GEORGIAN MATHEMATICAL JOURNAL: Vol. 4, No. 4, 1997, 313-331

NON-ABELIAN COHOMOLOGY OF GROUPS

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ABSTRACT. Following Guin's approach to non-abelian cohomology [4] and, using the notion of a crossed bimodule, a second pointed set of cohomology is defined with coefficients in a crossed module, and Guin's six-term exact cohomology sequence is extended to a nine-term exact sequence of cohomology up to dimension 2.

INTRODUCTION

In this and forthcoming papers [1] we discuss the cohomology $H^*(G,A)$ of a group G with coefficients in a G-group A.When A is abelian this cohomology is the well-known classical cohomology of groups which can be defined as derived functors either of the functor $\mathrm{Hom}_{\mathbf{Z}[G]}(-,A)$ in the category of $\mathbf{Z}[G]$-modules or of the functor $\mathrm{Der}(-,A)$ in the category of groups acting on A. When A is non-abelian, a functorial pointed set of cohomology $H^1(G,A)$ not equipped with a group structure was defined in a natural way in [2]. Guin defined, in [3]–[4], a first cohomology group when the coefficient group is a crossed G-module and obtained a six-term exact sequence of cohomology for any short exact coefficient sequence of crossed G-modules.

Our approach to a non-abelian cohomology of groups follows Guin's cohomology theory of groups [3]–[4] which differs from the classical first non-abelian cohomology pointed set [2] and from the setting of various papers on non-abelian cohomology [5]–[7] extending the classical exact non-abelian cohomology sequence from lower dimensions [2] to higher dimensions.

Let G and R be groups and let (A,μ) be a crossed R-module. We introduce the notion of a crossed $G-R$-bimodule signifying an action of G on the crossed R-module (A,μ) and generalizing the notion of a crossed G-module.The group of derivations $\mathrm{Der}(G,(A,\mu))$ from G to (A,μ) is defined to obtain a pointed set of cohomology $H^2(G,A)$ when A is a crossed G-module. The group $\mathrm{Der}(G,(A,\mu))$ and the pointed set $H^2(G,A)$ coincide

1991 *Mathematics Subject Classification.* 18G50, 18G55.

Key words and phrases. Crossed module, derivation, simplicial kernel, crossed bimodule.

respectively with the group $\mathrm{Der}_G(G, A)$ of Guin [4] when (A, μ) is a crossed G-module and with the usual cohomology group when A is abelian. A coefficient short exact sequence of crossed G-modules gives rise to a nine-term exact sequence of cohomology which extends the six-term exact cohomology sequence of Guin [4]. In [1] these results are generalized when the coefficients are crossed bimodules; in that case $H^1(G, (A, \mu))$ is equipped with a partial product, and, finally, in [1] the definition of a pointed set of cohomology $H^n(G, (A, \mu))$ of a group G with coefficients in a crossed $G-R$-bimodule (A, μ) for all $n \geq 1$ is given.

All considered groups will be arbitrary (not necessarily commutative). An action of a group G on a group A means an action on the left of G on A by automorphisms and will be denoted by ${}^g a, g \in G, a \in A$. We assume that G acts on itself by conjugation. The center of a group G will be denoted by $Z(G)$. If the groups G and R act on a group A then the notation ${}^{gr}a$ means ${}^g({}^r a)$, $g \in G$, $r \in R$, $a \in A$.

1. Crossed Bimodules

A precrossed G-module (A, μ) consists of a group G acting on a group A and a homomorphism $\mu : A \longrightarrow G$ such that

$$\mu({}^g a) = g\mu(a)g^{-1}, \quad g \in G, \quad a \in A.$$

If in addition we have

$${}^{\mu(a)}a' = aa'a^{-1}$$

for $a, a' \in A$, then (A, μ) is a crossed G-module.

Definition 1. Let G, R, and A be groups. It will be said that (A, μ) is a precrossed $G-R$-bimodule if

(1) (A, μ) is a precrossed R-module,

(2) G acts on R and A,

(3) the homomorphism $\mu : A \longrightarrow R$ is a homomorphism of G-groups,

(4) ${}^{({}^g r)}a = {}^{grg^{-1}} a$ (compatibility condition) for $g \in G$, $r \in R$, $a \in A$. If in addition (A, μ) is a crossed R-module then (A, μ) will be called a crossed $G-R$-bimodule. If conditions (1)–(3) hold it will be said that the group G acts on the precrossed (resp. crossed) R-module (A, μ).

It is easy to see that any precrossed (resp. crossed) G-module (A, μ) is in a natural way a precrossed (resp. crossed) $G-G$-bimodule. It is also clear that if (A, μ) is a crossed $G-R$-bimodule and $f : G' \longrightarrow G$ is a homomorphism of groups then (A, μ) is a crossed $G'-R$-bimodule induced by f, G' acting on A and R via f.

A homomorphism $f : (A, \mu) \longrightarrow (B, \lambda)$ of precrossed (crossed) $G-R$-bimodules is a homomorphism of groups $f : A \longrightarrow B$ such that

(1) $f({}^r a) = {}^r f(a)$, $r \in R$, $a \in A$,

(2) $f({}^g a) = {}^g f(a)$, $g \in G$, $a \in A$,
(3) $\mu = \lambda f$.

2. The Group Der $(G, (A, \mu))$

Consider a crossed $G-R$-bimodule (A, μ).

Definition 2. Denote by $\mathrm{Der}(G, (A, \mu))$ the set of pairs (α, r) where α is a crossed homomorphism from G to A, i.e.,

$$\alpha(xy) = \alpha(x)^x\alpha(y), \quad x, y \in G,$$

and r is an element of R such that

$$\mu\alpha(x) = r\,{}^x r^{-1}, \quad x \in G.$$

This set will be called the set of derivations from G to (A, μ).

We define in $\mathrm{Der}(G, (A, \mu))$ a product by

$$(\alpha, r)(\beta, s) = (\alpha * \beta, rs),$$

where $(\alpha * \beta)(x) = {}^r\beta(x)\alpha(x)$, $x \in G$.

Proposition 3. *Under the aforementioned product* $\mathrm{Der}(G, (A, \mu))$ *becomes a group which coincides with the group* $\mathrm{Der}_G(G, A)$ *of Guin when* (A, μ) *is a crossed* G*-module viewed as a crossed* $G-G$*-bimodule.*

Proof. We have to show that $(\alpha * \beta, rs) \in \mathrm{Der}(G, (A, \mu))$. Put $\gamma = \alpha * \beta$. At first we prove that γ is a crossed homomorphism. In effect, we have

$$\begin{aligned}\gamma(xy) = {}^r\beta(xy)\alpha(xy) = {}^r(\beta(x)^x\beta(x))\alpha(x)\,{}^x\alpha(y) = \\ = {}^r\beta(x)^{rx}\beta(y)\alpha(x)^x\alpha(y).\end{aligned}$$

On the other hand,

$$\begin{aligned}\gamma(x)\,{}^x\gamma(y) = {}^r\beta(x)\alpha(x)\,{}^x({}^r\beta(y)\alpha(y)) = \\ = {}^r\beta(x)\alpha(x)\,{}^{xr}\beta(y)\,{}^x\alpha(y).\end{aligned}$$

For any $a \in A$ and $(\alpha, r) \in \mathrm{Der}(G, (A, \mu))$ the equality

$$\alpha(x)\,{}^{xr}a = {}^{rx}a\alpha(x), \quad x \in G, \tag{1}$$

holds, since $\alpha(x)\,{}^{xr}a\alpha(x)^{-1} = {}^{\mu\alpha(x)\cdot xr}a = {}^{r\cdot{}^x r^{-1}}({}^{xr}a) = {}^{rxr^{-1}\cdot x^{-1}\cdot xr}a = {}^{rx}a$.

It follows that $\gamma(xy) = \gamma(x)\,{}^x\gamma(y)$. Further,we have

$$\begin{aligned}\mu\gamma(x) = \mu({}^r\beta(x)\alpha(x)) = {}^r\mu\beta(x)\mu\alpha(x) = \\ = {}^r(s\,{}^x s^{-1})r\,{}^x r^{-1} = {}^r s\,{}^r({}^x s^{-1})r\,{}^x r^{-1} = {}^r s r^x s^{-1}\,{}^x r^{-1} = \\ = rs^x(rs)^{-1}.\end{aligned}$$

Therefore $(\alpha * \beta, rs) \in \mathrm{Der}(G,(A,\mu))$.

It is evident that this product is associative. It is also obvious that $(\alpha_0, 1) \in \mathrm{Der}(G,(A,\mu))$, where $\alpha_0(x) = 1$ for all $x \in G$, and $(\alpha_0, 1)$ is the unit of $\mathrm{Der}(G,(A,\mu))$.

Now we will show that for $(\alpha, r) \in \mathrm{Der}(G,(A,\mu))$ we have

$$^{r^{-1}\cdot x}a\, ^{r^{-1}}\alpha(x)^{-1} = {}^{r^{-1}}\alpha(x)^{-1}\, ^{xr^{-1}}a, \quad x \in G, \ \ a \in A. \tag{2}$$

Since $\mu(^{r^{-1}}\alpha(x)^{-1}) = r^{-1} \cdot \mu\alpha(x)^{-1} \cdot r = r^{-1}\, ^{x}r$, this implies

$$^{\mu(^{r^{-1}}\alpha(x)^{-1})}(^{xr^{-1}}a) = {}^{r^{-1x}r}(^{xr^{-1}}a) = ^{r^{-1}xrx^{-1}xr^{-1}}a = ^{r^{-1}x}a.$$

On the other hand,

$$^{\mu(^{r^{-1}}\alpha(x)^{-1})}(^{xr^{-1}}a) = r^{-1}\alpha(x) \cdot ^{xr^{-1}} a \cdot ^{r^{-1}} \alpha(x)$$

and equality (2) is proved.

For $(\alpha, r) \in \mathrm{Der}(G,(A,\mu))$ take the pair $(\overline{\alpha}, r^{-1})$ where $\overline{\alpha}(x) = ^{r^{-1}} \alpha(x)^{-1}$, $x \in G$. It will be shown that $(\overline{\alpha}, r^{-1}) \in \mathrm{Der}(G,(A,\mu))$. We have

$$\begin{aligned}\overline{\alpha}(xy) = {}& ^{r^{-1}}\alpha(xy)^{-1} = {}^{r^{-1}}(^{x}\alpha(y)^{-1} \cdot \alpha(x)^{-1}) = \\ = {}& ^{r^{-1}x}\alpha(y)^{-1}\, ^{r^{-1}}\alpha(x)^{-1}\end{aligned}$$

and $\overline{\alpha}(x) \cdot^{x} \overline{\alpha}(y) = {}^{r^{-1}}\alpha(x)^{-1}\, ^{xr^{-1}}\alpha(y)^{-1}$.

By (2) one gets $\overline{\alpha}(xy) = \overline{\alpha}(x)\, ^{x}\overline{\alpha}(y)$, i.e., $\overline{\alpha}$ is a crossed homomorphism.

We also have

$$\mu\overline{\alpha}(x) = \mu(^{r^{-1}} \alpha(x)^{-1}) = r^{-1}\mu\alpha(x)^{-1}r = r^{-1x}r \cdot r^{-1} \cdot r = r^{-1x}r.$$

Therefore $(\overline{\alpha}, r^{-1}) \in \mathrm{Der}(G,(A,\mu))$.

It is easy to check that

$$(\alpha, r)(\overline{\alpha}, r^{-1}) = (\overline{\alpha}, r^{-1})(\alpha, r) = (\alpha_0, 1).$$

We conclude that $\mathrm{Der}(G,(A,\mu))$ is a group. If (A,μ) is a crossed G-module and $(\alpha, g) \in \mathrm{Der}(G,(A,\mu))$ then $\mu\alpha(x) = g\, ^{x}g^{-1} = gxg^{-1}x^{-1}$.

In $\mathrm{Der}_G(G,A)$ this product was defined by Guin [4] and it follows that the group $\mathrm{Der}(G,(A,\mu))$ coincides with $\mathrm{Der}_G(G,A)$ when (A,μ) is a crossed G-module. □

If (A,μ) is a precrossed R-module and (B,λ) is a crossed R-module then (B,λ) is a crossed $A-R$-bimodule induced by μ and the group $\mathrm{Der}_G(A,B)$ of Guin [4] is the group $\mathrm{Der}(A,(B,\lambda))$.

It is clear that a homomorphism of $G-R$-bimodules $f : (A,\mu) \longrightarrow (B,\lambda)$ induces a homomorphism

$$f^* : \mathrm{Der}(G,(A,\mu)) \longrightarrow \mathrm{Der}(G,(B,\lambda))$$

given by $(\alpha, r) \longmapsto (f\alpha, r)$.

There is an action of G on $\mathrm{Der}(G,(A,\mu))$ defined by

$$^{g}(\alpha, r) = (\widetilde{\alpha}, {}^{g}r), \quad g \in G, \ \ r \in R,$$

with $\widetilde{\alpha}(x) = {}^{g}\alpha({}^{g^{-1}}x)$, $x \in G$.

In effect,we have

$$\widetilde{\alpha}(xy) = {}^{g}\alpha({}^{g^{-1}}(xy)) = {}^{g}\alpha({}^{g^{-1}}x\,{}^{g^{-1}}y) = {}^{g}\alpha({}^{g^{-1}}x)\,{}^{xg}\alpha({}^{g^{-1}}y) = \\ = \widetilde{\alpha}(x)\,{}^{x}\widetilde{\alpha}(y)$$

and $\mu\widetilde{\alpha}(x) = \mu({}^{g}\alpha({}^{g^{-1}}x)) = {}^{g}\mu\alpha({}^{g^{-1}}x) = {}^{g}(r\ {}^{({}^{g^{-1}}x)}r^{-1}) = {}^{g}r\,{}^{xg}r^{-1}$, whence $(\widetilde{\alpha}, {}^{g}r) \in \mathrm{Der}(G,(A,\mu))$. It is easy to verify that one gets an action of G on the group $\mathrm{Der}(G,(A,\mu))$. In effect,

$$^{g}((\alpha, r)(\beta, s)) = {}^{g}(\alpha * \beta, rs) = (\widetilde{\alpha * \beta}, {}^{g}(rs)),$$

where $(\widetilde{\alpha \times \beta})(x) = {}^{g}(\alpha * \beta)({}^{g^{-1}}x) = {}^{g}({}^{r}\beta({}^{g^{-1}}x)) \cdot \alpha({}^{g^{-1}}x) = {}^{gr}\beta({}^{g^{-1}}x)) \cdot {}^{g}\alpha({}^{g^{-1}}x)$ and ${}^{g}(\alpha, r)\,{}^{g}(\beta, s) = (\widetilde{\alpha}, {}^{g}r)(\widetilde{\beta}, {}^{g}s) = (\widetilde{\alpha * \beta}, {}^{g}(rs))$ where $(\widetilde{\alpha} * \widetilde{\beta})(x) = {}^{{}^{g}r}({}^{g}\beta({}^{g^{-1}}x))\,{}^{g}\alpha({}^{g^{-1}}x) = {}^{grg^{-1}}({}^{g}\beta({}^{g^{-1}}x))\,{}^{g}\alpha({}^{g^{-1}}x) = {}^{gr}\beta({}^{g^{-1}}x)$ ${}^{g}\alpha({}^{g^{-1}}x)$.

Thus, ${}^{g}((\alpha, r)(\beta, s)) = {}^{g}(\alpha, r)\,{}^{g}(\beta, s)$ and it is clear that ${}^{gg'}(\alpha, r) = {}^{g}({}^{g'}(\alpha, r))$. This action on the group $\mathrm{Der}_G(A, B)$ is defined in [4].

Let (A, μ) be a crossed $G-R$-bimodule. If R acts on G and the compatibility condition

$$^{({}^{r}g)}a = {}^{rgr^{-1}}a, \quad {}^{({}^{r}g)}r' = {}^{rgr^{-1}}r' \ \text{ for } \ r, r' \in R, \ \ g \in G, \ \ a \in A, \tag{3}$$

holds, then there is also an action of R on $\mathrm{Der}(G,(A,\mu))$ given by

$$^{r}(\alpha, s) = (\widetilde{\alpha}, {}^{r}s),$$

where $\widetilde{\alpha}(x) = {}^{r}\alpha({}^{r^{-1}}x)$, $x \in G$.

A calculation similar to the case of the action of G on $\mathrm{Der}(G,(A,\mu))$ shows that $(\widetilde{\alpha}, {}^{r}s)$ is an element of $\mathrm{Der}(G,(A,\mu))$.

Let G and R be groups acting on each other and on themselves by conjugation. It is known [8] that these actions are said to be compatible if

$$^{({}^{g}r)}g' = {}^{grg^{-1}}g', \quad {}^{({}^{r}g)}r' = {}^{rgr^{-1}}r'$$

for $g, g' \in G$ and $r, r' \in R$.

Definition 4. It will be said that the groups G and R act on a group A compatibly if

$$^{({}^{g}r)}a = {}^{grg^{-1}}a, \quad {}^{({}^{r}g)}a = {}^{rgr^{-1}}a$$

for $g \in G$, $r \in R$, $a \in A$.

Proposition 5. *Let (A,μ) be a crossed $G-R$-bimodule. Let the groups G and R act on each other and on A compatibly. Under the aforementioned actions of G and R on* $\mathrm{Der}(G,(A,\mu))$ *and the homomorphism* $\gamma:\mathrm{Der}(G,(A,\mu))\longrightarrow R$ *given by* $(\alpha,r)\longmapsto r$, *the pair* $(\mathrm{Der}(G,(A,\mu)),\gamma)$ *is a precrossed $G-R$-bimodule.*

Proof. We have only to show that

$$^{(^g r)}(\alpha,s)={}^{grg^{-1}}(\alpha,s),$$

for $g\in G$, $r\in R$.

In effect,

$$^{(^g r)}(\alpha,s)=(\beta,{}^{(^g r)}s),$$

where $\beta(x)={}^{(^g r)}\alpha({}^{(^g r^{-1})}x)={}^{grg^{-1}}\alpha({}^{gr^{-1}g^{-1}}x)$, $x\in G$.

On the other hand,

$$^{grg^{-1}}(\alpha,s)=(\gamma,{}^{grg^{-1}}s),$$

where $\gamma(x)={}^{grg^{-1}}\alpha({}^{gr^{-1}g^{-1}}x)$ and

$$^{grg^{-1}}s={}^g(r^{g^{-1}}sr^{-1})={}^g rs\,{}^g r^{-1}={}^{(^g r)}s.$$

Therefore $^{(^g r)}(\alpha,s)={}^{grg^{-1}}(\alpha,s)$. $\square$

3. The Pointed Set $H^2(G,A)$

We will use the group of derivations in a crossed bimodule to define $H^2(G,A)$ when A is a crossed G-module.

We start by the following characterization of $H^2(G,A)$ when A is a $\mathbf{Z}[G]$-module.

Consider the diagram

$$M \underset{l_1}{\overset{l_0}{\rightrightarrows}} F \xrightarrow{\ \tau\ } G \qquad (4)$$

where F is a free group, τ is a surjective homomorphism, M is the set of pairs (x,y), $x,y\in F$, such that $\tau(x)=\tau(y)$ and l_0, $l_1$are canonical projections, $l_0(x,y)=x$, $l_1(x,y)=y$. Thus, (M,l_0,l_1) is the simplicial kernel of τ. Put $\Delta=\{(x,x),\,x\in F\}\subset M$.

Let f be a map from an arbitrary group C to a group D. Then in what follows by $f^{-1}:C\longrightarrow D$ will always be denoted a map with $f^{-1}(c)=f(c)^{-1}$, $c\in C$.

Let A be a $\mathbf{Z}[G]$-module. It is clear that A is a M-module via τl_0 and a F-module via τ. Denote by $Z^1(M,A)$ (resp. $Z^1(F,A)$) the abelian group of crossed homomorphims from M to A (resp. from F to A). Let $\widetilde{Z^1}(M,A)$

be a subgroup of $Z^1(M,A)$ consisting of all elements α such that $\alpha(\Delta)=1$. There is a homomorphism

$$\kappa : Z^1(F,A) \longrightarrow \widetilde{Z^1}(M,A)$$

defined by $\beta \longmapsto \beta l_0 \beta l_1^{-1}$.

Proposition 6. *$H^2(G,A)$ is canonically isomorphic to* Coker κ.

Proof. It is sufficient to show that Coker κ is isomorphic to $Opext(G,A,\varphi)$ where $\varphi : G \longrightarrow Aut(A)$ denotes the action of G on A.

Let $\alpha \in \widetilde{Z^1}(M,A)$ and introduce in the semi-direct product $A \bowtie F$ the relation

$$(a,x) \sim (a^{'},x^{'}) \Longleftrightarrow \tau(x)=\tau(x^{'})$$

and $a \cdot \alpha(x,x^{'}) = a^{'}$.

It is easy to see that this relation is an equivalence; use the fact that if $(x,x^{'},x^{''})$ is a triple of elements of F such that $\tau(x)=\tau(x^{'})=\tau(x^{''})$ then $\alpha(x,x^{''})=\alpha(x,x^{'})\alpha(x^{'},x^{''})$. Denote this equivalence by ρ and take the quotient set $(A \bowtie F)/\rho$. We will show that ρ is in fact a congruence and therefore $C=(A \bowtie F)/\rho$ is a group.

Let $(a,x) \sim (a^{'},x^{'})$ and $(b,y) \sim (b^{'},y^{'})$. Then $\tau(x)=\tau(x^{'})$, $\tau(y)=\tau(y^{'})$, $a\alpha(x,x^{'})=a^{'}$, $b\alpha(y,y^{'})=b^{'}$.

Further, $(a,x)(b,y)=(a\,{}^{x}b,xy)$, $(a^{'},x^{'})(b^{'},y^{'})=(a^{'}\,{}^{x^{'}}b^{'},x^{'}y^{'})$.

We have

$${}^{x}b\,{}^{x}\alpha(y,y^{'}) = {}^{x}b^{'} = {}^{x^{'}}b^{'},$$

whence $a\cdot\alpha(x,x^{'})\,{}^{x}b\,{}^{x}\alpha(y,y^{'})=a^{'}\,{}^{x^{'}}b^{'}$. Since $\alpha(xy,x^{'}y^{'})=\alpha(x,x^{'})\,{}^{x}\alpha(y,y^{'})$, it follows that

$$a\,{}^{x}b\alpha(xy,x^{'}y^{'}) = a^{'}\,{}^{x^{'}}b^{'}\ .$$

One gets a commutative diagram

$$\begin{array}{ccccc} M & \overset{l_0}{\underset{l_1}{\rightrightarrows}} & F & \overset{\tau}{\longrightarrow} & G \\ \downarrow \alpha & & \downarrow \beta & & \| \\ A & \overset{\sigma}{\longrightarrow} & C & \overset{\psi}{\longrightarrow} & G \end{array}$$

where $\sigma(a)=[(a,1)]$, $\psi[(a,x)]=\tau(x)$, $\beta(x)=[(1,x)]$. Denote by E the exact sequence

$$0 \longrightarrow A \overset{\sigma}{\longrightarrow} C \overset{\psi}{\longrightarrow} G \longrightarrow 1$$

which gives an element of $Opext(G,A,\varphi)$.

Define a map

$$\vartheta : \text{Coker}\,\kappa \longrightarrow Opext(g,A,\varphi)$$

given by $[\alpha] \longmapsto [E]$.

By standard calculations it can be easily proved that ϑ is a correctly defined homomorphism which is bijective. □

Let (A,μ) be a crossed G-module. Then (A,μ) is a crossed $M-G$-bimodule induced by τl_0 (or by τl_1) and a crossed $F-G$-bimodule induced by τ (see diagram (4)).

Consider the group $\mathrm{Der}(M,(A,\mu))$ and let $\widetilde{\mathrm{Der}}(M,(A,\mu))$ be the subgroup of $\mathrm{Der}(M,(A,\mu))$ consisting of elements (α,g) such that $\alpha(\Delta)=1$. If $(\alpha,g)\in\widetilde{\mathrm{Der}}(M,(A,\mu))$ this implies $g\in Z(G)$. Then we have $\mu\alpha(m)=1$ for any $m\in M$ and $\alpha(M)\subset Z(A)$. Denote by $\widetilde{Z^1}(M,(A,\mu))$ a subset of $\widetilde{\mathrm{Der}}(M,(A,\mu))$ consisting of all elements of the form $(\alpha,1)$.

Define, on the set $\widetilde{Z^1}(M,(A,\mu))$, a relation

$$(\alpha^{'},1)\sim(\alpha,1)\Leftrightarrow\exists(\beta,h)\in\mathrm{Der}(F,(A,\mu))$$

such that

$$(\alpha^{'},1)=(\beta l_0,h)(\alpha,1)(\beta l_1,h)^{-1}$$

in the group $\mathrm{Der}(M,(A,\mu))$.

We see that if $(\alpha^{'},1)\sim(\alpha,1)$ one has

$$\alpha^{'}(x)=\beta l_1(x)^{-1\,h}\alpha(x)\beta l_0(x),\quad x\in M,$$

for some $(\beta,h)\in\mathrm{Der}(F,(A,\mu))$.

Proposition 7. *The relation $\sim$ defined on $\widetilde{Z^1}(M,(A,\mu))$ is an equivalence.*

Proof. The reflexivity is clear. If $(\alpha^{'},1)\sim(\alpha,1)$, i.e., $(\alpha^{'},1)=(\beta l_0,h)(\alpha,1)$ $(\beta l_1,h)^{-1}$ where $(\beta,h)\in\mathrm{Der}(F,(A,\mu))$, then $(\alpha,1)=(\beta l_0,h)^{-1}(\alpha^{'},1)$ $(\beta l_1,h)$ where $(\beta l_0,h)^{-1}=(\widetilde{\beta}l_0,h^{-1})$ and $(\beta l_1,h)=(\widetilde{\beta}l_1,h^{-1})^{-1}$ with $(\widetilde{\beta},h^{-1})=(\beta,h)^{-1}\in\mathrm{Der}(F,(A,\mu))$. Thus the relation $\sim$ is symmetric.

Let $(\alpha^{'},1)\sim(\alpha,1)$ and $(\alpha^{''},1)\sim(\alpha^{'},1)$; then one has

$$(\alpha^{'},1)=(\beta l_0,h)(\alpha,1)(\beta l_1,h)^{-1},$$
$$(\alpha^{''},1)=(\beta^{'}l_0,h^{'})(\alpha^{'},1)(\beta^{'}l_1,h^{'})^{-1},$$

where $(\beta,h),(\beta^{'},h^{'})\in\mathrm{Der}(F,(A,\mu))$.

It follows that

$$(\alpha^{''},1)=(\beta^{'}l_0,h^{'})(\beta l_0,h)(\alpha,1)(\beta l_1,h)^{-1}(\beta^{'}l_1,h^{'})^{-1}=$$
$$=((\beta^{'}*\beta)l_0,h^{'}h)(\alpha,1)((\beta^{'}*\beta)l_1,h^{'}h)^{-1},$$

where $(\beta^{'}*\beta,h^{'}h)=(\beta^{'},h^{'})(\beta,h)\in\mathrm{Der}(F,(A,\mu))$. This means that $(\alpha^{''},1)\sim(\alpha,1)$ and the relation $\sim$ is an equivalence. □

Proposition 8. *Let (A, μ) be a crossed G-module. Then the quotient set $\widetilde{Z^1}(M, (A, \mu))/\sim$ is independent of the diagram* (4) *and is unique up to bijection.*

We need the

Lemma 9. *Let A be a G-group and let $\alpha : M \longrightarrow A$ be a crossed homomorphism such that $\alpha(\Delta) = 1$. Then there exists a map $q : F \longrightarrow A$ such that*

$$\alpha(y) = ql_1(y)^{-1}ql_0(y), \quad y \in M.$$

Proof. Observe that if $(x, x^{''}), (x^{'}, x^{''}) \in M$, then $\alpha(x, x^{''}) = \alpha(x^{'}, x^{''})$ $\alpha(x, x^{'})$. In effect,the equality $(x, x^{''}) = (1, x^{''}x^{'^{-1}})(x, x^{'})$ implies $\alpha(x, x^{''}) = \alpha(1, x^{''}x^{'^{-1}})\ \alpha(x, x^{'})$. But $(x^{'}, x^{''}) = (1, x^{''}x^{'^{-1}})(x^{'}, x^{'})$. Thus $\alpha(x^{'}, x^{''}) = \alpha(1, x^{''}x^{'^{-1}})\ \alpha(x^{'}, x^{'}) = \alpha(1, x^{''}x^{'^{-1}})$ and we get the desired equality.

In particular, applying this equality one gets $\alpha(x, x) = \alpha(x^{'}, x) \cdot \alpha(x, x^{'})$ for $(x, x), (x^{'}, x) \in M$. Therefore $\alpha(x^{'}, x) = \alpha(x, x^{'})^{-1}$ for any $(x, x^{'}) \in M$.

Take a section $\eta : G \longrightarrow F$, $\tau\eta = 1_G$ and define a map $q : F \longrightarrow A$ by

$$q(x) = \alpha(x, \eta\tau(x)), \quad x \in F.$$

For $(x, x^{'}) \in M$ one has

$$ql_1(x, x^{'})^{-1}ql_0(x, x^{'}) = q(x^{'})^{-1}q(x) = (\alpha(x^{'}, \eta\tau(x^{'}))^{-1}\alpha(x, \eta\tau(x)) =$$
$$= \alpha(\eta\tau(x^{'}), x^{'})\alpha(x, \eta\tau(x)).$$

On the other hand, since $\alpha(x, x^{'}) = \alpha(1, x^{'}x^{-1})$ for all $(x, x^{'}) \in M$, one has $\alpha(\eta\tau(x^{'}), x^{'}) = \alpha(1, x^{'}\eta\tau(x^{'})^{-1})$ and $\alpha(x, \eta\tau(x)) = \alpha(1, \eta\tau(x)x^{-1})$.

But $(1, x^{'}\eta\tau(x^{'})^{-1})(1, \eta\tau(x)x^{-1}) = (1, x^{'}x^{-1})$. Therefore, $\alpha(x, x^{'}) = \alpha(1, x^{'}\eta\tau(x^{'})^{-1})\alpha(1, \eta\tau(x)x^{-1}) = ql_1(x, x^{'})^{-1}ql_0(x, x^{'})$. $\square$

Proof of Proposition 8. Consider a commutative diagram

$$\begin{array}{ccccc} M^{'} & \overset{l_0^{'}}{\underset{l_1^{'}}{\rightrightarrows}} & F^{'} & \overset{\tau^{'}}{\longrightarrow} & G \\ \overline{\gamma_1} \downarrow \overline{\gamma_2} & & \gamma_1 \downarrow \gamma_2 & & \| \\ M & \overset{l_0}{\underset{l_1}{\rightrightarrows}} & F & \overset{\tau}{\longrightarrow} & G \end{array}$$

where (M, l_0, l_1) and $(M^{'}, l_0^{'}, l_1^{'})$ are the simplicial kernels of $\tau_1$and τ_2 respectively, $l_i\overline{\gamma_1} = \gamma_1 l_i^{'}$, $l_i\overline{\gamma_2} = \gamma_2 l_i^{'}$, $i = 0, 1$, $\tau\gamma_1 = \tau\gamma_2 = \tau^{'}$.

The pair $(\gamma_i, \overline{\gamma_i})$ induces a homomorphism

$$\mathrm{Der}(M, (A, \mu)) \longrightarrow \mathrm{Der}(M^{'}, (A, \mu))$$

given by $(\alpha, g) \longmapsto (\alpha\overline{\gamma_i}, g)$, $i = 1, 2$.

If $(\alpha^{'},1)\sim(\alpha,1)$, i.e.,

$$(\alpha^{'},1)=(\beta l_0,h)(\alpha,1)(\beta l_1,h)^{-1}$$

with $(\beta,h)\in \mathrm{Der}(F,(A,\mu))$, then

$$\alpha^{'}\overline{\gamma_i}(y)=\beta\gamma_i l_1^{'}(y)^{-1h}\alpha\overline{\gamma_i}(y)\beta\gamma_i l_0^{'}(y),\quad y\in M^{'}.$$

Thus $(\alpha^{'}\overline{\gamma_i},1)\sim(\alpha\overline{\gamma_i},1)$, $i=1,2$, and one gets a natural map

$$\epsilon_i:\widetilde{Z^1}(M,(A,\mu))/\sim\longrightarrow\widetilde{Z^1}(M^{'},(A,\mu))/\sim$$

induced by the pair $(\gamma_i,\overline{\gamma_i})$ and given by $[(\alpha,1)]\longmapsto[(\alpha\overline{\gamma_i},1)]$, $i=1,2$.

We will show that $\epsilon_1=\epsilon_2$. By Lemma 9 there is a map $q:F\longrightarrow A$ such that

$$\alpha(y)=ql_1(y)^{-1}ql_0(y),\quad y\in M.$$

Consider the homomorphism $s:F^{'}\longrightarrow M$ given by

$$s(x^{'})=(\gamma_1(x^{'}),\gamma_2(x^{'})),\quad x^{'}\in F^{'}.$$

It is clear that $(\alpha s,1)\in \mathrm{Der}(F^{'},(A,\mu))$.

Further we have

$$((\alpha sl_1)^{-1}\alpha\overline{\gamma_2}\alpha sl_0^{'})(x_0^{'},x_1^{'})=\alpha s(x_1^{'})^{-1}\ \alpha\overline{\gamma_2}(x_0^{'},x_1^{'})\alpha s(x_0^{'})=$$
$$=\alpha(\gamma_1(x_1^{'}),\gamma_2(x_1^{'}))^{-1}\alpha\overline{\gamma_2}(x_0^{'},x_1^{'})\ \alpha(\gamma_1(x_0^{'}),\gamma_2(x_0^{'}))=q\gamma_1(x_1^{'})^{-1}q\gamma_2(x_1^{'})=$$
$$=q\gamma_2(x_1^{'})^{-1}q\gamma_2(x_0^{'})q\gamma_2(x_0^{'})^{-1}q\gamma_1(x_0^{'})=q\gamma_1(x_1^{'})^{-1}q\gamma_1(x_0^{'})=\alpha\overline{\gamma_1}(x_0^{'},x_1^{'})$$

for $(x_0^{'},x_1^{'})\in M^{'}$.

Therefore $(\alpha\overline{\gamma_1},1)\sim(\alpha\overline{\gamma_2},1)$ with $(\alpha s,1)\in \mathrm{Der}(F^{'},(A,\mu))$ and one gets $\epsilon_1=\epsilon_2$.

The rest of the proof of the uniqueness is standard. □

Let (A,μ) be a crossed $G-R$-bimodule. Denote by $I\,\mathrm{Der}(G,(A,\mu))$ a subgroup of $\mathrm{Der}(G,(A,\mu))$ consisting of elements of the form (α,r) with $r\in H^0(G,R)$. If (A,μ) is a crossed G-module viewed as a crossed $G-G$-bimodule then

$$I\,\mathrm{Der}(G,(A,\mu))=\{(\alpha,r),g\in Z(G)\}.$$

Consider the diagram

$$M_G\ \underset{l_1}{\overset{l_0}{\rightrightarrows}}\ F_G\xrightarrow{\tau_G}G \tag{5}$$

where F_G is the free group generated by G, τ_G is the canonical homomorphism and (M_G,l_0,l_1) is the simplicial kernel of τ_G.

Proposition 10. *Let* (A,μ) *be a crossed* G*-module. Then:*
(i) *there is a canonical surjective map*

$$\vartheta^{'} : H^2(G, Ker\mu) \longrightarrow \widetilde{Z^1}(M_G, (A,\mu))/\sim$$

given by the composite map $[E] \overset{\vartheta^{-1}}{\longmapsto} [\alpha] \longmapsto [(\alpha,1)]$;

(ii) *if we assume* $\mathrm{Der}(F_G,(A,\mu)) = I\,\mathrm{Der}(F_G,(A,\mu))$ (*in particular, it is so if either* μ *is the trivial map or* G *is abelian*) *we can introduce, in the pointed set* $\widetilde{Z^1}(M_G,(A,\mu))/\sim$, *an abelian group structure defined by*

$$[(\alpha,1)][(\beta,1)] = [(\alpha * \beta,1)]$$

where (A,μ) *is viewed as a crossed* $F_G - G$*-bimodule induced by* τ_G*and* $[(\alpha,1)]$ *denotes the equivalence class containing* $(\alpha,1)$. *Under this product the map* $\vartheta^{'}$ *becomes an isomorphism.*

Proof. To prove (i) we have only to show the correctness of $[\alpha] \longmapsto [(\alpha,1)]$ where $\alpha : M_G \longrightarrow A$ is a crossed homomorphism with $\alpha(\Delta) = 1$ and $\alpha(M_G) \subset \mathrm{Ker}\,\mu$.

Let $\alpha^{'} \in [\alpha]$, i.e.,

$$\alpha^{'}(x) = \beta l_1^{-1}(x)\alpha(x)\beta l_0(x), x \in M_G,$$

where $\beta : F_G \longrightarrow Ker\ \mu$ is a crossed homomorphism. Then $(\beta,1) \in \mathrm{Der}(F_G,(A,\mu))$ and we have

$$(\alpha^{'},1) = (\beta l_0,1)(\alpha,1)(\beta l_1,1)^{-1}.$$

The surjectivity of $\vartheta^{'}$ is clear.

(ii) Let $(\alpha^{'},1) \sim (\alpha,1)$. Then

$$\alpha^{'}(x) = \eta l_1(x)^{-1\,g}\alpha(x)\eta l_0(x), \quad x \in M_G,$$

for some $(\eta,g) \in \mathrm{Der}(F_G,(A,\mu))$. By assumption, $g \in Z(G)$. Thus $\mu\eta(x) = gxg^{-1}x^{-1} = 1$, $x \in M_G$. It follows that $[\alpha^{'}] = [{}^g\alpha]$ with $g \in Z(G)$. But it is known that $Z(G)$ acts trivially on $H^2(G, Ker\ \mu)$. Therefore we have $[\alpha^{'}] = [\alpha]$. Hence there is a crossed homomorphism $\gamma : F_G \longrightarrow Ker\ \mu$ such that

$$\alpha^{'}(x) = \gamma l_1^{-1}(x)\alpha(x)\gamma l_0(x), \quad x \in M_G.$$

It follows that if $(\alpha,1) \sim (\alpha^{'},1)$ and $(\beta,1) \sim (\beta^{'},1)$ then $(\alpha * \beta,1) \sim (\alpha^{'} * \beta^{'},1)$. We conclude that the product is correctly defined and the map $\vartheta^{'}$ is an isomorphism when $\mathrm{Der}(F_G,(A,\mu)) = I\,\mathrm{Der}(F_G,(A,\mu))$. $\square$

Proposition10 motivates the following definition of the second cohomology of groups with coefficients in crossed modules.

Definition 11. Let (A,μ) be a crossed G-module. One denotes by $H^2(G,A)$ the quotient set $\widetilde{Z^1}(M,(A,\mu))\,/\sim$ which will be called the second set of cohomology of G with coefficients in the crossed G-module (A,μ).

Remark. Using diagram (4) it is possible to define the second cohomology of G with coefficients in a crossed G-module (A,μ) by a different "less abelian" way. Consider the set $\widetilde{Z^1}(M,A)$ of all crossed homomorphims $\alpha: M \longrightarrow A$ with $\alpha(\Delta)=1$ (the equality $\mu\alpha = 1$ is not required and therefore $\alpha(M)$ is not necessarily contained in $Z(A)$). Introduce in $\widetilde{Z^1}(M,A)$ a relation $\sim$ of equivalence as follows:

$\alpha^{'} \sim \alpha$ if $\exists\,(\beta,h)\,\mathrm{Der}(F,(A,\mu))$ such that $\alpha^{'}(x) = \beta l_1(x)^{-1}\,{}^h\alpha(x)\,\beta l_0(x)$, $x \in M$.

Define $\overline{H^2}(G,A) = \widetilde{Z^1}(M,A)\,/\sim$. It is obvious that $H^2(G,A) \subset \overline{H^2}(G,A)$. But it seems the exact cohomology sequence (Theorem 13) does not hold for $\overline{H^2}(G,A)$.

It is clear $H^2(G,A)$ is a pointed set with $[(\alpha_0,1)]$ as a distinguished element where $\alpha_0(y)=1$ for all $y \in M$.

A homomorphism of crossed G-modules $f:(A,\mu) \longrightarrow (B,\lambda)$ induces a map of pointed sets

$$f^2 : H^2(G,A) \longrightarrow H^2(G,B), \quad f^2([(\alpha,1)]) = [(f\alpha,1)].$$

There is an action of G on F_G (see diagram (5)) defined as follows:

$${}^g(\mid g_1\mid^{\epsilon}\cdots\mid g_n\mid^{\epsilon}) = \mid{}^g g_1\mid^{\epsilon}\cdots\mid{}^g g_n\mid^{\epsilon}, \quad g, g_1, \dots, g_n \in G,$$

where $\epsilon = \pm 1$.

This action induces an action of G on M_G by

$${}^g(x,x^{'}) = ({}^g x, {}^g x^{'}), \quad g \in G, \quad (x,x^{'}) \in M_G.$$

Let (A,μ) be a crossed G-module. Then we have an action of G on $\mathrm{Der}(M_G,(A,\mu))$ given by

$${}^g(\alpha,h) = (\widetilde{\alpha}, {}^g h)$$

where $\widetilde{\alpha}(m) = {}^g\alpha({}^{g^{-1}}m)$, $g \in G$, $m \in M_G$.

Proposition 12. *Let (A,μ) be a crossed G-module. There is an action of G on $H^2(G,A)$ induced by the above-defined action of G on $\mathrm{Der}(M_G,(A,\mu))$ under which the center $Z(G)$ acts trivially.*

Proof. Obviously, the action of G on $\mathrm{Der}(M_G,(A,\mu))$ induces an action of G on $\widetilde{\mathrm{Der}}(M_G,(A,\mu))$. Thus one gets an action of G on $\widetilde{Z^1}(M_G,(A,\mu))$.

If $(\alpha^{'},1) \sim (\alpha,1)$, where $(\alpha,1)$, $(\alpha^{'},1) \in \widetilde{Z^1}(M_G,(A,\mu))$, we have

$$\alpha^{'}(y) = \beta l_1(y)^{-1}\,{}^h\alpha(y)\beta l_0(y), \quad y \in M_G,$$

for some $(\beta, h) \in \mathrm{Der}(F_G, (A, \mu))$.

This implies

$$ {}^g\alpha'({}^{g^{-1}}y) = {}^g\beta l_1({}^{g^{-1}}y)^{-1}\, {}^{gh}\alpha({}^{g^{-1}}y)\, {}^g\beta l_0({}^{g^{-1}}y), \quad y \in M_G. $$

Hence $\widetilde{\alpha}'(y) = {}^g\beta({}^{g^{-1}}l_1(y))^{-1}\, {}^{ghg^{-1}}\widetilde{\alpha}({}^{g^{-1}}y)\, {}^g\beta({}^{g^{-1}}l_0(y))$ with $(\widetilde{\beta}, {}^g h) \in \mathrm{Der}(F_G, (A, \mu))$. Therefore $(\widetilde{\alpha^1}, 1) \sim (\widetilde{\alpha}, 1)$. It is obvious that the above-defined map $\vartheta' : H^2(G, \ker\mu) \longrightarrow H^2(G, A)$ is a G-map. Since ϑ' is surjective (see Proposition 11) and $Z(G)$ acts trivially on $H^2(G, \ker\mu)$, it follows that $Z(G)$ acts trivially on $H^2(G, A)$ too. $\square$

4. An Exact Cohomology Sequence

For any G-group A denote by $H^0(G, A)$ a subgroup of A consisting of all invariant elements under the action of G on A.

Theorem 13. *Let*

$$ 1 \longrightarrow (A, 1) \xrightarrow{\varphi} (B, \mu) \xrightarrow{\psi} (C, \lambda) \longrightarrow 1 \tag{6} $$

be an exact sequence of crossed G-modules. Then there is an exact sequence

$$ 1 \longrightarrow H^0(G, A) \xrightarrow{\varphi^0} H^0(G, B) \xrightarrow{\psi^0} H^0(G, C) \xrightarrow{\delta^0} H^1(G, A) \xrightarrow{\varphi^1} $$
$$ \xrightarrow{\varphi^1} H^1(G, B) \xrightarrow{\psi^1} H^1(G, C) \xrightarrow{\delta^1} H^2(G, A) \xrightarrow{\varphi^2} H^2(G, B) \xrightarrow{\psi^2} H^2(G, C) $$

where φ^0, ψ^0, δ^0, φ^1, ψ^1 are group homomorphisms, δ^1 is a crossed homomorphism under the action of $H^1(G, C)$ on $H^2(G, A)$ induced by the action of G on A, and φ^2, ψ^2 are maps of pointed G-sets.

Proof. The exactness of

$$ 1 \longrightarrow H^0(G, A) \xrightarrow{\varphi^0} H^0(G, B) \xrightarrow{\psi^0} H^0(G, C) \xrightarrow{\delta^0} H^1(G, A) \xrightarrow{\varphi^1} $$
$$ \xrightarrow{\varphi^1} H^1(G, B) \xrightarrow{\psi^1} H^1(G, C) \xrightarrow{\delta^1} H^2(G, A) $$

is proved in [4].

We have only to show the exactness of

$$ H^1(G, C) \xrightarrow{\delta^1} H^2(G, A) \xrightarrow{\varphi^2} H^2(G, B) \xrightarrow{\psi^2} H^2(G, C). $$

Let $[(\alpha, g)] \in H^1(G, C)$ and $\delta^1[(\alpha, g)] = [\gamma]$. Then one has a commutative diagram

$$ \begin{array}{ccccc} M & \overset{l_0}{\underset{l_1}{\rightrightarrows}} & F & \xrightarrow{\tau} & G \\ \downarrow \gamma & & \downarrow \beta & & \downarrow \alpha \\ A & \xrightarrow{\varphi} & B & \xrightarrow{\psi} & C \end{array} $$

where $\varphi\gamma(y) = \beta l_1(y)^{-1}\beta l_0(y)$, $y \in M$, and β is a crossed homomorphism, B being an F-group via τ. The existence of such β follows from the following assertion: if we have a surjective homomorphism $\psi : B \longrightarrow C$ of F-groups and $f : F \longrightarrow C$ is a crossed homomorphism where F is a free group, then there is a crossed homomorphism $\beta : F \longrightarrow B$ such that $\psi\beta = f$. In effect, take the semi-direct product $B \bowtie F$ and consider a subgroup Y of $B \bowtie F$ consisting of all elements (b, x) such that $\psi(b) = f(x)$. Then we have a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{pr_2} & F \\ \downarrow pr_1 & & \downarrow f \\ B & \xrightarrow{\psi} & C \end{array}$$

where pr_i is the projection, $i = 1, 2$, and pr_2 is surjective. Thus, since F is a free group, there is a homomorphism $f' : F \longrightarrow Y$ such that $ff' = 1_F$. Then $pr_1 f'$ is the required crossed homomorphism.

It will be shown that $(\beta, g) \in \mathrm{Der}(F, (B, \mu))$. One has

$$g\tau(x)g^{-1}\tau(x)^{-1} = \lambda\alpha\tau(x) = \lambda\psi\beta(x) = \mu\beta(x), \quad x \in F.$$

This means $(\beta, g) \in \mathrm{Der}(F, (B, \mu))$. It is clear that $\varphi^2\delta^1([\alpha, g]) = \varphi^2([\gamma]) = [(\varphi\gamma, 1)]$ and $(\varphi\gamma, 1) \sim (\alpha_0, 1)$ (use $(\beta, g) \in \mathrm{Der}(F, (B, \mu))$), where $\alpha_0(x) = 1$ for all $x \in F$. Therefore $\mathrm{Im}\,\delta^1 \subset \ker\varphi^2$.

Let $[\gamma] \in H^2(G, A)$ such that $\varphi^2([\gamma]) = [(\varphi\gamma, 1)] = [(\alpha_0, 1)]$. Then there exists $(\beta, h) \in \mathrm{Der}(F, (B, \mu))$ such that

$$\varphi\gamma(y) = \beta l_1(y)^{-1} \cdot \beta l_0(y), \quad y \in M,$$

whence $\psi\beta l_0(y) = \psi\beta l_1(y)$, $y \in M$. It follows that there is a crossed homomorphism $\alpha : G \longrightarrow C$ such that $\alpha\tau = \psi\beta$. We have to show $(\alpha, h) \in \mathrm{Der}_G(G, C)$. In effect, $h\ \tau(x)\ h^{-1}\ \tau(x)^{-1} = \mu\beta(x) = \lambda\psi\beta(x) = \lambda\alpha\tau(x)$, $x \in F$. This implies $(\alpha, h) \in \mathrm{Der}_G(G, C)$. It is clear that $\delta^1([\alpha, h]) = [\gamma]$. Therefore, $\ker\varphi^2 \subset \mathrm{Im}\,\delta^1$.

It is obvious that $\mathrm{Im}\,\varphi^2 \subset \ker\psi^2$.

Let $[(\alpha, 1)] \in H^2(G, B)$ such that $\psi^2([\alpha, 1]) = [(\psi\alpha, 1)] = [(\alpha_0, 1)]$. Then there exists $(\beta, h) \in \mathrm{Der}(F, (C, \lambda))$ such that

$$\psi\alpha(y) = \beta l_1(y)^{-1}\beta l_0(y), \quad y \in M.$$

It follows that there is a crossed homomorphism $\beta' : F \longrightarrow B$ such that $\psi\beta' = \beta$. One gets the following commutative diagram:

$$\begin{array}{ccccc} M & \overset{l_0}{\underset{l_1}{\rightrightarrows}} & F & \xrightarrow{\tau} & G \\ & \searrow \alpha & \downarrow \beta' & \searrow \beta & \\ A & \xrightarrow{\varphi} & B & \xrightarrow{\psi} & C \end{array} .$$

Thus, $\psi\alpha(y)=\psi\beta' l_1(y)^{-1}\ \psi\beta' l_0(y)$, $y\in M$, so that $\psi(\beta' l_1(y)\alpha(y)\,\beta' l_0(y)^{-1})$ $= 1$, $y \in M$. One gets $\beta' l_1(y)\alpha(y)\beta' l_0(y)^{-1} \in \varphi(A)$, $y \in M$. Denote by $\gamma : M \longrightarrow A$ the crossed homomorphism given by $\gamma(y) = \varphi^{-1}(\beta' l_1(y)\alpha(y)\beta'$ $l_0(y)^{-1})$. Then ${}^{h^{-1}}\gamma(y) = \varphi^{-1}({}^{h^{-1}}\beta' l_1(y){}^{h^{-1}}\alpha(y)\cdot{}^{h^{-1}}\beta' l_0(y)^{-1})$, $y \in M$.

In the group $\mathrm{Der}(F,(B,\mu))$ consider the product

$$(\beta' l_0, h)^{-1}(\alpha,1)(\beta' l_1, h) = (\eta, 1)$$

where $\eta(y) = {}^{h^{-1}}\beta' l_1(y)\,{}^{h^{-1}}\alpha(y)\,{}^{h^{-1}}\beta' l_0(y)^{-1}$, $y \in M$.

This implies that the map given by

$$y \longmapsto^{h^{-1}} \gamma(y), \quad y \in M,$$

is a crossed homomorphism.

Take $[\varphi^{-1}\eta] \in H^2(G,A)$. Then we have $\varphi^2([\varphi^{-1}\eta]) = [(\eta,1)]$. But $(\eta,1) \sim (\alpha,1)$ by the above equality with $(\beta', h)^{-1} \in \mathrm{Der}(F,(B,\mu))$. Therefore, $\ker\,\psi^2 \subset \mathrm{Im}\,\varphi^2$. $\square$

Any crossed G-module (A,μ) induces the following short exact sequence of crossed G-modules:

$$1 \longrightarrow (\ker\mu, 1) \xrightarrow{\varphi} (A,\mu) \xrightarrow{\psi} (\mathrm{Im}\,\mu, \sigma) \longrightarrow 1$$

where $\sigma : \mathrm{Im}\,\mu \longrightarrow G$ denotes the inclusion and G acts on $\mathrm{Im}\,\mu$ by conjugation.

Corollary 14. *If (A,μ) is a crossed G-module there is an exact sequence*

$$\begin{gathered}1 \longrightarrow H^0(G,\ker\mu) \xrightarrow{\varphi^0} H^0(G,A) \xrightarrow{\psi^0} H^0(G,\mathrm{Im}\,\mu) \xrightarrow{\delta^0} H^1(G,\ker\mu) \xrightarrow{\varphi^1} \\ \xrightarrow{\varphi^1} H^1(G,A) \xrightarrow{\psi^1} H^1(G,\mathrm{Im}\,\mu) \xrightarrow{\delta^1} H^2(G,\ker\mu) \xrightarrow{\vartheta'} H^2(G,A) \\ \longrightarrow 1.\end{gathered}$$

For the exact sequence (6) a connecting map

$$\delta^2 : H^2(G,C) \longrightarrow H^3(G,A)$$

will be defined, and for this we will use the equivalence of functors $H^{n+1}(-,A) \approx L_n\,\mathrm{Der}(-,A)$, $n \geq 1$ [9] when A is a $\mathbf{Z}[G]$-module, where $L_n\,\mathrm{Der}(-,A)$ is the non-abelian nth derived functor of the contravariant functor $\mathrm{Der}(-,A)$ from the category $\underline{D}_G$ of groups acting on A to the category of abelian groups.

Consider the following canonical free simplicial resolution of G in the category $\underline{D}_G$:

$$\vdots\; F_3 \xrightarrow{\tau_3} M_2 \overset{\iota_0^2}{\underset{\iota_3^2}{\vdots}} F_2 \xrightarrow{\tau_2} M_1 \overset{\iota_0^1}{\underset{\iota_2^1}{\rightrightarrows}} F_1 \xrightarrow{\tau_1} M_0 \overset{l_0^0}{\underset{l_1^0}{\rightrightarrows}} F_0 \xrightarrow{\tau_0} G \qquad (7)$$

where $F_0 = F_G$, $F_i = F_{M_{i-1}}$, $i \geq 1$, τ_i is the canonical homomorphism, and $(M_i, l_0^i, \ldots, l_{i+1}^i)$ is the simplicial kernel of $(l_0^{i-1}\tau_i, \ldots, l_i^{i-1}\tau_i)$, $i \geq 0$ (see [10]).

There is an action of $\mathrm{Der}(F_0, (C, \lambda))$ on $H^3(G, A)$ defined as follows.

Let $[f] \in H^3(G, A)$, where $f : F_2 \longrightarrow A$ is a crossed homomorphism such that $\prod_{i=0}^{3} (f l_i^1 \tau_3)^\epsilon = 1$, where $\epsilon = (-1)^i$, and let $(\alpha, g) \in \mathrm{Der}(F_0, (C, \lambda))$.

Define

$$^{(\alpha,g)}[f] = [^g f].$$

We have first to show that $^g f$ is a crossed homomorphism.

In the group $\mathrm{Der}(F_2, (B, \mu))$ ((B, μ) being a crossed F_2-G-bimodule induced by $\tau_0 \partial_0^1 \partial_0^2$ where $\partial_0^1 = l_0^0 \tau_1$, $\partial_0^2 = l_0^1 \tau_2$) take the product

$$(\beta \partial_0^1 \partial_0^2, g)(\varphi f, 1)(\beta \partial_0^1 \partial_0^2, g)^{-1} = (\widetilde{f}, 1),$$

where $\widetilde{f}$is a crossed homomorphism and $\beta : F_0 \longrightarrow B$ is a crossed homomorphism too such that $\psi\beta = \alpha$ (such β exists, since F_0 is a free group and ψ is surjective).

One has

$$\widetilde{f}(x) = \beta \partial_0^1 \partial_0^2 (x)^{-1\, g} \varphi f(x) \beta \partial_0^1 \partial_0^2 (x) =^g \varphi f(x), \quad x \in F_2.$$

We show now the correctness. If $f \sim f'$ then there is a crossed homomorphism $\eta : F_1 \longrightarrow A$ such that

$$f'(x) = f(x) \prod_{i=0}^{2} (\eta l_i^1 \tau_2(x))^\epsilon, \quad x \in F_2,$$

where $\epsilon = (-1)^i$. It can be shown in the same manner as for f that $^g\eta$ is a crossed homomorphism if $(\alpha, g) \in \mathrm{Der}(F_0, (C, \lambda))$.

Thus one gets

$$^g f' = {}^g f \prod_{i=0}^{2} {}^g(\eta l_i^1 \tau_2)^\epsilon,$$

where $\epsilon = (-1)^i$. This implies

$$[^g f'] = [^g f].$$

Therefore the action of $\mathrm{Der}(F_0, (C, \lambda))$ on $H^3(G, A)$ is correctly defined.

Let $(\alpha, 1) \in \widetilde{\mathrm{Der}}(M_0, (C, \lambda))$ and let $\beta : F_1 \longrightarrow B$ be a crossed homomorphism (F_1 acts on B via $\tau_0 l_0^0 \tau_1 = \tau_0 l_0^1 \tau_1$) such that $\psi\beta = \alpha\tau_1$. Then we have $\beta(F_1) \subset Z(B)$ and define a crossed homomorphism $\overline{\beta} = \prod_{i=0}^{2} (\beta l_i^1)^\epsilon :$

$M_1 \longrightarrow B$, $\epsilon = (-1)^i$. Hence $\psi\overline{\beta}(y) = \prod_{i=0}^{2} \psi\beta l_i^1(y)^\epsilon = \prod_{i=0}^{2} \alpha\tau_1 l_i^1(y)^\epsilon$, $y \in M_1$, $\epsilon = (-1)^i$.

It is easy to see that

$$\tau_1 l_0^1(y)\tau_1 l_1^1(y)^{-1}\tau_1 l_2^1(y) \in \Delta, \quad y \in M_1.$$

Since $\alpha(\Delta) = 1$, there is a crossed homomorphism $\gamma : F_2 \longrightarrow A$ such that $\varphi\gamma = \overline{\beta}\tau_2$.

Theorem 15. *Let* (6) *be an exact sequence of crossed G-modules. If either the action of* $\mathrm{Der}(F_0, (C, \lambda))$ *on* $H^3(G, A)$ *is trivial* (*in particular, if G acts trivially on A*) *or* $\mathrm{Der}(F_0, (C, \lambda)) = I\,\mathrm{Der}(F_0, (C, \lambda))$ (*in particular, if either $\lambda = 1$ or G is abelian*) *then the connecting map* $\delta^2 : H^2(G, C) \longrightarrow H^3(G, A)$ *is defined by*

$$\delta^2([\alpha, 1]) = [\gamma], (\alpha, 1) \in \widetilde{\mathrm{Der}}(M_0, (C, \lambda)),$$

and the sequence

$$H^2(G, B) \xrightarrow{\psi^2} H^2(G, C) \xrightarrow{\delta^2} H^3(G, A)$$

is exact.

Proof. We have to show that δ^2 is correctly defined.

Let $\mathrm{Der}(F_0, (C, \lambda))$ act trivially on $H^3(G, A)$. If $(\alpha', 1) \in [(\alpha, 1)] \in H^2(G, A)$, we have

$$\alpha'(x) = \eta l_1^0(x)^{-1\,g}\alpha(x)\eta l_0^0(x), \quad x \in F_0,$$

with $(\eta, g) \in \mathrm{Der}(F_0, (C, \lambda))$.

Take a crossed homomorphism $\eta' : F_0 \to B$ such that $\psi\eta' = \eta$. Recall that $\psi\beta = \alpha\tau_1$. Consider the product

$$(\eta' l_0^0\tau_1, g)(\beta, 1)(\eta' l_1^0\tau_1, g)^{-1} = (\beta', 1)$$

in the group $\mathrm{Der}(F_1, (B, \mu))$. Then

$$\beta'(x) = \eta' l_1^0\tau_1(x)^{-1\,g}\beta(x)\eta' l_0^0\tau_1(x), \quad x \in F_1.$$

Thus $\psi\beta' = \alpha'\tau_1$ and $\beta'(F_1) \subset Z(B)$. Note that this implies $\eta' l_1^0(x)^{-1}\,\eta' L_0^0(x) \in Z(B)$ for all $z \in M_0$.

Further,

$$\begin{aligned}\beta'(l_0^1(y))\beta'(l_1^1(y)^{-1})\beta'(l_2^1(y)) = {}^g\beta(l_0^1(y))\eta' l_1^0(l_0^1(y))^{-1}\eta' l_0^0(l_0^1(y))\\ {}^g\beta(l_1^1(y))^{-1}\eta' l_1^0(l_1^1(y)^{-1})^{-1}\eta' l_0^0(l_1^1(y)^{-1\,g}\beta(l_2^1(y))\eta' l_1^0(l_2^1(y))^{-1}\\ \eta' l_0^0(l_2^1(y)) = {}^g\overline{\beta}(y).\end{aligned}$$

Therefore $\overline{\beta'}\tau_2 = {}^g(\overline{\beta}\tau_2)$ and $[\gamma'] = [{}^g\gamma] = [\gamma]$.

If $\mathrm{Der}(F_0,(C,\lambda)) = I\,\mathrm{Der}(F_0(C,\lambda))$ and $\alpha',1) \sim (\alpha,1)$ then there is an element $(\eta,g) \in \mathrm{Der}(F_0,(C,\lambda))$ such that

$$\alpha'(x) = \eta l_1^0(x)^{-1}\alpha(x)\eta l_0^0(x), \quad x \in F_0,$$

(see the proof of Proposition 10 (ii)) and it is clear that in both cases δ^2 is correctly defined.

We will now prove the exactness. Let $[(\alpha,1)] \in H^2(G,B)$. Then $\delta^2\psi^2[(\alpha,1)] = \delta^2([(\psi\alpha,1)]) = [\gamma]$, where $\varphi\gamma = \overline{\beta}\tau_2$ and $\overline{\beta}$ is taken such that $\beta = \alpha\tau_1$. Thus

$$\overline{\beta}(y) = \alpha\tau_1 l_0^1(y)\alpha\tau_1 l_1^1(y)^{-1}\alpha\tau_1 l_2^1(y), \quad y \in M_1.$$

Since $\tau_1 L_0^1(y)\,\tau_1 L_1^1(y)^{-1}\,\tau_1 L_2^1(y) \in \Delta$, $y \in M_1$ and $\alpha(\Delta) = 1$, this implies $\overline{\beta} = 1$. Thus we have $\mathrm{Im}\,\psi^2 \subset \ker\delta^2$.

Let $[(\alpha,1)] \in H^2(G,C)$ such that $\delta^2([(\alpha,1)]) = 1$. Then we have $[\gamma] = 1$ with $\varphi\gamma = \overline{\beta}\tau_2$, where $\overline{\beta} : M_+1 \to B$ is a crossed homomorphism such that $\overline{\beta} = \beta l_0^1(\beta l_1^1)^{-1}\beta l_2^1$ with $\psi\beta = \alpha\tau_1$ and $\beta : F_1 \to B$ a crossed homomorphism.

It follows that there exists a crossed homomorphism $\eta : F_1 \to A$ such that $\gamma = (\eta l_0^1(\eta l_1^1)^{-1}\eta l_2^1)\tau_2$.

Thus we have $\overline{\beta}\tau_2 = \varphi(\eta l_0^1(\eta l_1^1)^{-1}\eta l_2^1)\tau_2$, whence $\overline{\beta}(y) = \varphi\eta l_0^1(y)\varphi\eta l_1^1(y)^{-1}$ $\varphi\eta l_2^1$, $y \in M_1$.

For $y_1, y_2 \in F_1$ such that $\tau_1(y_1) = \tau_1(y_2) = x$, since $(y_1, y_2, s_0^0 l_1^0(x)) \in M_1$, one gets $\beta(y_1)\beta(y_2)^{-1}\beta s_0^0 l_1^0(x) = \varphi\eta(y_1)\varphi\eta(y_2)^{-1}\varphi\eta(s_0^0 l_1^0(x))$, where $s_0^0 : F_0 \to F_1$ is the degeneracy map. In particular, if $y_1 = y_2$ we have

$$\beta(s_0^0 l_1^0(x)) = \varphi\eta(s_0^0 l_1^0(x)), \quad x \in M_0.$$

Therefore $\beta(y_1)\varphi\eta(y_1)^{-1} = \beta(y_2)\varphi\eta(y_2)^{-1}$ if $\tau_1(y_1) = \tau_1(y_2)$.

This implies a crossed homomorphism $\beta' : M_0 \to B$ given by

$$\beta'(x) = \beta(y)\varphi\eta(y)^{-1}, \quad x \in M_0,$$

where $\tau_1(y) = x$. Thus $\beta'\tau_1 = \beta$ and $\beta'(M_0) \subset Z(B)$. If $x \in \Delta \subset M_0$ then $\tau_1 s_0^0 l_1^0(x) = x$. Therefore we have

$$\beta'(x) = \beta(s_0^0 l_1^0(x)) = \varphi\eta(s_0^0 l_1^0(x))^{-1}, \quad x \in \Delta.$$

Whence $\beta'(x) = 1$ if $x \in \Delta$.

On the other hand,

$$\mu\beta'(x)\mu\beta(y)\mu\varphi\eta(y)^{-1} = \mu\beta(y) = \lambda\psi\beta(y) = \lambda\alpha\tau_1(y) = 1.$$

We conclude that $(\beta',1) \in \widetilde{\mathbf{Z}}^1(M_0,(B,\mu))$ and it is clear that $\psi^2([(\beta',1)])$ $= [(\alpha,1)]$. □

Acknowledgement

The research described in this publication was made possible in part by Grant MXH200 from the International Science Foundation and by INTAS Grant No 93-2618.

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(Received 2.06.1995)

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