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# ON THE ABSOLUTE CONVERGENCE OF FOURIER SERIES

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ABSTRACT. The necessary and sufficient conditions of the absolute convergence of a trigonometric Fourier series are established for continuous $2\pi$-periodic functions which in $[0,2\pi]$ have a finite number of intervals of convexity, and whose $n$th Fourier coefficients are $O(\omega(1/n;f)/n)$, where $\omega(\delta;f)$ is the continuity modulus of the function $f$.

Let $\omega$ be an arbitrary modulus of continuity, i.e., a nondecreasing function continuous on $[0,1]$, $\omega(0)=0$ and $\omega(\delta_1+\delta_2)\le\omega(\delta_1)+\omega(\delta_2)$. As usual, denote by $H^\omega$ the class of all functions $f$ continuous on $[0,2\pi]$ for which

$$\omega(\delta;f)=\sup_{|x_1-x_2|\le\delta}|f(x_1)-f(x_2)|=O(\omega(\delta)),\quad 0\le\delta\le 1$$

(see, for instance, [5, Ch. 3, pp. 150, 157]).

Let $M$ be the class of all continuous $2\pi$-periodic functions $f$ for which there exists a partitioning of the segment $[0,2\pi]$ by the points $0=x_1(f)<\cdots<x_{m+1}(f)=2\pi$ such that $f$ is convex or concave on each segment $[x_k(f),x_{x+1}(f)]$, $k=1,\dots m$.

The Fourier coefficients of a function $f$ with respect to the trigonometric system will be denoted by $a_n=a_n(f)$, $b_n=b_n(f)$.

Problems pertaining to the absolute convergence of Fourier series have been studied quite completely (see, for instance, the monographs of Bari [2, Ch. 9], Zygmund [3, Ch. 6], Kahane [4, Ch. 2], and the survey by Guter and Ulyanov [5, p. 391]).

This paper deals with some problems of the absolute convergence of trigonometric Fourier series of a function from the class $M$.

The following facts are well known:

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(1) The Fourier series of any $2\pi$-periodic continuous even function, convex on $[0, 2\pi]$, converges absolutely ([4, Ch. 2]).

(2) Let $f$ be an odd function convex on $[0, +\infty)$. Then $f \in A^{loc}$ if and only if $\int_0^1 f(t)\frac{dt}{t} < \infty$ [4, Ch. 2]. Here $A^{loc}$ is the set of all functions $f$, continuous on $(-\infty, +\infty)$, for which every point can be encircled by an interval on which $f = g$, where $g$ is a function, continuous on $[0, 2\pi]$, whose Fourier series converges absolutely.

We have obtained the following results:

**Theorem 1.** *If $f \in M$, then for the absolute convergence of the Fourier series of the function $f$ it is necessary and sufficient that*

$$\sum_{n=1}^{\infty} \Big| f\Big(x_k(f) + \frac{1}{n}\Big) - f\Big(x_k(f) - \frac{1}{n}\Big)\Big| \frac{1}{n} < +\infty, \quad k = 1, \dots, m.$$

**Theorem 2.**

(*a*) *Let $f \in M$; then*

$$a_n(f) = O\Big(\omega\Big(\frac{1}{n}; f\Big)\frac{1}{n}\Big), \quad b_n(f) = O\Big(\omega\Big(\frac{1}{n}; f\Big)\frac{1}{n}\Big).$$

(*b*) *If $\sum_{n=1}^{\infty} \omega(\frac{1}{n})\frac{1}{n} = +\infty$, then in the class $H^\omega \cap M$ there exists a a function whose Fourier series does not converge absolutely.*

**Theorem 3.** *Let $f \in M$ and at least one of the following conditions be fulfilled:*

(1) *for any adjacent intervals $(x_k(f), x_{k+1}(f))$ and $(x_{k+1}(f), x_{k+2}(f))$, the function $f$ is convex on one of them and concave on the other;*

(2) *each point $x_k(f)$ can be encircled by an interval where the function $f$ is monotonous;*

(3) *for any $x_k(f)$, at least one of the two series*

$$\sum_{n=1}^{\infty} \Big| f\Big(x_k(f) + \frac{1}{n}\Big) - f(x_k(f))\Big| \frac{1}{n} \quad \text{and} \quad \sum_{n=1}^{\infty} \Big| f\Big(x_k(f) - \frac{1}{n}\Big) - f(x_k(f))\Big| \frac{1}{n}$$

*converges.*

*Then the convergence of the series $\sum_{n=1}^{\infty} \omega(\frac{1}{n}; f)\frac{1}{n}$ is the necessary and sufficient condition for the Fourier series of the function $f$ to converge absolutely.*

*Proof of Theorem* 1. Let $f_1$, $f_2$, $f$ be continuous $2\pi$-periodic functions defined as follows: $f_1$ is convex or concave on a segment $[0, \pi]$, $f_1(0) = f_1(\pi) = 0$, linear on $[1, \pi]$, $f_1(x) = 0$, for $x \in [-\pi, 0]$; $f_2$ is convex or concave on $[-\pi, 0]$, $f_2(-\pi) = f_2(0) = 0$, linear on $[-\pi, -1]$, $f_1(x) = 0$ for $x \in [0, \pi]$; $f = f_1 + f_2$.

The theorem will be proved by showing that for the Fourier series of $f$ to converge it is necessary and sufficient that

$$\sum_{n=1}^{\infty}\Big|f\Big(\frac{1}{n}\Big)-f\Big(-\frac{1}{n}\Big)\Big|\frac{1}{n}<+\infty.$$

This follows from Wiener's theorem and from the following facts: If the function $f$ is convex or concave on a segment $[a,b]$, then $f\in \operatorname{Lip}1$ on any segment $[c,d]$ entirely lying inside $[a,b]$, and the Fourier series of the functions $f(x)$ and $f(x+c)$ simultaneously converge or diverge absolutely.

The function $f_1$ is convex on $[0,\pi]$ and continuous, which means that it is absolutely continuous so that one can apply integration by parts and the Newton–Leibniz formulas to obtain $a_n(f)=a_n(f_1)+a_n(f_2)$.

$$a_n(f_1)=\frac{1}{\pi}\int_0^{2\pi} f_1(t)\cos nt\,dt=\frac{1}{\pi}\int_0^{2\pi} f_1(t)d\frac{\sin nt}{n}=\frac{1}{\pi}\Big(f_1(t)\frac{\sin nt}{n}\Big)_0^{2\pi}-$$

$$-\frac{1}{\pi n}\int_0^{2\pi} f_1'(t)\sin nt\,dt=\frac{-1}{\pi n}\int_0^{2\pi} f_1'(t)\sin nt\,dt=\frac{-1}{\pi n}\int_0^{\pi} f_1'(t)\sin nt\,dt=$$

$$=-\frac{1}{\pi n}\int_0^{1/n} f_1'(t)\sin nt\,dt-\frac{1}{\pi n}\int_{1/n}^{1} f_1'(t)\sin nt\,dt-\frac{1}{\pi n}\int_1^{\pi} f_1'(t)\sin nt\,dt.$$

The derivative $f'$ of the convex or concave function $f$ is monotonous and therefore, applying the second theorem of the mean value, we obtain

$$\Big|\frac{1}{\pi n}\int_{1/n}^{1} f_1'(t)\sin nt\,dt\Big|=\Big|\frac{1}{\pi n}f_1'\Big(\frac{1}{n}+0\Big)\int_{1/n}^{\xi}\sin nt+$$

$$+\frac{1}{\pi n}f_1'(1-0)\int_{\xi}^{1}\sin nt\,dt\Big|\le\frac{1}{\pi n^2}\Big|f_1'\Big(\frac{1}{n}+0\Big)\Big|+\frac{1}{\pi n^2}\big|f_1'(1-0)\big|$$

with $1/n<\xi<1$.

Wherever we come across expressions of the form $f'(x\pm 0)$, the left and right limits are considered with respect to the set at whose points the derivative $f'$ exists.

For the convex (concave) function $f$ we have the relation

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}\ge f'(x_2\pm 0)\ge\frac{f(x_3)-f(x_2)}{x_3-x_2}$$

$$\Big(\frac{f(x_2)-f(x_1)}{x_2-x_1}\le f'(x_2\pm 0)\le \frac{f(x_3)-f(x_2)}{x_3-x_2}\Big)$$

where $x_1<x_2<x_3$. Therefore

$$\Big|f_1'\Big(\frac{1}{n}+0\Big)\Big|\le \frac{f_1\big(\frac{1}{n}\big)-f_1\big(\frac{1}{n+1}\big)}{1/n-1/(n+1)}\le (n+1)^2\Big(f_1\Big(\frac{1}{n}\Big)-f_1\Big(\frac{1}{n+1}\Big)\Big).$$

Hence

$$\sum_{n=1}^{\infty}\Big|f_1'\Big(\frac{1}{n}+0\Big)\Big|\frac{1}{n^2}\le 2\sum_{n=1}^{\infty}\Big(f_1\Big(\frac{1}{n}\Big)-f_1\Big(\frac{1}{n+1}\Big)\Big)<+\infty.$$

Since $f_1\in \operatorname{Lip}1$ on the segment $[\varepsilon,\pi]$, we have $|f_1'(1-0)|\le M$ and $\sum_{n=1}^{\infty}|f_1'(1-0)|/n^2\le\sum_{n=1}^{\infty}M/n^2<+\infty$.

The function $f_1$ is linear on the segment $[1,\pi]$, i.e., $f_1'(t)=\cos nt=c$, so that $1/n\big|\int_1^{\pi}f_1'(t)\sin nt\big|\le c/n^2$.

Finally, $a_n(f_1)=-\frac{1}{\pi n}\int_0^{1/n}f_1'(t)\sin nt\,dt+\gamma_n$, where $\sum_{n=1}^{\infty}|\gamma_n|<+\infty$.

If we introduce the notation $I_n=\frac{-1}{\pi n}\int_0^{1/n}f_1'(t)\sin nt\,dt$, then $a_n(f_1)=I_n+\gamma_n$, $I_n=a_n(f_1)-\gamma_n$.

Since the function $f_1$ has a bounded variation, we have

$$f_1(x)=\frac{a_n(f_1)}{2}+\sum_{n=1}^{\infty}a_n(f_1)\cos nx+b_n(f_1)\sin nx.$$

By substituting here $x=0$ we obtain $\sum_{n=1}^{\infty}a_n(f_1)<\infty$. Therefore $\sum_{n=1}^{\infty}I_n=\sum_{n=1}^{\infty}(a_n(f_1)-\gamma_n)<\infty$.

One can easily verify that the values $I_n$ do not change their sign for sufficiently large $n$. Thus $\sum_{n=1}^{\infty}|I_n|<+\infty$. Since $|a_n(f_1)|\le|I_n|+|\gamma_n|$, we obtain $\sum_{n=1}^{\infty}|a_n(f_1)|<+\infty$.

In a similar manner we shall show that $\sum_{n=1}^{\infty}|a_n(f_2)|<\infty$. We have $|a_n(f)|=|a_n(f_1)+a_n(f_2)|\le|a_n(f_1)|+|a_n(f_2)|$ and $\sum_{n=1}^{\infty}|a_n(f)|<+\infty$.

Now consider the coefficients $b_n(f)$. We have $b_n(f)=b_n(f_1)+b_n(f_2)$,

$$b_n(f_1)=\frac{1}{\pi}\int_0^{2\pi}f_1(t)\sin nt\,dt=\frac{1}{\pi}\int_0^{2\pi}f_1(t)d\frac{\cos nt}{n}=\frac{1}{\pi}\Big(f_1(t)\frac{\cos nt}{n}\Big)\Big|_0^{2\pi}-$$

$$-\frac{1}{\pi n}\int_0^{2\pi}f_1'(t)\cos nt\,dt=-\frac{1}{\pi n}\int_0^{2\pi}f_1'(t)\cos nt\,dt=-\frac{1}{\pi n}\int_0^{\pi}f_1'(t)\cos nt\,dt=$$

$$=-\frac{1}{\pi n}\int_0^{1/n}f_1'(t)\cos nt\,dt-\frac{1}{\pi n}\int_{1/n}^{1}f_1'(t)\cos nt\,dt-\frac{1}{\pi n}\int_1^{\pi}f_1'(t)\cos nt\,dt.$$

The function $f_1$ is linear on the segment $[1, \pi]$, i.e., $f_1'(t) = const = C$, so that

$$\frac{1}{n}\Big|\int_1^\pi f_1'(t)\cos nt\,dt\Big| \le \frac{C}{n^2}.$$

Again applying the theorem of the mean, we obtain (with $1/n < \xi < 1$)

$$\Big|\frac{1}{n}\int_{1/n}^1 f_1'(t)\cos nt\,dt\Big| = \frac{1}{n}\Big|f_1'\Big(\frac{1}{n}+0\Big)\int_{1/n}^\xi \cos nt\,dt +$$

$$+f_1'(1-0)\int_\xi^1 \cos nt\,dt\Big| \le \frac{1}{n^2}\Big|f_1'\Big(\frac{1}{n}+0\Big)\Big| + \frac{1}{n^2}|f_1'(1-0)| < +\infty.$$

Therefore $b_n(f_1) = -\frac{1}{\pi n}\int_0^{1/n} f_1'(t)\cos nt\,dt + \gamma_n$, $\sum_{n=1}^\infty |\gamma_n| < +\infty$.

$$-\frac{1}{\pi n}\int_0^{1/n} f_1'(t)\cos nt\,dt = \frac{1}{\pi n}\int_0^{1/n} f_1'(t)(1-\cos nt - 1)dt =$$

$$= \frac{-1}{\pi n}\int_0^{1/n} f_1'(t)dt + \frac{1}{\pi n}\int_0^{1/n} f_1'(t)(1-\cos nt)dt =$$

$$= \frac{-1}{\pi n} f_1\Big(\frac{1}{n}\Big) + \frac{1}{\pi n}\int_0^{1/n} f_1'(t) 2\sin^2\frac{nt}{2}dt,$$

$$\Big|\frac{1}{\pi n}\int_0^{1/n} f_1'(t) 2\sin^2\frac{nt}{2}dt\Big| \le \frac{2}{\pi n}\int_0^{1/n} |f_1'(t)|\,|\sin nt|dt = 2|I_n|.$$

As we have seen above, $\sum_{n=1}^\infty |I_n| < +\infty$ and therefore

$$b_n(f_1) = -\frac{1}{\pi n} f_1\Big(\frac{1}{n}\Big) + C_n = \frac{-1}{\pi n} f\Big(\frac{1}{n}\Big) + C_n,$$

where $\sum_{n=1}^\infty |C_n| < +\infty$. In a similar manner it will be shown that

$$b_n(f_2) = \frac{1}{\pi n} f\Big(-\frac{1}{n}\Big) + P_n, \quad \text{where} \quad \sum_{n=1}^\infty |P_n| < +\infty.$$

Since $b_n(f) = b_n(f_1) + b_n(f_2)$, we have

$$b_n(f) = \frac{-1}{\pi n}\Big\{f\Big(\frac{1}{n}\Big) - f\Big(\frac{-1}{n}\Big)\Big\} + \gamma_n, \quad \sum_{n=1}^\infty |\gamma_n| < +\infty. \quad \square$$

*Proof of Theorem* 2.

(a) It is the well-known fact that estimates of Fourier coefficients can be derived using the integral modulus of continuity (see, for instance [3, Ch. 2])

$$|a_n| \le \sup_{|h|\le\frac{1}{n}} \frac{1}{\pi}\int\limits_0^{2\pi} \big|f(x+h)+f(x-h)-2f(x)\big|dx.$$

Applying the above inequality, we obtain

$$|a_n| \le \sup_{|h|\le\frac{1}{n}} \frac{1}{\pi}\int\limits_0^{2\pi} \big|f(x+h)+f(x-h)-2f(x)\big|dx =$$

$$= \sup_{|h|\le\frac{1}{n}} \frac{1}{\pi}\int\limits_0^{2\pi} \big|f(x+kh)+f(x+(k-2)h)-2f(x+(k-1)h)\big|dx =$$

$$= \frac{1}{\pi n}\sup_{|h|\le\frac{1}{n}} \int\limits_0^{2\pi} \sum_{k=1}^{n} \big|f(x+kh)+f(x+(k-2)h)-$$

$$-2f(x+(k-1)h)\big|dx = \frac{1}{\pi n}\sup_{|h|\le\frac{1}{n}} \int\limits_0^{2\pi} \Big(\sum_{k=1}^{n}|u_k-u_{k-1}|\Big)dx,$$

where $u_k = f(x+kh)-f(x+(k-1)h)$.

Convexity (concavity) of a function $f$ on some segment $[a,b]$ implies that

$$f(x+kh)+f(x-(k-2)h)-2f(x+(k-1)h) = u_k-u_{k-1} \le 0 \ \ (u_k-u_{k-1}\ge 0)$$

for $x+(k-2)h$, $x+(k-1)h$, $x+kh \in [a,b]$. Therefore on the segment $[0,2\pi]$ the values $u_k-u_{k-1}$ change their sign a finite number of times. Thus

$$|a_n(f)| \le \frac{1}{\pi n}\sup_{|h|\le\frac{1}{n}} \int\limits_0^{2\pi} \Big(\sum_{k=1}^{n}|u_k-u_{k-1}|\Big)dx \le$$

$$\le \frac{1}{\pi n}\sup_{|h|\le\frac{1}{n}} \int\limits_0^{2\pi} \Big|\sum_{k=1}^{n}u_k-u_{k-1}\Big|dx + \frac{C(f)}{n}\omega\Big(\frac{1}{n},f\Big) \le \frac{C_1(f)}{n}\omega\Big(\frac{1}{n},f\Big).$$

The proof of the estimate for $b_n(f)$ is similar.

(b) Let $\sum_{n=1}^{\infty}\omega(\frac{1}{n})\frac{1}{n} = +\infty$. By Stechkin's lemma (see [6]) there exists a convex modulus of continuity $\omega'(\delta)$ such that $H^{\omega}=H^{\omega'}$. Hence $\omega(\delta)$ can be regarded as convex function.

Consider a continuous function

$$f_0(t) = \begin{cases} \omega(t), & t \in [0,1], \\ \text{linear for} & [t \in [1, 2\pi], \quad f_0(t+2\pi) = f_0(t). \end{cases}$$

Clearly, $f_0 \in H^\omega \cap M$.

On the interval $[1, 2\pi]$ the function $f_0(t)$ is linear, $f(2\pi) = f(0) = 0$. Therefore $f_0\Big(-\frac{1}{n}\Big) = f\Big(2\pi - \frac{1}{n}\Big) = \frac{c_0}{n}$, where $c_0$ is some number . We have

$$\sum_{n=1}^{\infty} \Big| f_0\Big(\frac{1}{n}\Big) - f_0\Big(\frac{-1}{n}\Big)\Big| \frac{1}{n} \geq \sum_{n=1}^{\infty} \Big| f_0\Big(\frac{1}{n}\Big)\Big| - \Big| f_0\Big(\frac{-1}{n}\Big)\Big| \frac{1}{n} =$$

$$= \sum_{n=1}^{\infty} \Big| f_0\Big(\frac{1}{n}\Big)\Big| \frac{1}{n} - \sum_{n=1}^{\infty} \Big| f_0\Big(-\frac{1}{n}\Big)\Big| \frac{1}{n} = \sum_{n=1}^{\infty} \omega\Big(\frac{1}{n}\Big)\frac{1}{n} - \sum_{n=1}^{\infty} \frac{C_0}{n^2} = +\infty.$$

By virtue of Theorem 1 we see that the necessary condition for Fourier series to be absolutely convergent is not fulfilled. The Fourier series of $f_0$ does not converge absolutely. □

*Proof of Theorem* 3. The sufficiency follows from part (a) of the proof of Theorem 2.

First note that if the function is convex (concave) on $[a, b]$, then its modulus of continuity on $[a, b]$ equals

$$\max\big\{|f(a+\delta) - f(a)|, |f(b-\delta) - f(b)|\big\}.$$

Therefore

$$\omega(1/n, f) \leq \max_{1 \leq k \leq m} \big\{|f(x_k + 1/n) - f(x_k)| + |f(x_k - 1/n) - f(x_k)|\big\},$$

where $x_k \equiv x_k(f)$.

Hence it follows that if $\sum_{n=1}^{\infty} \omega(\frac{1}{n}, f)\frac{1}{n} = +\infty$, then for some $x_k$ we have

$$\sum_{n=1}^{\infty} \Big| f\Big(x_k + \frac{1}{n}\Big) - f(x_k)\Big| \frac{1}{n} = +\infty \quad \text{or} \quad \sum_{n=1}^{\infty} \Big| f\Big(x_k - \frac{1}{n}\Big) - f(x_k)\Big| \frac{1}{n} = +\infty.$$

For convenience we assume that

$$\sum_{n=1}^{\infty} \Big| f\Big(x_k + \frac{1}{n}\Big) - f(x_k)\Big| \frac{1}{n} = +\infty.$$

If the conditions (2) are fulfilled, then

$$\sum_{n=1}^{\infty} \Big| f\Big(x_k + \frac{1}{n}\Big) - f\Big(x_k - \frac{1}{n}\Big)\Big| \frac{1}{n} \geq \sum_{n=1}^{\infty} \Big| f\Big(x_k + \frac{1}{n}\Big) - f(x_k)\Big| \frac{1}{n} = +\infty,$$

i.e., by Theorem 1 the Fourier series of the function $f$ is not absolutely convergent.

If the conditions (3) are fulfilled, then

$$\sum_{n=1}^{\infty}\Big|f\Big(x_k+\frac{1}{n}\Big)-f\Big(x_k-\frac{1}{n}\Big)\Big|\frac{1}{n}=$$
$$=\sum_{n=1}^{\infty}\Big|f\Big(x_k+\frac{1}{n}\Big)-f(x_k)+f(x_k)-f\Big(x_k-\frac{1}{n}\Big)\Big|\frac{1}{n}\geq$$
$$\geq\sum_{n=1}^{\infty}\Big|f\Big(x_k+\frac{1}{n}\Big)-f(x_k)\Big|\frac{1}{n}-\sum_{n=1}^{\infty}\Big|f\Big(x_k-\frac{1}{n}\Big)-f(x_k)\Big|\frac{1}{n}=+\infty.$$

which proves the theorem under conditions (3).

If , however, the conditions of (1) are fulfilled, then, as one can easily verify, the conditions of (2) or (3) are fulfilled too. □

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