GEORGIAN MATHEMATICAL JOURNAL: Vol. 4, No. 4, 1997, 341-354

ON THE SOLVABILITY OF THE MULTIDIMENSIONAL VERSION OF THE FIRST DARBOUX PROBLEM FOR A MODEL SECOND-ORDER DEGENERATING HYPERBOLIC EQUATION

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ABSTRACT. A multidimensional version of the first Darboux problem is considered for a model second order degenerating hyperbolic equation. Using the technique of functional spaces with a negative norm, the correct formulation of this problem in the Sobolev weighted space is proved.

In a space of variables x_1, x_2, t let us consider a second-order degenerating hyperbolic equation of the type

$$Lu \equiv u_{tt} - |x_2|^m u_{x_1x_1} - u_{x_2x_2} + a_1 u_{x_1} + a_2 u_{x_2} + a_3 u_t + a_4 u = F, \quad (1)$$

where a_i, $i = 1, \dots, 4$, F are given real functions and u is an unknown real function, $m = \text{const} > 0$.

Denote by $D : x_2 < t < 1 - x_2$, $0 < x_2 < \frac{1}{2}$, an unbounded domain lying in a space $x_2 > 0$ and bounded by characteristic surfaces $S_1 : t - x_2 = 0$, $0 < x_2 < \frac{1}{2}$, $S_2 : t + x_2 - 1 = 0$, $0 < x_2 < \frac{1}{2}$, of equation (1) and by a plane surface $S_0 : x_2 = 0$, $0 < t < 1$, of time type with an equation degenerating on it. The coefficients a_i, $i = 1, \dots, 4$, of equation (1) in the domain D are assumed to be bounded functions of the class $C^1(\overline{D})$.

For equation (1) let us consider a multidimensional version of the first Darboux problem formulated as follows: find in the domain D a solution $u(x_1, x_2, t)$ of equation (1) satisfying the boundary condition

$$u\big|_{S_0 \cup S_1} = 0. \quad (2)$$

1991 *Mathematics Subject Classification.* 35L80.

Key words and phrases. Degenerating hyperbolic equation, multidimensional version of the first Darboux problem, Sobolev's weighted space, functional space with negative norm.

1072-947X/97/0700-0341$12.50/0

The problem in the domain D for the equation

$$L^*v \equiv v_{tt} - x_2^m v_{x_1x_1} - v_{x_2x_2} - (a_1v)_{x_1} - (a_2v)_{x_2} - (a_3v)_t + a_4v = F \quad (3)$$

is posed analogously by the boundary condition

$$v\big|_{S_0\cup S_2} = 0, \quad (4)$$

where L^* is a formally conjugate operator of L.

Note that similar problems for $m = 0$, when equation (1) is not degenerating and contains, in its principal part, a wave operator, were studied in [1–6]. Other versions of multidimensional Darboux problems can be found in [7–9].

Denote by E, E^* the classes of functions from the Sobolev space $W_2^2(D)$ satisfying respectively the boundary condition (2) and the boundary condition (4). Let $W_+(W_+^*)$ be the weighted Hilbert space obtained by closing the space $E(E^*)$ in the norm

$$\|u\|_{1,+}^2 = \int\limits_D (u_t^2 + x_2^m u_{x_1}^2 + u_{x_2}^2 + u^2)dD.$$

Denote by $W_-(W_-^*)$ the space with a negative norm constructed with respect to $L_2(D)$ and $W_+(W_+^*)$ [10].

Consider the condition

$$M = \sup_{\overline{D}} \big|x_2^{-\frac{m}{2}} a_1(x_1, x_2, t)\big| < \infty \quad (5)$$

imposed an the lowest coefficient a_1 in equation (1).

The uniqueness theorem required for solving problem (1), (2) of the class $W_2^2(D)$ follows from

Lemma 1. *Let condition* (5) *be fulfilled. Then for every* $u \in W_2^2(D)$ *satisfying the homogeneous boundary condition*

$$u\big|_{S_0} = 0 \quad (6)$$

the a priori estimate

$$\|u\|_{1,+} \le c\big(\|f\|_{1,*} + \|F\|_{L_2(D)}\big) \quad (7)$$

is valid, where the positive constant c *does not depend on* u*;* $f = u|_{S_1}$, $F = Lu$,

$$\|f\|_{1,*}^2 = \int\limits_{S_1} \Big[x_2^m f_{x_1}^2 + \Big(\frac{\partial f}{\partial N}\Big)^2\Big] ds,$$

and $\dfrac{\partial}{\partial N} = -\dfrac{1}{2}\Big(\dfrac{\partial}{\partial x_2} + \dfrac{\partial}{\partial t}\Big)$ *is the derivative with respect to the conormal which is the inner differential operator on the characteristic surface* S_1.

Proof. Let $n = (\nu_1, \nu_2, \nu_0)$ be the outer unit vector to ∂D, i.e., $\nu_1 = \cos(\widehat{n, x_1})$, $\nu_2 = \cos(\widehat{n, x_2})$, $\nu_0 = \cos(\widehat{n, t})$. For the function $u \in W_2^2(D)$ satisfying the boundary condition (6) and $\lambda = \text{const} > 0$ a simple integration by parts gives

$$2\int\limits_D e^{-\lambda t} u_{tt} u_t \, dD = \int\limits_{\partial D} e^{-\lambda t} u_t^2 \nu_0 \, ds + \int\limits_D \lambda e^{-\lambda t} u_t^2 \, dD, \tag{8}$$

$$\begin{gathered} -2\int\limits_D e^{-\lambda t}(x_2^m u_{x_1x_1} u_t + u_{x_2x_2} u_t) dD = \\ = -2\int\limits_{\partial D} e^{-\lambda t}(x_2^m u_{x_1} u_t \nu_1 + u_{x_2} u_t \nu_2) ds + \\ + \int De^{-\lambda t}(x_2^m u_{x_1}^2 + u_{x_2}^2)\nu_0 \, ds + \int\limits_D e^{-\lambda t}(\lambda x_2^m u_{x_1}^2 + \lambda u_{x_2}^2) dD. \end{gathered} \tag{9}$$

It can be easily seen that

$$\begin{gathered} u\big|_{S_0} = u_t\big|_{S_0} = \nu_0\big|_{S_0} = 0, \quad n\big|_{S_1} = \Big(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\Big), \\ \nu_0\big|_{S_2} > 0, \quad (\nu_0^2 - x_2^m \nu_1^2 - \nu_2^2)\big|_{S_1 \cup S_2} = 0. \end{gathered} \tag{10}$$

Taking into account (8)–(10), multiplying both parts of equation (1) by $2e^{-\lambda t} u_t$, where $F = Lu$, and integrating the obtained expression with respect to D, we obtain

$$\begin{gathered} 2(Lu, e^{-\lambda t} u_t)_{L_2(D)} = \int\limits_D e^{-\lambda t} \big[\lambda(u_t^2 + x_2^m u_{x_1}^2 + u_{x_2}^2) + \\ +2(a_1 u_{x_1} + a_2 u_{x_2} + a_3 u_t + a_4 u) u_t\big] dD + \int\limits_{S_1 \cup S_2} e^{-\lambda t} \nu_0^{-1} \big[x_2^m(\nu_0 u_{x_1} - \\ -\nu_1 u_t)^2 + (\nu_0 u_{x_2} - \nu_2 u_t)^2 + (\nu_0^2 - x_2^m \nu_1^2 - \nu_2^2) u_t^2\big] ds \geq \\ \geq \int\limits_D e^{-\lambda t} \big[\lambda(u_t^2 + x_2^m u_{x_1}^2 + u_{x_2}^2) + 2(a_1 u_{x_1} + a_2 u_{x_2} + a_3 u_t + \\ +a_4 u) u_t\big] dD - \sqrt{2} \int\limits_{S_1} e^{-\lambda t} \Big[x_2^m u_{x_1}^2 + \Big(\frac{\partial u}{\partial N}\Big)^2\Big] ds. \end{gathered} \tag{11}$$

Owing to condition (6) and the structure of D, one can easily verify that the inequality

$$\int_D u^2 dD \le c_0 \int_D u_{x_2}^2 dD \tag{12}$$

is valid for some $c_0 = \text{const} > 0$ not depending on $u \in W_2^2(D)$.

Using inequality (5), we can show that

$$|2a_1 u_{x_1} u_t| \le 2M(x_2^{\frac{m}{2}} u_{x_1}) u_t \le M(x_2^m u_{x_1}^2 + u_t^2). \tag{13}$$

By virtue of (12) and (13), from (11) for sufficiently large λ we get

$$2(Lu, e^{-\lambda t} u_t)_{L_2(D)} \ge c_1 \int_D (u_t^2 + x_2^m u_{x_1}^2 + u_{x_2}^2 + u^2) dD - \\ - c_2 \int_{S_1} \Big[x_2^m u_{x_1}^2 + \Big(\frac{\partial u}{\partial N}\Big)^2 \Big] ds, \tag{14}$$

where the positive constants c_1 and c_2 do not depend on u; note that depending on λ, the constant c_1 can be chosen arbitrarily large. Therefore estimate (7) follows obviously from (14). $\square$

Remark 1. Since for the principal part of the operator L the derivative with respect to the conormal $\frac{\partial}{\partial N} = \nu_0 \frac{\partial}{\partial t} - x_2^m \nu_1 \frac{\partial}{\partial x_1} - \nu_2 \frac{\partial}{\partial x_2}$ is the inner differential operator on the characteristic surfaces of equation (1), by (2) and (4) we have

$$\frac{\partial u}{\partial N}\Big|_{S_1} = 0, \quad \frac{\partial v}{\partial N}\Big|_{S_2} = 0 \tag{15}$$

for the functions $u \in E$ and $v \in E^*$.

Lemma 2. *Let condition* (5) *be fulfilled. Then for all functions* $u \in E$, $v \in E^*$ *the inequalities*

$$\|Lu\|_{W_-^*} \le c_1 \|u\|_{W_+}, \tag{16}$$

$$\|L^* v\|_{W_-} \le c_2 \|v\|_{W_+^*} \tag{17}$$

are fulfilled, where the positive constants c_1 *and* c_2 *do not depend respectively on* u *and* v, $\|\cdot\|_{W_+} = \|\cdot\|_{W_+^*} = \|\cdot\|_{1,+}$.

Proof. According to the definition of a negative norm and because of (2), (4), and (15), we have

$$\|Lu\|_{W^*_-} = \sup_{v\in W^*_+} \|v\|^{-1}_{W^*_+}(Lu,v)_{L_2(D)} = \sup_{v\in E^*} \|v\|^{-1}_{W^*_+}(Lu,v)_{L_2(D)} =$$

$$= \sup_{v\in E^*} \|v\|^{-1}_{W^*_+} \int_{\partial D} [u_t v\nu_0 - x_2^m u_{x_1} v\nu_1 - u_{x_2} v\nu_2]ds + \sup_{v\in E^*} \|v\|^{-1}_{W^*_+} \int_D [-u_t v_t +$$

$$+x_2^m u_{x_1} v_{x_1} + u_{x_2} v_{x_2} + a_1 u_{x_1} v + a_2 u_{x_2} v + a_3 u_t v + a_4 uv]dD =$$

$$= \sup_{v\in E^*} \|v\|^{-1}_{W^*_+} \int_{S_1\cup S_2} \frac{\partial}{\partial N} v\, ds + \sup_{v\in E^*} \|v\|^{-1}_{W^*_+} \int_D [-u_t v_t + x_2^m u_{x_1} v_{x_1} +$$

$$+u_{x_2} v_{x_2} + a_1 u_{x_1} v + a_2 u_{x_2} v + a_3 u_t v + a_4 uv]dD = \sup_{v\in E^*} \|v\|^{-1}_{W^*_+} \int_D [-u_t v_t +$$

$$+x_2^m u_{x_1} v_{x_1} + u_{x_2} v_{x_2} + a_1 u_{x_1} v + a_2 u_{x_2} v + a_3 u_t v + a_4 uv]dD. \quad (18)$$

Due to condition (5) and the Schwartz inequality we obtain

$$\Big|\int_D [-u_t v_t + x_2^m u_{x_1} v_{x_1} + u_{x_2} v_{x_2}]dD\Big| \le 3\Big[\int_D (u_t^2 + x_2^m u_{x_1}^2 + u_{x_2}^2)dD\Big]^{\frac12} \times$$

$$\times\Big[\int_D (v_t^2 + x_2^m v_{x_1}^2 + v_{x_2}^2)dD\Big]^{\frac12} \le 3\|u\|_{W_+}\|v\|_{W^*_+}, \quad (19)$$

$$\Big|\int_D (a_1 u_{x_1} v + a_2 u_{x_2} v + a_3 u_t v + a_4 uv]dD\Big| \le M\Big(\int_D x_2^m u_{x_1}^2 dD)^{\frac12}\|v\|_{L_2(D)} +$$

$$+\sup_D |a_2|\ \|u_{x_2}\|_{L_2(D)}\|v\|_{L_2(D)} + \sup_D |a_3|\ \|u_t\|_{L_2(D)}\|v\|_{L_2(D)} +$$

$$+\sup_D |a_4|\ \|u\|_{L_2(D)}\|v\|_{L_2(D)} \le \Big(M + \sum_{i=1}^4 \sup_D |a_i|\Big)\|u\|_{W_+}\|v\|_{W^*_+}. \quad (20)$$

From (18)–(20) it follows that

$$\|Lu\|_{W^*_-} \le \Big(3 + M + \sum_{i=2}^4 \sup_D |a_i|\Big) \sup_{v\in E^*} \|v\|^{-1}_{W^*_+}\|u\|_{W_+}\|v\|_{W^*_+} =$$

$$= \Big(3 + M + \sum_{i=2}^4 \sup_D |a_i|\Big)\|u\|_{W_+},$$

which proves inequality (16). Thus Lemma 2 is completely proved, since the proof of inequality (17) repeats that of the inequality (16). □

Remark 2. By inequalities (16) and (17), the operator $L : W_+ \to W_-^*(L : W_+^* \to W_-)$ with a dense domain of definition $E(E^*)$ admits a closure which is a continuous operator from $W_+(W_+^*)$ to $W_-^*(W_-)$. Denoting this closure as previously by $L(L^*)$, we note that it is defined on the whole Hilbert space $W_+(W_+^*)$.

Lemma 3. *Problems* (1), (2) *and* (3), (4) *are self-conjugate, i.e., the equality*

$$(Lu, v) = (u, L^*v) \tag{21}$$

holds for every $u \in W_+$ *and* $v \in W_+^*$.

Proof. By Remark 2 it suffices to prove equality (21) when $u \in E$ and $v \in E^*$. We have

$$\begin{aligned}
(Lu, v) &= (Lu, v)_{L_2(D)} = \int\limits_{\partial D} [u_t v\nu_0 - x_2^m u_{x_1} v\nu_1 - u_{x_2} v\nu_2] ds + \\
&+ \int\limits_{\partial D} [a_1\nu_1 + a_2\nu_2 + a_3\nu_0] uv\, ds + \int\limits_{D} \big[- u_t v_t + x_2^m u_{x_1} v_{x_1} + u_{x_2} v_{x_2} - \\
&- u(a_1 v)_{x_1} - u(a_2 v)_{x_2} - u(a_3 v)_t + a_4 uv\big] dD = \int\limits_{\partial D} [u_t v\nu_0 - x_2^m u_{x_1} v\nu_1 - \\
&- u_{x_2} v\nu_2] ds + \int\limits_{\partial D} [a_1\nu_1 + a_2\nu_2 + a_3\nu_0] uv\, ds - \int\limits_{\partial D} [uv_t\nu_0 - x_2^m uv_{x_1}\nu_1 - \\
&- uv_{x_2}\nu_2] ds + \int\limits_{D} \big[uv_{tt} - x_2^m uv_{x_1x_1} - uv_{x_2x_2} - u(a_1 v)_{x_1} - \\
&- u(a_2 v)_{x_2} - u(a_3 v)_t + a_4 uv\big] dD = \int\limits_{\partial D} \Big[\Big(v\frac{\partial u}{\partial N} - u\frac{\partial v}{\partial N}\Big) + \\
&+ (a_1\nu_1 + a_2\nu_2 + a_3\nu_3] uv\Big] ds + (u, L^*v)_{L_2(D)}.
\end{aligned} \tag{22}$$

By (2), (4), and (15), equality (21) follows directly from (22), which proves Lemma 3. □

Lemma 4. *Let condition* (5) *be fulfilled. Then for every* $u \in W_+$ *we have the inequality*

$$c\|u\|_{L_2(D)} \le \|Lu\|_{W_-^*} \tag{23}$$

with the positive constant c *not depending on* u.

Proof. By Remark 2 it suffices to prove the inequality (23) when $u \in E$. If $u \in E$, then it can be easily verified that the function

$$v(x_1,x_2,t) = \int\limits_{t}^{\varphi_2(x_1,x_2)} e^{-\lambda\tau} u(x_1,x_2,\tau)d\tau, \quad \lambda = \text{const} > 0,$$

where $t = \varphi_2(x_1,x_2)$ is the equation of the characteristic surface S_2, belongs to the space E^*, and the equalities

$$v_t(x_1,x_2,t) = -e^{-\lambda t} u(x_1,x_2,t), \quad u(x_1,x_2,t) = -e^{\lambda t} v_t(x_1,x_2,t) \qquad (24)$$

are valid. By (10), (15) and (24) we have

$$(Lu,v)_{L_2(D)} = \int\limits_{D} \Big[v\frac{\partial u}{\partial N} + (a_1\nu_1 + a_2\nu_2 + a_3\nu_0)uv \Big] ds +$$

$$+ \int\limits_{D} [-u_t v_t + x_2^m u_{x_1} v_{x_1} + u_{x_2} v_{x_2} - u a_{1x_1} v - u a_1 v_{x_1} - u a_{2x_2} v -$$

$$-u a_2 v_{x_2} - u a_{3t} v - u a_3 v_t + a_4 uv]dD = \int\limits_{D} e^{-\lambda t} u_t u \, dD +$$

$$+ \int\limits_{D} e^{\lambda t} [-x_2^m v_{x_1 t} v_{x_1} - v_{x_2 t} v_{x_2} + a_{1x_1} v_t v + a_1 v_t v_{x_1} + a_{2x_2} v_t v +$$

$$+a_2 v_t v_{x_2} + a_{3t} v_t v + a_3 v_t^2 - a_4 v_t v]dD. \qquad (25)$$

Analogously to (8) and (9), because of (2) we have

$$\int\limits_{D} e^{-\lambda t} u_t u \, dD = \frac{1}{2} \int\limits_{\partial D} e^{-\lambda t} u^2 \nu_0 \, ds + \frac{1}{2} \int\limits_{D} e^{-\lambda t} \lambda u^2 \, dD =$$

$$= \frac{1}{2} \int\limits_{S_2} e^{-\lambda t} u^2 \nu_0 \, ds + \frac{1}{2} \int\limits_{D} e^{\lambda t} \lambda v_t^2 \, dD =$$

$$= \frac{1}{2} \int\limits_{S_2} e^{\lambda t} v_t^2 \nu_0 \, ds + \frac{1}{2} \int\limits_{D} e^{\lambda t} \lambda v_t^2 \, dD, \qquad (26)$$

$$\int\limits_{D} e^{\lambda t} [-x_2^m v_{x_1 t} v_{x_1} - v_{x_2 t} v_{x_2}]dD = -\frac{1}{2} \int\limits_{\partial D} e^{\lambda t} [x_2^m v_{x_1}^2 + v_{x_2}^2] \nu_0 \, ds +$$

$$+\frac{1}{2} \int\limits_{D} e^{\lambda t} \lambda [x_2^m v_{x_1}^2 + v_{x_2}^2] dD. \qquad (27)$$

Since $v\big|_{S_2}=0$, for some α on S_2 we have

$$v_t=\alpha\nu_0,\quad v_{x_1}=\alpha\nu_1,\quad v_{x_2}=\alpha\nu_2.$$

Therefore, since the surface S_2 is characteristic, we have

$$(v_t^2-x_2^m v_{x_1}^2-v_{x_2}^2)\big|_{S_2}=\alpha^2(\nu_0^2-x_2^m\nu_1^2-\nu_2^2)\big|_{S_2}=0. \tag{28}$$

Due to the fact that $\nu_0\big|_{S_0}=0$, $\nu_0\big|_{S_1}<0$ and owing to equalities (4) and (28), we find that

$$\begin{aligned}
&\frac{1}{2}\int\limits_{S_2} e^{\lambda t}v_t^2\nu_0\,ds-\frac{1}{2}\int\limits_{\partial D} e^{\lambda t}[x_2^m v_{x_1}^2+v_{x_2}^2]\nu_0\,ds=\\
&=\frac{1}{2}\int\limits_{S_2} e^{\lambda t}v_t^2\nu_0\,ds-\frac{1}{2}\int\limits_{S_1} e^{\lambda t}[x_2^m v_{x_1}^2+v_{x_2}^2]\nu_0\,ds-\\
&-\frac{1}{2}\int\limits_{S_2} e^{\lambda t}[x_2^m v_{x_1}^2+v_{x_2}^2]\nu_0\,ds\geq\frac{1}{2}\int\limits_{S_2} e^{\lambda t}v_t^2\nu_0\,ds-\frac{1}{2}\int\limits_{S_2} e^{\lambda t}[x_2^m v_{x_1}^2+\\
&+v_{x_2}^2]\nu_0\,ds=\frac{1}{2}\int\limits_{S_2} e^{\lambda t}[v_t^2-x_2^m v_{x_1}^2-v_{x_2}^2]\nu_0\,ds=0.
\end{aligned}\tag{29}$$

Taking into consideration (26), (27), and (29), from (25) we get

$$\begin{aligned}
&(Lu,v)_{L_2(D)}=\frac{1}{2}\int\limits_{S_2} e^{\lambda t}v_t^2\nu_0\,ds+\frac{1}{2}\int\limits_{D} e^{\lambda t}\lambda v_t^2\,dD-\\
&-\frac{1}{2}\int\limits_{\partial D} e^{\lambda t}[x_2^m v_{x_1}^2+v_{x_2}^2]\nu_0\,ds+\frac{1}{2}\int\limits_{D} e^{\lambda t}\lambda[x_2^m v_{x_1}^2+v_{x_2}^2]dD+\\
&+\int\limits_{D} e^{\lambda t}\big[a_1v_tv_{x_1}+a_2v_tv_{x_2}+a_3v_t^2+(a_{1x_1}+a_{2x_2}+a_{3t}-a_4)v_tv\big]dD\geq\\
&\geq\frac{\lambda}{2}\int\limits_{D} e^{\lambda t}[v_t^2+x_2^m v_{x_1}^2+v_{x_2}^2]dD-\Big|\int\limits_{D} e^{\lambda t}[a_1v_tv_{x_1}+\\
&+a_2v_tv_{x_2}+a_3v_t^2+(a_{1x_1}+a_{2x_2}+a_{3t}-a_4)v_tv]dD\Big|.
\end{aligned}\tag{30}$$

Putting

$$\mu=\max\big(\sup_D|a_2|,\sup_D|a_3|,\sup_D|a_{1x_1}+a_{2x_2}+a_{3t}-a_4|\big)$$

and taking into account (5), we find that

$$\Big|\int\limits_D e^{\lambda t}[a_1 v_t v_{x_1} + a_2 v_t v_{x_2} + a_3 v_t^2 + (a_{1x_1} + a_{2x_2} + a_{3t} - a_4) v_t v] dD\Big| \leq$$

$$\leq \int\limits_D e^{\lambda t}\Big[M\frac{1}{2}(x_2^m v_{x_1}^2 + v_t^2) + \frac{\mu}{2}(v_{x_2}^2 + v_t^2) + \mu v_t^2 + \frac{\mu}{2}(v^2 + v_t^2)\Big] dD =$$

$$= \int\limits_D e^{\lambda t}\Big[\Big(\frac{1}{2}M + 2\mu\Big)v_t^2 + \frac{1}{2}M x_2^m v_{x_1}^2 + \frac{\mu}{2}v_{x_2}^2 + \frac{\mu}{2}v^2\Big] dD \leq$$

$$\leq \Big(\frac{1}{2}M + 2\mu\Big)\int\limits_D e^{\lambda t}[v_t^2 + x_2^m v_{x_1}^2 + v_{x_2}^2 + v^2] dD. \qquad (31)$$

Since the function $e^{\frac{\lambda}{2}t} v\big|_{S_0} = 0$, by virtue of inequality (12) we have

$$\int\limits_D e^{\lambda t} v^2 dD \leq c_0 \int\limits_D e^{\lambda t} v_{x_2}^2 dD \leq c_0 \int\limits_D e^{\lambda t}[v_t^2 + x_2^m v_{x_1}^2 + v_{x_2}^2] dD$$

and, consequently,

$$\int\limits_D e^{\lambda t}[v_t^2 + x_2^m v_{x_1}^2 + v_{x_2}^2] dD \geq \frac{1}{1+c_0}\int\limits_D e^{\lambda t}[v_t^2 + x_2^m v_{x_1}^2 + v_{x_2}^2 + v^2] dD. \quad (32)$$

By (31), (32), and (24) from (30) we obtain

$$(Lu, v)_{L_2(D)} \geq \frac{\lambda}{2(1+c_0)}\int\limits_D e^{\lambda t}[v_t^2 + x_2^m v_{x_1}^2 + v_{x_2}^2 + v^2] dD -$$

$$-\Big(\frac{1}{2}M + 2\mu\Big)\int\limits_D e^{\lambda t}[v_t^2 + x_2^m v_{x_1}^2 + v_{x_2}^2 + v^2] dD =$$

$$= \Big(\frac{\lambda}{2(1+c_0)} - \frac{1}{2}M - 2\mu\Big)\int\limits_D e^{\lambda t}[v_t^2 + x_2^m v_{x_1}^2 + v_{x_2}^2 + v^2] dD \geq$$

$$\geq \sigma\Big[\int\limits_D e^{\lambda t} v_t^2 dD\Big]^{\frac{1}{2}}\Big[\int\limits_D [v_t^2 + x_2^m v_{x_1}^2 + v_{x_2}^2 + v^2] dD\Big]^{\frac{1}{2}} =$$

$$= \sigma\Big[\int\limits_D e^{-\lambda t} u^2 dD\Big]^{\frac{1}{2}} \|v\|_{W_+^*} \geq \sigma \cdot \inf_D e^{-\lambda t}\|u\|_{L_2(D)}\|v\|_{W_+^*}, \qquad (33)$$

where $\sigma = \Big(\frac{\lambda}{2(1+c_0)} - \frac{1}{2}M - 2\mu\Big) > 0$ for sufficiently large λ, while $\inf_D e^{-\lambda t} = \text{const} > 0$ owing to the structure of the domain D.

If now we apply the generalized Schwartz inequality

$$(Lu, v)_{L_2(D)} \leq \|Lu\|_{W_-^*}\ \|v\|_{W_-^*}$$

to the left-hand side of (33), then after reduction by $\|v\|_{W_+^*}$ we obtain inequality (23), where $c = \sigma \inf_D e^{-\lambda t} = \text{const} > 0$. □

Lemma 5. *Let condition* (5) *be fulfilled. Then for every* $v \in W_+^*$ *the inequality*

$$c\|v\|_{L_2(D)} \leq \|L^* v\|_{W_-} \tag{34}$$

is valid for some $c = \text{const} > 0$ *not depending on* $v \in W_+^*$.

Proof. As in Lemma 4, by Remark 2 it suffices to prove the validity of inequality (34) for $v \in E^*$. Let $v \in E^*$ and introduce into consideration the function

$$u(x_1, x_2, t) = \int\limits_{\varphi_1(x_1,x_2)}^{t} e^{\lambda\tau} v(x_1, x_2, \tau) d\tau, \quad \lambda = \text{const} > 0,$$

where $t = \varphi_1(x_1, x_2)$ is the equation of the characteristic surface S_1. It is easily seen that the function $u(x_1, x_2, t)$ belongs to the class E, and we have the equalities

$$u_t(x_1, x_2, t) = e^{\lambda t} v(x_1, x_2, t), \quad v(x_1, x_2, t) = e^{-\lambda t} u_t(x_1, x_2, t). \tag{35}$$

Because of (10), (15), and (35) we have

$$(L^* v, u)_{L_2(D)} = \int\limits_{\partial D} \Big[u \frac{\partial v}{\partial N} - (a_1\nu_1 + a_2\nu_2 + a_3\nu_0)uv\Big] ds +$$

$$+ \int\limits_{D} [-v_t u_t + x_2^m v_{x_1} u_{x_1} + v_{x_2} u_{x_2} + a_1 v u_{x_1} + a_2 v u_{x_2} + a_3 v u_t + a_4 uv] dD =$$

$$= -\int\limits_{D} e^{\lambda t} v_t v\, dD + \int\limits_{D} e^{-\lambda t}[x_2^m u_{x_1 t} u_{x_1} + u_{x_2 t} u_{x_2}] dD +$$

$$+ \int\limits_{D} e^{-\lambda t}[a_1 u_{x_1} + a_2 u_{x_2} + a_3 u_t + a_4 u] u_t\, dD. \tag{36}$$

Similarly to (26)–(29) we can prove the equalities

$$-\int\limits_{D} e^{\lambda t} v_t v\, dD = -\frac{1}{2}\int\limits_{\partial D} e^{\lambda t} v^2 \nu_0\, ds + \frac{1}{2}\int\limits_{D} e^{\lambda t} \lambda v^2\, dD =$$

$$= -\frac{1}{2}\int\limits_{S_1} e^{-\lambda t} u_t^2 \nu_0 \, ds + \frac{1}{2}\int\limits_{D} e^{-\lambda t}\lambda u_t^2 \, dD, \tag{37}$$

$$\int\limits_{D} e^{-\lambda t}[x_2^m u_{x_1 t} u_{x_1} + u_{x_2 t} u_{x_2}] dD = \frac{1}{2}\int\limits_{\partial D} e^{-\lambda t}[x_2^m u_{x_1}^2 + u_{x_2}^2]\nu_0 \, ds +$$

$$+\frac{1}{2}\int\limits_{D} e^{-\lambda t}\lambda[x_2^m u_{x_1}^2 + u_{x_2}^2] dD, \tag{38}$$

$$(u_t^2 - x_2^m u_{x_1}^2 - u_{x_2}^2)\big|_{S_1} = 0, \tag{39}$$

$$-\frac{1}{2}\int\limits_{S_1} e^{-\lambda t} u_t^2 \nu_0 \, ds + \frac{1}{2}\int\limits_{\partial D} e^{-\lambda t}[x_2^m u_{x_1}^2 + u_{x_2}^2]\nu_0 \, ds =$$

$$= -\frac{1}{2}\int\limits_{S_1} e^{-\lambda t} u_t^2 \nu_0 \, ds + \frac{1}{2}\int\limits_{S_1} e^{-\lambda t}[x_2^m u_{x_1}^2 + u_{x_2}^2]\nu_0 \, ds +$$

$$+\frac{1}{2}\int\limits_{S_2} e^{-\lambda t}[x_2^m u_{x_1}^2 + u_{x_2}^2]\nu_0 \, ds \geq$$

$$\geq -\frac{1}{2}\int\limits_{S_1} e^{-\lambda t}[u_t^2 - x_2^m u_{x_1}^2 - u_{x_2}^2]\nu_0 \, ds = 0. \tag{40}$$

To obtain inequality (40) we have used the fact that $\nu_0|_{S_2} > 0$.

Owing to (37)–(40), from (36) we get

$$(L^* v, u)_{L_2(D)} \geq \frac{1}{2}\int\limits_{D} e^{-\lambda t}\lambda[u_t^2 + x_2^m u_{x_1}^2 + u_{x_2}^2] dD +$$

$$+\int\limits_{D} e^{-\lambda t}[a_1 u_{x_1} + a_2 u_{x_2} + a_3 u_t + a_4 u]u_t \, dD \geq \frac{\lambda}{2}\int\limits_{D} e^{-\lambda t}[u_t^2 + x_2^m u_{x_1}^2 +$$

$$+u_{x_2}^2] dD - \Big|\int\limits_{D} e^{-\lambda t}[a_1 u_{x_1} + a_2 u_{x_2} + a_3 u_t + a_4 u]u_t \, dD\Big|.$$

whence, as in obtaining inequality (33) from (30), we have

$$(L^* v, u)_{L_2(D)} \geq \Big[\frac{\lambda}{2(1+c_0)} - \Big(\frac{1}{2}M + \max_{i=2,3,4}\sup_D |a_i|\Big)\Big] \times$$
$$\times \inf_D e^{-\lambda t}\|v\|_{L_2(D)}\|u\|_{W_+}.$$

Inequality (34) follows directly from the above inequality for sufficiently large λ. □

Definition 1. If $F \in L_2(D)$, then the function u will be called a strong generalized solution of problem (1), (2) of the class W_+ if $u \in W_+$, and there exists a sequence of functions $u_n \in E$ such that $u_n \to u$ and $Lu_n \to F$ respectively in the spaces W_+ and W_-^* as $n \to \infty$, i.e.,

$$\lim_{n\to\infty} \|u_n - u\|_{W_+} = 0. \quad \lim_{n\to\infty} \|Lu_n - F\|_{W_-^*} = 0.$$

Definition 2. If $F \in W_-^*$, then the function u will be called a strong generalized solution of problem (1), (2) of the class L_2 if $u \in L_2(D)$, and there exists a sequence of functions $u_n \in E$ such that $u_n \to u$ and $Lu_n \to F$ respectively in the spaces $L_2(D)$ and W_-^* as $n \to \infty$, i.e.,

$$\lim_{n\to\infty} \|u_n - u\|_{L_2(D)} = 0. \quad \lim_{n\to\infty} \|Lu_n - F\|_{W_-^*} = 0.$$

According to the results of [11], the theorems below are consequences of Lemmas 2–5.

Theorem 1. *Let condition* (5) *be fulfilled. Then for every* $F \in W_-^*$ *there exists a unique strong generalized solution* u *of problem* (1), (2) *of the class* L_2 *for which the estimate*

$$\|u\|_{L_2(D)} \le c\|F\|_{W_-^*}, \tag{41}$$

with a positive constant c *not depending on* F, *is valid.*

Theorem 2. *Let condition* (5) *be fulfilled. Then for every* $F \in L_2(D)$ *there exists a unique strong generalized solution* u *of problem* (1), (2) *of the class* W_+ *for which estimate* (41) *is valid.*

Proof. The existence of a solution of problem (1), (2) in Theorem 2 follows, for example, from the arguments as follows. By virtue of inequality (34), the functional $(F, v)_{L_2(D)}$ can be regarded as a linear continuous functional of L^*v, where $v \in E^*$, $F \in L_2(D)$. Indeed, using this inequality, we have

$$|(F, v)_{L_2(D)}| \le \|F\|_{L_2(D)} \|v\|_{L_2(D)} \le c^* \|L^*v\|_{W_-}, \quad c^* = \text{const} > 0.$$

By the Khan–Banach theorem, this functional can be linearly and continuously extended into the whole space W_-. Following the theorem on a general type of a linear continuous functional over W_-, there exists a function $u \in W_+$ such that

$$(u, L^*)_{L_2(D)} = (F, v)_{L_2(D)}, \quad v \in E^*. \tag{42}$$

Equality (42) means that u is a weak generalized solution of the problem (1), (2). Let us now show that this solution is also a strong generalized solution of problem (1), (2) of the class W_+.

Since the space E is dense in W_+, there exists a sequence $u_n \in E$ of functions such that

$$\lim_{n\to\infty} \|u_n - u\|_{W_+} = 0. \tag{43}$$

Using equalities (21) and (42), we have

$$(u_n - u, L^* v)_{L_2(D)} = (Lu_n - F, v)_{L_2(D)}. \tag{44}$$

Now, according to the generalized Schwartz inequality,

$$\left|(u_n - u, L^* v)_{L_2(D)}\right| \le \|u_n - u\|_{W_+} \|L^* v\|_{W_-}. \tag{45}$$

It follows from (43)–(45) that in the space W_-^* the sequence Lu_n of functions converges weakly to the function F. But since this sequence, because of (16) and (43), converges in the norm of the space W_-^*, we obtain

$$\lim_{n\to\infty} \|Lu_n - F\|_{W_-^*} = 0.$$

Consequently, the function u is a strong generalized solution of problem (1), (2) of the class W_+.

This fact can be proved in a different way. Indeed, using equalities (21) and (42) and inequality (17), we have

$$\begin{aligned}
\|Lu_n - F\|_{W_-^*} &= \sup_{v\in W_+^*} \|v\|_{W_+^*}^{-1} (Lu_n - F, v)_{L_2(D)} = \\
&= \sup_{v\in E^*} \|v\|_{W_+^*}^{-1} \left[(Lu_n, v)_{L_2(D)} - (F, v)_{L_2(D)}\right] = \\
&= \sup_{v\in E^*} \|v\|_{W_+^*}^{-1} \left[(u_n, L^* v)_{L_2(D)} - (u, L^* v)_{L_2(D)}\right] = \\
&= \sup_{v\in E^*} \|v\|_{W_+^*}^{-1} (u_n - u, L^* v)_{L_2(D)} \le \\
&\le \sup_{v\in E^*} \|v\|_{W_+^*}^{-1} \|u_n - u\|_{W_+} \|L^* v\|_{W_-} \le \\
&\le \sup_{v\in E^*} \|v\|_{W_+^*}^{-1} \|u_n - u\|_{W_+} c_2 \|v\|_{W_+^*} = c_2 \|u_n - u\|_{W_+},
\end{aligned}$$

whence $\lim_{n\to\infty} \|Lu_n - F\|_{W_-^*} = 0$.

The uniqueness of a strong generalized solution of problem (1), (2) of the class W_+ in Theorem 2 as well as estimate (41) follow from inequality (23).

As for Theorem 1, it can be proved as follows. Since the space $L_2(D)$ is dense in the space W_-^*, for every element $F \in W_-^*$ there exists a sequence $F_n \in L_2(D)$ of functions such that $\lim\limits_{n\to\infty} \|F_n - F\|_{W_-^*} = 0$. According to Theorem 2, for every function $F_n \in L_2(D)$ there exists a unique strong generalized solution u_n of problem (1), (2) of the class W_+. Furthermore, using inequality (23) and passing to the limit, we obtain the existence and the uniqueness of a strong generalized solution of problem (1), (2) of the class L_2 as well as estimate (41). □

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(Received 23.11.1994)

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