a} \frac{\|x(t) - c_0\|}{h(t)} = 0. \tag{2}$$

It is assumed everywhere that  $f : C([a, b]; R^n) \rightarrow L_{loc}(]a, b[; R^n)$  is a Volterra, generally speaking, singular operator satisfying the local Carathéodory conditions,  $c_0 \in R^n$ , and  $h : [a, b] \rightarrow [0, +\infty[$  is a continuous function nondecreasing and positive on  $]a, b[$ .

We will separately consider the case where  $h(a) > 0$  so that condition (2) takes the form

$$x(a) = c_0. \tag{2_1}$$

The vector differential equation with delay

$$\frac{dx(t)}{dt} = f_0(t, x(t), x(\tau_1(t)), \dots, x(\tau_m(t))) \tag{3}$$

is the important particular case of the functional differential equation (1).

Below, whenever equation (3) is discussed, it will be assumed that the vector function  $f_0 : ]a, b[ \times R^{(m+1)n} \rightarrow R^n$  satisfies the local Carathéodory conditions, i.e.,  $f_0(t, \cdot, \dots, \cdot) : R^{(m+1)n} \rightarrow R^n$  is continuous for almost all  $t \in ]a, b[$ ,  $f_0(\cdot, x_0, x_1, \dots, x_m) : ]a, b[ \rightarrow R^n$  is measurable for all  $x_k \in R^n$  ( $k = 0, 1, \dots, m$ ), and on the set  $]a, b[ \times R^{(m+1)n}$  there holds the inequality

$$\|f_0(t, x_0, x_1, \dots, x_m)\| \leq \gamma\left(t, \sum_{k=0}^m \|x_k\|\right),$$

where  $\gamma : ]a, b[ \times R_+ \rightarrow R_+$  does not decrease with respect to the second argument and  $\gamma(\cdot, \rho) \in L_{loc}(]a, b[; R_+)$  for  $\rho \in R_+$ . As for  $\tau_i : [a, b] \rightarrow [a, b]$  ( $i = 1, \dots, m$ ), they are measurable and

$$\tau_i(t) \leq t \quad \text{for } a \leq t \leq b \quad (i = 1, \dots, m).$$

**Definition 3.** If  $b_0 \in ]a, b[$ , then:

(i) For any  $x \in C([a, b_0]; R^n)$ , by  $f(x)$  is understood the vector function given by the equality  $f(x)(t) = f(\bar{x})(t)$  for  $a \leq t \leq b_0$ , where

$$\bar{x}(t) = \begin{cases} x(t) & \text{for } a \leq t \leq b_0 \\ x(b_0) & \text{for } b_0 < t \leq b \end{cases};$$

(ii) a continuous vector function  $x : [a, b_0] \rightarrow R^n$  is called a solution of equation (1) on the segment  $[a, b_0]$  if  $x$  is absolutely continuous on each segment contained in  $]a, b_0[$  and satisfies equation (1) almost everywhere on  $]a, b_0[$ ;

(iii) a vector function  $x : [a, b_0] \rightarrow R^n$  is called a solution of equation (1) on the semi-open segment  $[a, b_0[$  if for each  $b_1 \in ]a, b_0[$  the restriction of  $x$  on  $[a, b_1]$  is a solution of this equation on the segment  $[a, b_1]$ ;

(iv) a solution  $x$  of equation (1) satisfying the initial condition (2) is called a solution of problem (1), (2).

**Definition 4.** Problem (1), (2) is said to be globally solvable if it has at least one solution on the segment  $[a, b]$ .

**Definition 5.** A solution  $x$  of equation (1) defined on the segment  $[a, b_0] \subset [a, b]$  (on the semi-open segment  $[a, b_0[ \subset [a, b]$ ) is called *continuable* if for some  $b_1 \in ]b_0, b]$  ( $b_1 \in [b_0, b]$ ) equation (1) has, on the segment  $[a, b_1]$ , a solution  $y$  satisfying the condition  $x(t) = y(t)$  for  $a \leq t \leq b_0$ . Otherwise, the solution  $x$  is called *noncontinuable*.

**Definition 6.** An operator  $\varphi : C([a, b]; R_+) \rightarrow L_{loc}(]a, b[; R)$  is called nondecreasing if the inequality  $\varphi(u)(t) \leq \varphi(v)(t)$  is fulfilled almost everywhere on  $]a, b[$  for any  $u$  and  $v \in C([a, b]; R_+)$  satisfying the condition  $u(t) \leq v(t)$  when  $a \leq t \leq b$ .

**Theorem 1.** *Let there exist a positive number  $\rho$ , summable functions  $p$  and  $q : [a, b] \rightarrow R_+$ , and a continuous nondecreasing Volterra operator  $\varphi : C([a, b]; R_+) \rightarrow L_{loc}([a, b]; R_+)$  such that*

$$\limsup_{t \rightarrow a} \left( \frac{1}{h(t)} \int_a^t p(s) ds \right) < 1, \quad \lim_{t \rightarrow a} \left( \frac{1}{h(t)} \int_a^t q(s) ds \right) = 0 \quad (4)$$

and the inequalities

$$f(c_0 + hy)(t) \cdot \operatorname{sgn}(y(t)) \leq \varphi(\|y\|)(t), \quad (5)$$

$$\varphi(u)(t) \leq p(t)v(u)(a, t) + q(t) \quad (6)$$

are fulfilled for any  $y \in C([a, b]; R^n)$  and  $u \in C_\rho([a, b]; R_+)$  almost everywhere on  $]a, b[$ . Let furthermore the problem

$$\frac{dv(t)}{dt} = \varphi\left(\frac{v}{h}\right)(t); \quad \lim_{t \rightarrow a} \frac{v(t)}{h(t)} = 0 \quad (7)$$

have an upper solution on the segment  $[a, b]$ . Then problem (1), (2) is globally solvable and each of its noncontinuable solutions is defined on  $[a, b]$ .

*Proof.* By virtue of Theorem 3.2 from [1] conditions (4)–(6) imply the existence of a noncontinuable solution of problem (1), (2). Let  $x$  be an arbitrary noncontinuable solution of problem (1), (2) defined on the segment  $I_0$ . Our aim is to prove that  $I_0 = [a, b]$ .

By (4) there exist  $b_0 \in ]a, b[$  and  $\alpha \in ]0, 1[$  such that

$$\int_a^t p(s) ds \leq \alpha h(t), \quad h_0(t) \leq \rho h(t) \quad \text{for } a \leq t \leq b_0, \quad (8)$$

where  $h_0(a) = 0$ , and

$$h_0(t) = \frac{h(t)}{1 - \alpha} \sup \left\{ \frac{1}{h(s)} \int_a^s q(\xi) d\xi : a < s \leq t \right\} \quad \text{for } a < t \leq b_0. \quad (9)$$

By Lemma 2.3 and Corollary 3.1 from [1], conditions (4)–(6) and (8) imply that  $I_0 \supset [a, b_0]$  and

$$\|x(t) - c_0\| \leq h_0(t) \quad \text{for } a \leq t \leq b_0. \quad (10)$$

Let  $v^*$  be an upper solution of problem (7). Then by conditions (4), (6), (8) and Lemma 2.3 from [1]

$$v^*(t) \leq h_0(t) \quad \text{for } a \leq t \leq b_0. \quad (11)$$

By (9) the function

$$\varepsilon(t) = \begin{cases} \frac{h_0(t)}{h(t)} + \frac{t-a}{b_0-t} & \text{for } a < t < b_0 \\ 0 & \text{for } t = a \end{cases} \quad (12)$$

is continuous on  $[a, b_0[$ .

For any  $u \in C([a, b]; R)$  we set

$$\chi(u)(t) = \begin{cases} \varepsilon(t) & \text{for } u(t) > \varepsilon(t), \quad a \leq t < b_0 \\ [u(t)]_+ & \text{for } u(t) \leq \varepsilon(t), \quad a \leq t < b_0 \\ [u(t)]_+ & \text{for } b_0 \leq t \leq b \end{cases} \quad (13)$$

and

$$\bar{\varphi}(u)(t) = \varphi\left(\chi\left(\frac{u}{h}\right)\right)(t). \quad (14)$$

Obviously,  $\chi : C([a, b]; R) \rightarrow C([a, b]; R_+)$  and  $\bar{\varphi} : C([a, b]; R) \rightarrow L_{loc}([a, b]; R_+)$  are continuous nondecreasing Volterra operators and the inequality

$$\bar{\varphi}(hu)(t) \cdot \text{sgn}(u(t)) \leq p(t)\nu(u)(a, t) + q(t) \quad (15)$$

holds for any  $u \in C_\rho([a, b]; R)$  almost everywhere on  $]a, b[$ .

By Corollary 1.3 from [2] the problem

$$\frac{dv(t)}{dt} = \bar{\varphi}(v)(t); \quad \lim_{t \rightarrow a} \frac{v(t)}{h(t)} = 0 \quad (16)$$

has a noncontinuable upper solution  $\bar{v}$  defined on some interval  $I$ . On the other hand, by conditions (11)–(14),

$$\varphi\left(\frac{v^*}{h}\right)(t) \equiv \bar{\varphi}(v^*)(t)$$

and therefore  $v^*$  is a solution of problem (16). Thus

$$v^*(t) \leq \bar{v}(t) \quad \text{for } t \in I. \quad (17)$$

By Lemma 2.3 and Corollary 3.1 from [1], conditions (4), (6), and (8) imply the estimate

$$0 \leq \bar{v}(t) \leq h_0(t) \quad \text{for } a \leq t \leq b_0$$

and hence by (12)–(14) we obtain

$$\bar{\varphi}(\bar{v})(t) \equiv \varphi\left(\frac{\bar{v}}{h}\right)(t).$$

Therefore  $\bar{v}$  is a solution of problem (7) so that

$$\bar{v}(t) \leq v^*(t) \quad \text{for } t \in I. \quad (18)$$

By (17) and (18) it is obvious that  $I = [a, b]$  and  $\bar{v}(t) \equiv v^*(t)$ . Thus  $v^*$  is an upper solution of problem (16).

Due to conditions (5), (10) and (12)–(14) we have

$$\frac{d}{dt}(\|x(t) - c_0\|) \leq \bar{\varphi}(\|x - c_0\|)(t) \tag{19}$$

almost everywhere on  $I_0$ .

By Theorem 1.3 from [2], (19) and (2) imply the estimate

$$\|x(t) - c_0\| \leq v^*(t) \quad \text{for } t \in I_0.$$

Hence by Corollary 3.1 from [1], we conclude that  $I_0 = [a, b]$ .  $\square$

For our further discussion we need

**Definition 7.** Let  $D \subset C([a, b]; R^k)$  and  $M \subset L_{loc}([a, b]; R^l)$ . An operator  $g : D \rightarrow M$  is called a strictly Volterra operator if there exists a continuous nondecreasing function  $\tau : [a, b] \rightarrow [a, b]$  such that

$$\tau(t) < t \quad \text{for } a < t \leq b$$

and the equality

$$g(x)(t) = g(y)(t)$$

holds almost everywhere on  $[a, t_0]$  for any  $t_0 \in ]a, b]$  and any vector functions  $x$  and  $y \in D$  satisfying the condition

$$x(t) = y(t) \quad \text{for } a \leq t \leq \tau(t_0).$$

**Corollary 1.** Let, for any  $y \in C([a, b]; R^n)$ , the inequality

$$f(c_0 + hy) \cdot \text{sgn}(y(t)) \leq p(\|y\|)(t)\varphi_0(\nu(y)(a, t)) + q(\|y\|)(t)$$

be fulfilled almost everywhere on  $]a, b[$ , where  $p$  and  $q : C([a, b]; R_+) \rightarrow L_{loc}([a, b]; R_+)$  are continuous, nondecreasing, strictly Volterra operators satisfying, for some  $\rho > 0$ , the conditions

$$\limsup_{t \rightarrow a} \left( \frac{1}{h(t)} \int_a^t p(\rho)(s) ds \right) < 1, \quad \lim_{t \rightarrow a} \left( \frac{1}{h(t)} \int_a^t q(\rho)(s) ds \right) = 0,$$

and  $\varphi_0 : R_+ \rightarrow R_+$  is a continuous, nondecreasing function such that

$$\limsup_{s \rightarrow 0} \frac{\varphi_0(s)}{s} < 1, \tag{20}$$

$$\varphi_0(s) > 0, \quad \int_s^{+\infty} \frac{d\xi}{\varphi_0(\xi)} = +\infty \quad \text{for } s > 0. \tag{21}$$

Then the conclusion of Theorem 1 is valid.

*Proof.* By (20) the number  $\rho > 0$  can be assumed to be so small that

$$\frac{\varphi_0(s)}{s} < \rho \quad \text{for } 0 < s \leq \rho.$$

We set

$$\varphi(u)(t) = p(u)(t)\varphi_0(\nu(u)(a, t)) + q(u)(t). \tag{22}$$

Then inequalities (5) and (6), where

$$p(t) \equiv p(\rho)(t), \quad q(t) \equiv q(\rho)(t),$$

are fulfilled almost everywhere on  $]a, b[$  for any  $y \in C([a, b]; R^n)$  and  $u \in C_\rho([a, b]; R_+)$ .

By virtue of Theorem 1, to prove the corollary it is sufficient to show that problem (7) has an upper solution on the segment  $[a, b]$ .

Choose  $b_0 \in ]a, b[$  and  $\alpha \in ]0, 1[$  such that inequalities (8) are fulfilled.

Let  $h_0, \varepsilon, \chi$  and  $\bar{\varphi}$  be the functions and operators given by equalities (9) and (12)–(14). As shown while proving Theorem 1, problem (7) is equivalent to problem (16).

Following Corollary 1.3 from [2], problem (16) has an upper solution  $v^*$  defined on some interval  $I$ . Since problems (7) and (16) are equivalent it remains for us only to show that  $I = [a, b]$ .

By virtue of Corollary 3.1 and Lemma 2.3 from [1],  $I \supset [a, b_0]$  and inequality (11) is fulfilled.

Now assume that  $I \neq [a, b]$ . Then, by Corollary 3.1 from [1] and the non-negativeness of the operator  $\bar{\varphi}$ , there exists  $b_1 \in ]b_0, b[$  such that  $I = [a, b_1[$  and

$$\lim_{t \rightarrow b_1} v^*(t) = +\infty. \tag{23}$$

By (8) and (11)–(13)

$$\frac{v^*(t)}{h(t)} < \rho \quad \text{for } 0 \leq t \leq b_0$$

and

$$\nu\left(\chi\left(\frac{v^*}{h}\right)\right)(a, t) \leq \rho + \frac{v^*(t)}{h(b_0)} \quad \text{for } b_0 \leq t < b_1,$$

due to which (14) and (22) imply

$$\bar{\varphi}(v^*)(t) \leq p_0(t)\varphi_0\left(\rho + \frac{v^*(t)}{h(b_0)}\right) \quad \text{for } b_0 \leq t < b_1,$$

where

$$p_0(t) = p\left(\chi\left(\frac{v^*}{h}\right)\right)(t) + \frac{q\left(\chi\left(\frac{v^*}{h}\right)\right)(t)}{\varphi_0(\rho)}.$$

On the other hand, since the operators  $p$  and  $q$  are the strictly Volterra ones, the function  $p_0 : [b_0, b_1] \rightarrow R_+$  is summable.

From the above discussion it follows that the inequality

$$\frac{dv^*(t)}{dt} / \varphi_0 \left( \rho + \frac{v^*(t)}{h(b_0)} \right) \leq p_0(t)$$

is fulfilled almost everywhere on  $]b_0, b_1[$ .

If we integrate both parts of this inequality from  $b_0$  to  $b_1$ , by (23) we will obtain

$$h(b_0) \int_{s_0}^{+\infty} \frac{ds}{\varphi_0(s)} \leq \int_{b_0}^{b_1} p_0(s) ds < +\infty,$$

which contradicts condition (21). The obtained contradiction proves the corollary.  $\square$

The next corollary is proved similarly.

**Corollary 2.** *Let, for any  $y \in C([a, b]; R^n)$ , the inequality*

$$f(c_0 + hy) \cdot \operatorname{sgn}(y(t)) \leq p(\|y\|)(t) \varphi_0(\nu(y)(a, t))$$

be fulfilled almost everywhere on  $]a, b[$ , where  $p : C([a, b]; R_+) \rightarrow L([a, b]; R_+)$  is a continuous, nondecreasing, strictly Volterra operator satisfying, for some  $\rho > 0$ , the condition

$$\lim_{t \rightarrow a} \left( \frac{1}{h(t)} \int_a^t p(\rho)(s) ds \right) = 0,$$

and  $\varphi_0 : R_+ \rightarrow ]0, +\infty[$  is a continuous, nondecreasing function such that

$$\int_0^{+\infty} \frac{ds}{\varphi_0(s)} = +\infty. \quad (24)$$

Then the conclusion of Theorem 1 is valid.

**Corollary 3.** *Let, for any  $y \in C([a, b]; R^n)$ , the inequality*

$$f(c_0 + y)(t) \cdot \operatorname{sgn}(y(t)) \leq \frac{\alpha}{b-a} \left( \frac{b-t}{b-a} \right)^\beta \exp [\nu(y)(a, \tau(t))] \quad (25)$$

be fulfilled almost everywhere on  $]a, b[$ , where

$$\tau(t) = b - (b-a) \left( 1 + \ln \frac{b-a}{b-t} \right)^{-1}, \quad (26)$$

$\alpha > 0$  and  $\beta$  are constants such that

$$\beta > [\alpha - 1]_+ - 1. \quad (27)$$



Then problem (1), (2) is globally solvable and each of its noncontinuable solutions is defined on the entire  $[a, b]$ .

*Proof.* By Theorem 1 it is sufficient to show that the problem

$$\frac{du(t)}{dt} = \frac{\alpha}{b-a} \left(\frac{b-t}{b-a}\right)^\beta \exp [\nu(u)(a, \tau(t))], \quad u(a) = 0$$

has an upper solution on the segment  $[a, b]$ . But this problem is equivalent to the problem

$$\frac{du(t)}{dt} = \frac{\alpha}{b-a} \left(\frac{b-t}{b-a}\right)^\beta \exp [u(\tau(t))], \quad u(a) = 0. \tag{28}$$

By Theorem 3.2 from [1] problem (28) has a noncontinuable solution. It is not difficult to verify that this solution is unique. We denote it by  $u$ . Let  $I$  be the definition interval of  $u$ . Our aim is to show that  $I = [a, b]$ .

Consider the function

$$v(t) = \alpha \left(\frac{b-a}{b-t} - 1\right) \quad \text{for } a \leq t < b.$$

By virtue of (26) and (27)

$$a \leq \tau(t) < t \quad \text{for } a < t < b$$

and

$$\begin{aligned} \frac{dv(t)}{dt} &= \frac{\alpha}{b-a} \left(\frac{b-t}{b-a}\right)^{\alpha-2} \exp [v(\tau(t))] \geq \\ &\geq \frac{\alpha}{b-a} \left(\frac{b-t}{b-a}\right)^\beta \exp [v(\tau(t))] \quad \text{for } a \leq t < b. \end{aligned}$$

Hence, by Corollary 1.8 from [2], we have

$$v(t) \geq u(t) \quad \text{for } t \in I.$$

However, by Corollary 3.1 from [2] this estimate implies that  $I \supset [a, b[$  and

$$u(t) \leq \alpha \left(\frac{b-a}{b-t} - 1\right) \quad \text{for } a \leq t < b. \tag{29}$$

On account of (27) there is  $\varepsilon \in ]0, 1[$  such that  $\beta > [\alpha - 1]_+ - \varepsilon$ . If, along with this inequality, we use equality (26) and estimate (29), then (28) will yield

$$u(t) \leq \frac{\alpha}{b-a} \int_a^t \left(\frac{b-s}{b-a}\right)^{[\alpha-1]_+ - \varepsilon} \exp [u(\tau(s))] ds \quad \text{for } a \leq t < b, \tag{30}$$

$$\begin{aligned} u(t) &\leq \frac{\alpha}{b-a} \int_a^t \left(\frac{b-s}{b-a}\right)^{\alpha-1-\varepsilon} \left(\frac{b-s}{b-a}\right)^{-\alpha} ds = \\ &= \alpha(b-a)^\varepsilon \int_a^t (b-s)^{-1-\varepsilon} ds < \frac{\alpha}{\varepsilon} \left(\frac{b-a}{b-t}\right)^\varepsilon \quad \text{for } a \leq t < b, \end{aligned}$$

and hence

$$u(t) \leq \alpha_1 + \frac{1-\varepsilon}{2} \left(\frac{b-a}{b-t}\right) \quad \text{for } a \leq t < b, \quad (31)$$

where

$$\alpha_1 = \frac{1}{\varepsilon} \left(\frac{2\alpha}{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}}.$$

By (26) and (31), from (30) we obtain

$$\begin{aligned} u(t) &\leq \frac{\alpha}{b-a} \exp(\alpha_1) \int_a^t \left(\frac{b-s}{b-a}\right)^{[\alpha-1]_+ - \varepsilon} \left(\frac{b-s}{b-a}\right)^{\frac{\varepsilon-1}{2}} ds \leq \\ &\leq \frac{\alpha}{b-a} \exp(\alpha_1) \int_a^t \left(\frac{b-s}{b-a}\right)^{-\frac{1+\varepsilon}{2}} ds \leq \alpha_2 \quad \text{for } a \leq t < b, \end{aligned}$$

where

$$\alpha_2 = \frac{2\alpha}{1-\varepsilon}.$$

Hence by Corollary 3.1 from [1] it follows that  $I = [a, b]$ .  $\square$

Corollaries 1–3 imply the following propositions for equation (3).

**Corollary 4.** *Let there exist  $m_0 \in \{1, \dots, m\}$  and a continuous nondecreasing function  $\tau : [a, b] \rightarrow [a, b]$  such that*

$$\tau_k(t) \leq \tau(t) < t \quad \text{for } a < t \leq b \quad (k = m_0, \dots, m). \quad (32)$$

Let furthermore on the set  $]a, b[ \times R^{(m+1)n}$  there hold the inequality

$$\begin{aligned} &f_0(t, c_0 + h(t)y_0, c_0 + h(\tau_1(t))y_1, \dots, c_0 + h(\tau_m(t))y_m) \cdot \operatorname{sgn}(y_0) \leq \\ &\leq p\left(t, \sum_{k=m_0}^m \|y_k\|\right) \varphi_0\left(\sum_{k=0}^{m_0-1} \eta_k \|y_k\|\right) + q\left(t, \sum_{k=m_0}^m \|y_k\|\right), \end{aligned}$$

where  $p$  and  $q : [a, b] \times R_+ \rightarrow R_+$  are functions summable with respect to the first argument and continuously nondecreasing with respect to the second argument, and satisfying, for some  $\rho > 0$ , the conditions

$$\limsup_{t \rightarrow a} \left( \frac{1}{h(t)} \int_a^t p(s, \rho) ds \right) < 1, \quad \lim_{t \rightarrow a} \left( \frac{1}{h(t)} \int_a^t q(s, \rho) ds \right) = 0,$$

$\varphi_0 : R_+ \rightarrow R_+$  is a continuous nondecreasing function satisfying conditions (20) and (21),  $\eta_k$  ( $k = 0, \dots, m_0$ ) are non-negative constants such that

$$\sum_{k=0}^{m_0} \eta_k = 1.$$

Then problem (3), (2) is globally solvable and each of its noncontinuable solutions is defined on  $[a, b]$ .

**Corollary 5.** Let the function  $h$  be nondecreasing and inequalities (32) be fulfilled, where  $m_0 \in \{1, \dots, m\}$  and  $\tau : [a, b] \rightarrow [a, b]$  is a continuous nondecreasing function. Let, furthermore, on the set  $]a, b[ \times R^{(m+1)^n}$  there hold the inequality

$$\begin{aligned} f_0(t, c_0 + h(t)y_0, c_0 + h(\tau_1(t))y_1, \dots, c_0 + h(\tau_m(t))y_m) \cdot \operatorname{sgn}(y_0) &\leq \\ &\leq p\left(t, \sum_{k=m_0}^m \|y_k\|\right) \varphi_0\left(\sum_{k=0}^{m_0-1} \|y_k\|\right), \end{aligned}$$

where  $p : [a, b] \times R_+ \rightarrow R_+$  is a function summable with respect to the first argument and continuously nondecreasing with respect to the second argument, and satisfying, for some  $\rho > 0$ , the condition

$$\lim_{t \rightarrow a} \left( \frac{1}{h(t)} \int_a^t p(s, \rho) ds \right) = 0,$$

$\varphi_0 : R_+ \rightarrow ]0, +\infty[$  is a continuous nondecreasing function satisfying condition (21). Then the conclusion of Corollary 4 is valid.

**Corollary 6.** Let on the set  $]a, b[ \times R^{(m+1)^n}$  and the segment  $]a, b[$  there hold respectively the inequalities

$$\begin{aligned} f_0(t, c_0 + y_0, c_0 + y_1, \dots, c_0 + y_m) \cdot \operatorname{sgn}(y_0) &\leq \\ &\leq \frac{\alpha}{b-a} \left( \frac{b-t}{b-a} \right)^\beta \exp\left(\sum_{k=1}^m \eta_k \|y_k\|\right) \end{aligned}$$

and

$$\tau_k(t) \leq b - (b-a) \left( 1 + \ln \frac{b-a}{b-t} \right)^{-1} \quad (k = 1, \dots, m),$$

where  $\alpha > 0$ ,  $\beta$  and  $\eta_k \geq 0$  ( $k = 1, \dots, m$ ) are constants satisfying condition (27) and

$$\sum_{k=1}^m \eta_k = 1.$$

Then problem (3),(2<sub>1</sub>) is globally solvable and each of its noncontinuable solutions is defined on the entire  $[a, b]$ .

**Example 1.** Consider problem (28) where  $\alpha > 1$ ,  $\beta = \alpha - 2$ ,  $\tau$  is the function given by equality (26). All conditions of Corollary 6 except (27) are fulfilled. Nevertheless it has the noncontinuable solution

$$u(t) = \alpha \left( \frac{b-a}{b-t} - 1 \right),$$

which is defined not on the segment  $[a, b]$  but on  $[a, b[$ . This example shows that in Corollaries 3 and 6 the strict inequality (27) cannot be replaced by the nonstrict inequality  $\beta \geq [\alpha - 1]_+ - 1$ .

**Theorem 2.** Let there exist  $\delta : ]a, b] \rightarrow ]0, \infty[$  and  $c : ]a, b] \rightarrow R^n$  such that  $\delta(s) < s - a$  for  $a < s \leq b$ , and for any numbers  $s \in ]a, b]$ ,  $\rho > 0$  and any vector function  $y \in C([a, s]; R^n)$  satisfying the condition

$$\|y(t)\| \leq \rho \quad \text{for } a \leq t \leq s - \delta(s), \quad (33)$$

let the inequality

$$f(c(s) + y)(t) \cdot \operatorname{sgn}(y(t)) \leq \varphi_{s,\rho}(\|y\|)(t) \quad (34)$$

hold almost everywhere on  $]s - \delta(s), s[$ , where  $\varphi_{s,\rho} : C([s - \delta(s), s]; R_+) \rightarrow L([s - \delta(s), s]; R_+)$  is a continuous nondecreasing Volterra operator. Let furthermore the problem

$$\frac{du(t)}{dt} = \varphi_{s,\rho}(u)(t), \quad u(s - \delta(s)) = \rho \quad (35)$$

have an upper solution on the interval  $[s - \delta(s), s]$  for any  $s \in ]a, b]$  and  $\rho > 0$ . Then each noncontinuable solution of equation (1) is defined on  $[a, b]$ .

*Proof.* Assume that the theorem is not valid. Then by virtue of Corollary 3.1 from [1] there exist  $s \in ]a, b]$  and a noncontinuable solution  $x : [a, s[ \rightarrow R^n$  of equation (1) such that

$$\limsup_{t \rightarrow s} \|x(t)\| = +\infty. \quad (36)$$

We set  $y(t) = x(t) - c(s)$  and choose  $\rho > 0$  such that inequality (33) is fulfilled. Then by (34) the inequality

$$\frac{d\|y(t)\|}{dt} \leq \varphi_{s,\rho}(\|y\|)(t)$$

is fulfilled almost everywhere on  $]s - \delta(s), s[$ .

Moreover,

$$\|y(s - \delta(s))\| \leq \rho.$$

By Corollary 1.7 from [2] the latter two inequalities yield the estimate

$$\|y(t)\| \leq u(t) \quad \text{for } s - \delta(s) \leq t < s.$$

Therefore

$$\limsup_{t \rightarrow s} \|x(t)\| \leq \|c(s)\| + \limsup_{t \rightarrow s} \|y(t)\| \leq \|c(s)\| + u(s) < +\infty,$$

which contradicts equality (36). The obtained contradiction proves the theorem.  $\square$

*Remark.* It is obvious that if the conditions of Theorem 2 are fulfilled, then the local solvability of problem (1), (2) guarantees its global solvability. Therefore if the conditions of Theorem 2, as well as of Theorem 2.1 from [1], are fulfilled, then problem (1), (2) is globally solvable and each of its noncontinuable solutions is defined on  $[a, b]$ .

**Corollary 7.** *Let there exist functions  $\delta : ]a, b] \rightarrow ]0, 1[$ ,  $c : ]a, b] \rightarrow R^n$ ,  $\alpha : ]a, b] \times R_+ \rightarrow R_+$ ,  $\beta : ]a, b] \times R_+ \rightarrow ]-1, 0]$ , and  $\lambda_k : ]a, b] \times R_+ \rightarrow [1, +\infty[$  ( $k = 1, \dots, m$ ) such that  $\delta(s) < s - a$  for  $a < s \leq b$ , and for any numbers  $s \in ]a, b]$ ,  $\rho > 0$  and any vector function  $y \in C([a, b]; R^n)$  satisfying condition (33), there hold, almost everywhere on  $]s - \delta(s), s[$ , the inequality*

$$\begin{aligned} & f(c(s) + y)(t) \cdot \operatorname{sgn}(y(t)) \leq \\ & \leq \alpha(s, \rho)(s - t)^{\beta(s, \rho)} \left( 1 + \sum_{k=1}^m [\nu(y)(s - \delta(s), \tau_{ks\rho}(t))]^{\lambda_k(s, \rho)} \right) \times \\ & \times \ln \left( 2 + \sum_{k=1}^m \nu(y)(s - \delta(s), \tau_{ks\rho}(t)) \right), \end{aligned} \tag{37}$$

where

$$\tau_{ks\rho}(t) = \begin{cases} s - \delta(s) & \text{for } s - \delta(s) \leq t \leq s - [\delta(s)]^{\lambda_k(s, \rho)} \\ s - (s - t)^{\frac{1}{\lambda_k(s, \rho)}} & \text{for } s - [\delta(s)]^{\lambda_k(s, \rho)} < t \leq s \end{cases}. \tag{38}$$

Then each noncontinuable solution of equation (1) is defined on  $[a, b]$ .

*Proof.* Let  $s \in ]a, b]$  and  $\rho > 0$  be arbitrarily fixed. We introduce  $\varepsilon \in ]0, \frac{1}{2}]$  such that

$$\beta(s, \rho) \geq 2\varepsilon - 1.$$

By virtue of Theorem 2 and inequality (37), to prove the corollary it is sufficient to establish that problem (35), where

$$\begin{aligned} \varphi_{s\rho}(u)(t) = & \alpha(s, \rho)(s-t)^{2\varepsilon-1} \left( 1 + \sum_{k=1}^m [\nu([u]_+)(s-\delta(s), \tau_{ks\rho}(t))]^{\lambda_k(s, \rho)} \right) \times \\ & \times \ln \left( 2 + \sum_{k=1}^m \nu([u]_+)(s-\delta(s), \tau_{ks\rho}(t)) \right), \end{aligned} \quad (39)$$

has an upper solution on the interval  $[s - \delta(s), s]$ .

By Corollary 1.3 from [2] and equalities (38) and (39), problem (35) has a noncontinuable upper solution  $u^*$  on some interval  $I \subset [s - \delta(s), s]$ . On the other hand, from Corollary 2 it immediately follows that  $I \supset ]s - \delta(s), s[$ . It remains for us to show that  $s \in I$ .

Choose numbers  $\delta_0 \in ]0, \delta(s)[$  and  $\rho_0 \in ]\rho, +\infty[$  such that

$$\frac{1}{\varepsilon} < \ln \frac{1}{\delta_0} < [\alpha(s, \rho)(m+1)(1 + \ln(2+m))]^{-1} \delta_0^{-\varepsilon}, \quad (40)$$

$$u^*(s - \delta_0) < \rho_0, \quad (41)$$

and for any  $k \in \{1, \dots, m\}$  and  $u \in C([s - \delta_0, s]; R)$  put

$$\tau_k^*(t) = \begin{cases} s - \delta_0 & \text{for } s - \delta_0 \leq t \leq s - \delta_0^{\lambda_k(s, \rho)} \\ s - (s-t)^{\frac{1}{\lambda_k(s, \rho)}} & \text{for } s - \delta_0^{\lambda_k(s, \rho)} < t \leq s \end{cases}, \quad (42)$$

$$\nu_k^*(u)(t) = \begin{cases} \rho_0 & \text{for } s - \delta_0 \leq t \leq s - \delta_0^{\lambda_k(s, \rho)} \\ [u(\tau_k^*(t))]_+ & \text{for } s - \delta_0^{\lambda_k(s, \rho)} < t \leq s \end{cases}, \quad (43)$$

$$\begin{aligned} \varphi^*(u)(t) = & \alpha(s, \rho)(s-t)^{2\varepsilon-1} \left( 1 + \sum_{k=1}^m [\nu_k^*(u)(t)]^{\lambda_k(s, \rho)} \right) \times \\ & \times \ln \left( 2 + \sum_{k=1}^m \nu_k^*(u)(t) \right). \end{aligned} \quad (44)$$

Then by virtue of (38) and (39) the inequality  $\varphi_{s, \rho}(u^*)(t) \leq \varphi^*(u^*)(t)$  holds almost everywhere on  $]s - \delta_0, s[$  and therefore

$$0 < \frac{du^*(t)}{dt} \leq \varphi^*(u^*)(t). \quad (45)$$

Let  $l$  be a natural number so large that

$$\delta_0^{-\frac{(l+\frac{1}{2})\varepsilon}{\lambda_k(s, \rho)}} > \rho_0 \quad (k = 1, \dots, m). \quad (46)$$

Setting  $v(t) = (s - t)^{-(l+\frac{1}{2})\varepsilon}$ , we obtain

$$v(s - \delta_0) > \rho_0. \tag{47}$$

Moreover, with (42), (43), and (46) taken into account we find

$$\nu_k^*(v)(t) = \begin{cases} \rho_0 & \text{for } s - \delta_0 \leq t \leq s - \delta_0^{\lambda_k(s,\rho)} \\ (s - t)^{-\frac{(l+\frac{1}{2})\varepsilon}{\lambda_k(s,\rho)}} & \text{for } s - \delta_0^{\lambda_k(s,\rho)} < t \leq s \end{cases},$$

and

$$\nu_k^*(v)(t) \leq (s - t)^{-\frac{(l+\frac{1}{2})\varepsilon}{\lambda_k(s,\rho)}} \text{ for } s_0 - \delta_0 \leq t < s \quad (k = 1, \dots, m). \tag{48}$$

By (44) and (48) the inequality

$$\begin{aligned} \varphi^*(v)(t) &\leq \alpha(s, \rho)(s - t)^{2\varepsilon-1} (1 + m(s - t)^{-(l+\frac{1}{2})\varepsilon}) \times \\ &\quad \times \ln(2 + m(s - t)^{-(l+\frac{1}{2})\varepsilon}) \leq \alpha(s, \rho)(m + 1) \times \\ &\quad \times (s - t)^{2\varepsilon-1-(l+\frac{1}{2})\varepsilon} \ln[(2 + m)(s - t)^{-(l+\frac{1}{2})\varepsilon}] \end{aligned}$$

holds almost everywhere on  $]s - \delta_0, s[$ .

On the other hand, using (40) we have

$$\begin{aligned} (s - t)^\varepsilon \ln \frac{1}{s - t} &\leq \delta_0^\varepsilon \ln \frac{1}{\delta_0} \leq \\ &\leq [\alpha(s, \rho)(m + 1)(1 + \ln(2 + m))]^{-1} \text{ for } s - \delta_0 \leq t < s \end{aligned}$$

and

$$\begin{aligned} \ln[(2 + m)(s - t)^{-(l+\frac{1}{2})\varepsilon}] &= \ln(2 + m) + \left(l + \frac{1}{2}\right)\varepsilon \ln \frac{1}{s - t} \leq \\ &\leq (1 + \ln(2 + m)) \left(l + \frac{1}{2}\right)\varepsilon \ln \frac{1}{s - t} \leq \\ &\leq \frac{(l + \frac{1}{2})\varepsilon}{\alpha(s, \rho)(m + 1)} (s - t)^{-\varepsilon} \text{ for } s - \delta_0 \leq t < s. \end{aligned}$$

Therefore

$$\varphi^*(v)(t) \leq \left(l + \frac{1}{2}\right)\varepsilon (s - t)^{\varepsilon-1-(l+\frac{1}{2})\varepsilon} \tag{49}$$

and

$$\frac{dv(t)}{dt} > \varphi^*(v)(t). \tag{50}$$

By Theorem 1.4 from [2] inequalities (41), (45), (50) imply the estimate

$$0 < u^*(t) < (s - t)^{-(l+\frac{1}{2})\varepsilon} \text{ for } s - \delta_0 \leq t < s,$$

by means of which we find from (45) and (49) that

$$0 \leq \frac{du^*(t)}{dt} \leq \rho_1(s-t)^{-(l-1+\frac{1}{2})\varepsilon-1},$$

where  $\rho_1 = (l + \frac{1}{2})\varepsilon$ . Assume now that the inequality

$$0 \leq \frac{du^*(t)}{dt} \leq \rho_k(s-t)^{-(l-k+\frac{1}{2})\varepsilon-1}, \quad (51)$$

where  $\rho_k$  is a positive constant, holds almost everywhere on  $]s - \delta_0, s[$  for some  $k \in \{1, \dots, l\}$ . Then

$$\begin{aligned} 0 < u^*(t) &\leq \rho_0 + \left(l - k + \frac{1}{2}\right)^{-1} \varepsilon^{-1} (s-t)^{-(l-k+\frac{1}{2})\varepsilon} \leq \\ &\leq \rho_{1k} (s-t)^{-(l-k+\frac{1}{2})\varepsilon} \quad \text{for } s - \delta_0 \leq t < s, \end{aligned}$$

where  $\rho_{1k} = \rho_0 + (l-k+\frac{1}{2})^{-1} \varepsilon^{-1} \rho_k$ . On account of this, estimates (42)–(46) yield

$$\begin{aligned} \nu_k^*(u^*)(t) &\leq \rho_0 + \rho_{1k} (s-t)^{-\frac{(l-k+\frac{1}{2})\varepsilon}{\lambda_k(s,\rho)}} \leq \\ &\leq (\rho_0 + \rho_{1k}) (s-t)^{-\frac{(l-k+\frac{1}{2})\varepsilon}{\lambda_k(s,\rho)}}, \\ 1 + \sum_{k=1}^m [\nu_k^*(u^*)(t)]^{\lambda_k(s,\rho)} &\leq 1 + \sum_{k=1}^m (\rho_0 + \rho_{1k})^{\lambda_k(s,\rho)} (s-t)^{-(l-k+\frac{1}{2})\varepsilon} \leq \\ &\leq \left[1 + \sum_{k=1}^m (\rho_0 + \rho_{1k})^{\lambda_k(s,\rho)}\right] (s-t)^{-(l-k+\frac{1}{2})\varepsilon}, \\ \ln \left(2 + \sum_{k=1}^m \nu_k^*(u)(t)\right) &\leq \ln \left[\left(2 + \sum_{k=1}^m (\rho_0 + \rho_{1k})\right) (s-t)^{-l}\right] \leq \\ &\leq \ln \left(2 + \sum_{k=1}^m (\rho_0 + \rho_{1k})\right) + l \ln \left(\frac{1}{s-t}\right) \leq \\ &\leq \left[\ln \left(2 + \sum_{k=1}^m (\rho_0 + \rho_{1k})\right) + l\right] \ln \frac{1}{s-t} \end{aligned}$$

and

$$\begin{aligned} 0 < \frac{du^*(t)}{dt} &\leq \rho_{k+1} (s-t)^{-(l-k-1+\frac{1}{2})\varepsilon} (s-t)^\varepsilon \ln \frac{1}{s-t} \leq \\ &\leq \rho_{k+1} (s-t)^{-(l-k-1+\frac{1}{2})\varepsilon} \delta_0^\varepsilon \ln \frac{1}{\delta_0} < \rho_{k+1} (s-t)^{-(l-k-1+\frac{1}{2})\varepsilon}, \end{aligned}$$



where

$$\rho_{k+1} = \alpha(s, \rho) \left[ 1 + \sum_{k=1}^m (\rho_0 + \rho_{1k})^{\lambda_k(s, \rho)} \right] \left[ \ln \left( 2 + \sum_{k=1}^m (\rho_0 + \rho_{1k}) \right) + l \right].$$

We have thus shown by induction that inequality (51) holds almost everywhere on  $]s - \delta_0, s[$  for each  $k \in \{1, \dots, l + 1\}$ . Therefore

$$0 < \frac{du^*(t)}{dt} \leq \rho_{l+1}(s - t)^{\frac{\varepsilon}{2} - 1}$$

and  $0 < u^*(t) < \rho^*$  for  $s - \delta_0 \leq t < s$ , where  $\rho^* = \rho_0 + \frac{2}{\varepsilon} \delta_0^{\frac{\varepsilon}{2}}$ . By Corollary 3.3, the latter estimate implies  $s \in I$ .  $\square$

The proved proposition immediately implies

**Corollary 8.** *Let there exist functions  $\delta : ]a, b[ \rightarrow ]0, 1[$ ,  $c : ]a, b[ \rightarrow R^n$ ,  $\alpha : ]a, b[ \times R_+ \rightarrow R_+$ ,  $\beta : ]a, b[ \rightarrow ]-1, 0[$ ,  $\lambda_k : ]a, b[ \rightarrow [1, +\infty[$  ( $k = 1, \dots, m$ ) and a number  $m_0 \in \{1, \dots, m\}$  such that the inequalities*

$$\begin{aligned} \tau_k(t) &\leq s - (s - t)^{\frac{1}{\lambda_k(s)}} \quad (k = 1, \dots, m_0), \\ \tau_k(t) &\leq s - \delta(s) \quad (k = m_0 + 1, \dots, m), \end{aligned}$$

and

$$\begin{aligned} &f_0(t, c(s) + y_0, \dots, c(s) + y_m) \cdot \operatorname{sgn}(y_0) \leq \\ &\leq \alpha \left( s, \sum_{k=m_0+1}^m \|y_k\| \right) (s - t)^{\beta(s)} \left( 1 + \sum_{k=1}^{m_0} \|y_k\|^{\lambda_k(s)} \right) \times \\ &\quad \times \ln \left( 2 + \sum_{k=1}^{m_0} \|y_k\| \right) \end{aligned} \tag{52}$$

hold respectively on  $]s - \delta(s), s[$  and  $]s - \delta(s), s[ \times R^{(m+1)n}$ . Then each non-continuable solution of equation (3) is defined on  $[a, b]$ .

**Example 2.**<sup>1</sup> Let  $b - a \leq 1$ ,  $\lambda \geq 1$  and  $\varepsilon > 0$ . Then the differential equation

$$\frac{dx(t)}{dt} = \frac{\lambda}{\varepsilon} |x(b - (b - t)^{\frac{1}{\lambda}})|^{\lambda + \varepsilon}$$

has the noncontinuable solution  $x(t) = (b - t)^{-\frac{\lambda}{\lambda + \varepsilon}}$  defined on the interval  $[a, b[$ .

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<sup>1</sup>See [3].

This example shows that the index  $\lambda_k(s)$  on the right-hand side of (52) cannot be replaced by  $\lambda_k(s) + \varepsilon$  for any  $k \in \{1, \dots, m\}$  no matter how small  $\varepsilon > 0$  is.

To conclude, note that Corollaries 1, 2, 4, and 5 are analogues of the well-known theorem of A. Wintner ([4], Ch. III, §3.5) for problems (1), (2) and (3), (2), and Corollaries 7 and 8 are generalizations of the theorem of A. Myshkis and Z. Tsalyuk [3] (see also [5]).

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