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A MULTIDIMENSIONAL SINGULAR BOUNDARY VALUE PROBLEM OF THE CAUCHY–NICOLETTI TYPE

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Abstract. A two-point singular boundary value problem of the Cauchy–Nicoletti type is studied by introducing a two-point boundary value set and using the topological principle. The results on the existence of solutions whose graph lies in this set are proved. Applications and comparisons to the known results are given, too.

Introduction

Consider the system of ordinary differential equations

$$y' = f(x, y), \tag{1}$$

where $x \in I = (a, b)$, $-\infty \le a < b \le \infty$, $y \in \mathbb{R}^n$ and $n > 1$.

We will study the following singular boundary value problem of the Cauchy–Nicoletti type:

$$y_i(a+) = A_i \ \ (i = 1, \dots, m), \quad y_k(b-) = A_k \ \ (k = m + 1, \dots, n) \tag{2}$$

where A_i, $i = 1, \dots, n$, are some constants and $1 \le m < n$.

It is assumed that the vector-function $f \in C(\Omega, \mathbb{R}^n)$, where Ω is an open set such that $\Omega \cap \{(x^*, y) : y \in \mathbb{R}^n\} \neq \varnothing$ for each $x^* \in I$ and, moreover, f satisfies local Lipschitz condition in the variable y in Ω ($f \in L_{loc}(\Omega)$). In this case the solutions of system (1) are uniquely determined by the initial data in Ω.

We define the solution of problem (1), (2) as a vector-function $y = (y_1, \dots, y_n) \in C^1(I, \mathbb{R}^n)$ which satisfies system (1) on I, $(x, y_1(x), \dots, y_n(x)) \subset \Omega$ if $x \in I$ and $y_i(a+) = A_i$ ($i = 1, \dots, m$), $y_k(b-) = A_k$ ($k = m + 1, \dots, n$).

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In the paper certain sufficient conditions for the existence of solutions of problem (1), (2) will be given whose graph lies on the interval I in a *two-point boundary value set* Ω^0 defined in the following way.

Definition 1. Let $\Omega^0 \subset \Omega$ and $\Omega^0 \cap \{(x^*, y) : y \in \mathbb{R}^n\} \neq \varnothing$ for each $x^* \in I$. We will call the set Ω^0 a *two-point boundary value set* if each continuous curve $l = \{(x, y) : x \in I,\ y = y(x)\}$ defined on I, for which the relation $(x, y(x)) \subset \overline{\Omega}^0$ holds on I, has the following limit values:

$$\lim_{x \to a+} y_i(x) = A_i \quad (i = 1, \dots, m), \tag{3}$$

$$\lim_{x \to b-} y_k(x) = A_k \quad (k = m+1, \dots, n). \tag{4}$$

In the sequel $\Omega^0_{a,b}$ will denote such a type of set Ω^0.

Boundary value problems for systems of ordinary differential equations were considered by many authors (see [1]–[9], for example). Singular boundary value problems of such types were studied in [3]–[8], [10]–[16]. Our results are independent of the known ones. Some specific comparisons to the known results will be made in the paper. The main results are formulated as Theorems 2 and 3.

Main results

Let $\Omega^0 \subset \Omega$ be some open set with the boundary $\partial\Omega^0$. According to Ważewski ([1], [17]), a point $(x_0, y_0) \in \partial\Omega^0 \cap \Omega$ is a point of egress from Ω^0 with respect to system (1) and the set Ω^0 if, for the solution $y = y(x)$ of the problem $y(x_0) = y_0$, there exists $\varepsilon > 0$ such that $(x, y(x)) \in \int \Omega^0$ if $x \in [x_0 - \varepsilon, x_0)$. A point of egress is a point of strict egress from Ω^0 if, moreover, there exists $\varepsilon_1 > 0$ such that $(x, y(x)) \notin \overline{\Omega}^0$ if $x \in (x_0, x_0 + \varepsilon_1]$.

As usual, the set of all points of egress (strict egress) from Ω^0 will be denoted by Ω^0_e (Ω^0_{se}).

Theorem 1 ([1], [17]). *Let $\Omega^0 \subset \Omega$ be some open set such that $\Omega^0_e = \Omega^0_{se}$. Assume that S is a nonempty subset of $\Omega^0 \cup \Omega^0_e$ such that the set $S \cap \Omega^0_e$ is not a retract of S but is a retract of Ω^0_e.*

Then there is at least one point $(x_0, y_0) \in S \cap \Omega^0$ such that the graph of the solution $y(x)$ of the Cauchy problem $y(x_0) = y_0$ lies in Ω^0 on its right-hand maximal interval of existence.

In a further discussion we will suppose that all sets of the type Ω^0 satisfy all the conditions of Definition 1, i.e., $\Omega^0 = \Omega^0_{a,b}$.

Theorem 2. *Let $\Omega^0 = \Omega^0_{a,b}$ and $\Omega^0_e = \Omega^0_{se}$. Assume that there are nonempty subsets $S_i \subset \{(x, y) \in \Omega,\ x = x_i\} \cap (\Omega^0 \cup \Omega^0_e)$, $i = 1, 2, \dots,$ where $\{x_i\}$ is some decreasing sequence of real numbers with $x_i \in (a, b)$ and $\lim_{i \to \infty} x_i = a$ such that $S_i \cap \Omega^0_e$ is not a retract of S_i but is a retract of Ω^0_e.*

Then there is at least one solution $y = y(x)$ *of problem* (1), (2) *such that its graph lies in* $\overline{\Omega}^0$ *on the interval* (a, b).

Proof. Let the index i be fixed. Then, as follows from Theorem 1, there is at least one point $(x_i, y^i) \in S_i$ such that the graph of the solution $y^i(x)$ of the Cauchy problem $y^i(x_i) = y^i$ for (1) lies in Ω^0 on its right-hand maximal interval of existence, i.e., on the interval $[x_i, b)$. Further, we denote by M_i the set of all initial points from the set $\Omega_i = \{(x, y) \in \overline{\Omega}^0, \, x = x_i\}$ with the property that each point $(x_i, y^{*i}) \in \Omega_i$ defines a solution $y = y^*(x)$ such that its graph lies in $\overline{\Omega}^0$ on $[x_i, b)$. Obviously, $M_i \neq \varnothing$. The set M_i is closed in Ω_i (including the case where M_i consists of one point only) since otherwise we get the contrary with continuous dependence of solutions on the initial data. Let $\chi\{M_i, [x_i, b)\}$ be the set of all solutions of (1) on $[x_i, b)$ defined by the initial data from the set M_i. Then $M_i' \subset M_1$ where $M_i' \equiv \chi\{M_i, [x_i, b)\} \cap \Omega_1$, and if $i > 2$ then $M_i' \subset M_{i-1}'$. Then, as the sets M_i', $i = 1, 2 \ldots$, are compact, there is a nonzero set $M_0 = \cap_{i=1}^{\infty} M_i'$. If a point $(x_1, y_0) \in M_0$ then for the corresponding solution $y = y_0(x)$ we have $(x, y_0(x)) \subset \Omega^0$ on (a, b). As $\Omega^0 = \Omega^0_{a,b}$, by (3) and (4) $\lim_{x\to a+} y_{0i}(x) = A_i$, $i = 1, \ldots, m$, and $\lim_{x\to b-} y_{0i}(x) = A_i$, $i = m+1, \ldots, n$, i.e., the solution $y_0(x)$ is a solution of problem (1), (2) with appropriate properties. □

Now we will suppose that the open region Ω^0 can be described by the functions $n_i \in C^1(\Omega)$, $i = 1, \ldots, l$, and $p_j \in C^1(\Omega)$, $j = 1, \ldots, q$, as follows:

$$\Omega^0 = \Big\{(x, y) \in \Omega, \ x \in I, \ n_i < 0, \ i = 1, \ldots, l, \ p_j < 0, \ j = 1, \ldots, q\Big\}. \tag{5}$$

For $\alpha \in \{1, \ldots, l\}$ we denote

$$N_\alpha = \Big\{(x, y) \in \overline{\Omega}^0 \cap \Omega, \ n_\alpha = 0, \ n_i \le 0, \ i = 1, \ldots, l; \ i \neq \alpha, \\ p_j \le 0, \ j = 1, \ldots, q\Big\}$$

and for $\beta \in \{1, \ldots, q\}$

$$P_\beta = \Big\{(x, y) \in \overline{\Omega}^0 \cap \Omega, \ p_\beta = 0, \ n_i \le 0, \ i = 1, \ldots, l; \\ p_j \le 0, \ j = 1, \ldots, q, \ j \neq \beta\Big\}.$$

Definition 2 ([1]). The open set $\Omega^0 \subset \Omega$ given by (5) is called an (n, p)-*subset with respect to system* (1) if for derivatives of the functions n_α ($\alpha = 1, \ldots, l$) and p_β ($\beta = 1, \ldots, q$) along the trajectories of system (1)

$$dn_\alpha(x, y)/dx < 0, \quad \text{for} \quad (x, y) \in N_\alpha, \tag{6}$$

$$dp_\beta(x, y)/dx > 0, \quad \text{for} \quad (x, y) \in P_\beta. \tag{7}$$

Theorem 3. *Let $f \in C(\Omega, \mathbb{R}^n)$, $f \in L_{loc}(\Omega)$, $\Omega^0 = \Omega^0_{a,b}$ and Ω^0 be an (n,p)-subset with respect to system* (1). *Let us assume that there are nonempty subsets $S_i \subset \{(x,y) \in \Omega,\, x = x_i\} \cap (\Omega^0 \cup \Omega^0_e)$, $i = 1, 2, \dots,$ where $\{x_i\}$ is some decreasing sequence of numbers with $x_i \in (a,b)$ and $\lim_{i\to\infty} x_i = a$ such that $S_i \cap \Omega^0_e$ is not a retract of S_i but is a retract of Ω^0_e.*

Then there is at least one solution $y = y(x)$ of problem (1), (2) *such that its graph lies in Ω^0 on interval I, i.e., the inequalities*

$$n_i(x, y(x)) < 0, \quad i = 1, \dots, l, \tag{8}$$

$$p_j(x, y(x)) < 0, \quad j = 1, \dots, q, \tag{9}$$

hold on interval I.

Proof. From the known result in [1] (Lemma 3.1, §3, Chapter X) it follows that $\Omega^0_e = \Omega^0_{se} = \cap_{\beta=1}^{q} P_\beta \setminus \cap_{\alpha=1}^{l} N_\alpha$. Then Theorem 3 is a consequence of Theorem 2 and that result. In this case $(x, y(x)) \subset \Omega^0$ on I (instead of $(x, y(x)) \subset \overline{\Omega}^0$ on I) because in view of (6), (7) $\{(x,y) \in \Omega,\, x \in I,\, y = y(x)\} \cap \partial\Omega^0 = \varnothing$. □

Applications

(A) Let system (1) be of the form

$$y' = A(x)y + g(x,y), \tag{10}$$

where $A = \{a_{ij}\}_{i,j=1,\dots,n}$, $a_{ij} \in C(I, \mathbb{R})$, $g \in C(\Omega, \mathbb{R}^n)$ and $g \in L_{loc}(\Omega)$.

Let $\delta_i(x)$, $i = 1, \dots, n$ be some functions continuously differentiable and positive on the interval I with the property

$$\lim_{x\to a+} \delta_i(x) = 0 = \lim_{x\to b-} \delta_k(x) \quad (i = 1, \dots, m,\ k = m+1, \dots, n). \tag{11}$$

For some integers m_1, $0 \le m_1 \le m$ and n_1, $0 \le n_1 \le n - m$ and for $(x,y) \in \Omega$ we define the functions

$$N_k(x,y) \equiv N_k(x, y_k) \equiv (y_k - A_k)^2 - \delta_k^2(x), \tag{12}$$

where $k \in \{1, \dots, m_1\} \cup \{m+1, \dots, m+n_1\}$ and

$$P_r(x,y) \equiv P_r(x, y_r) \equiv (y_r - A_r)^2 - \delta_r^2(x), \tag{13}$$

where $r \in \{m_1+1, \dots, m\} \cup \{m+n_1+1, \dots, n\}$. If we put $l = m_1 + n_1$ and $q = n - l$ then by formulas (12), (13) the functions n_i $(i = 1, \dots, l)$ and p_j $(j = 1, \dots, q)$ are defined as follows:

$$n_i \equiv \begin{cases} N_i & \text{if} \quad i \in \{1, \dots, m_1\}, \\ N_{i-m_1+m} & \text{if} \quad i \in \{m_1+1, \dots, m_1+n_1\}, \end{cases} \tag{14}$$

$$p_j \equiv \begin{cases} P_{j+m_1} & \text{if } \ j \in \{1, \dots, m - m_1\}, \\ P_{j+m_1+n_1} & \text{if } \ j \in \{m - m_1 + 1, \dots, n - m_1 - n_1\}. \end{cases} \tag{15}$$

In such a case the sets Ω^0, N_α, $\alpha \in \{1, \dots, l\}$, and P_β, $\beta \in \{1, \dots, q\}$, have the following simpler form:

$$\Omega^0 = \Big\{ x \in I, \ |y_i - A_i| < \delta_i(x), \ i = 1, \dots, n \Big\}, \tag{16}$$

$$N_\alpha = \Big\{ x \in I, \ |y_\alpha - A_\alpha| = \delta_\alpha(x), \ |y_i - A_i| < \delta_i(x), \\ i = 1, \dots, n, \ i \neq \alpha \Big\}, \tag{17}$$

$$P_\beta = \Big\{ x \in I, \ |y_\beta - A_\beta| = \delta_\beta(x), \ |y_i - A_i| < \delta_i(x), \\ i = 1, \dots, n, \ i \neq \beta \Big\}, \tag{18}$$

In the proof of the next theorem we apply Theorem 3.

Theorem 4. *Assume that:*

(a) There are continuously differentiable and positive functions $\delta_i(x)$, $i = 1, \dots, n$, on the interval I with property (11).

(b) The inequality

$$\sum_{j=1, j \neq \alpha^0}^{n} |a_{\alpha^0 j}(x)| \delta_j(x) + \sum_{j=1}^{n} |a_{\alpha^0 j}(x) A_j| + |g_{\alpha^0}(x, y)| < \\ < \delta'_{\alpha^0}(x) - a_{\alpha^0 \alpha^0}(x) \delta_{\alpha^0}(x) \tag{19}$$

holds for each $\alpha^0 \in \{1, \dots, m_1\} \cup \{m+1, \dots, m+n_1\}$ and $(x, y) \in N_\alpha$, where $\alpha = \alpha^0$ if $\alpha^0 \in \{1, \dots, m_1\}$ and $\alpha = \alpha^0 + m_1 - m$ if $\alpha^0 \in \{m+1, \dots, m+n_1\}$.

(c) The inequality

$$\sum_{j=1, j \neq \beta^0}^{n} |a_{\beta^0 j}(x)| \delta_j(x) + \sum_{j=1}^{n} |a_{\beta^0 j}(x) A_j| + |g_{\beta^0}(x, y)| < \\ < a_{\beta^0 \beta^0}(x) \delta_{\beta^0}(x) - \delta'_{\beta^0}(x) \tag{20}$$

holds for each $\beta^0 \in \{m_1 + 1, \dots, m\} \cup \{m + n_1 + 1, \dots, n\}$ and $(x, y) \in P_\beta$, where $\beta = \beta^0 - m_1$ if $\beta^0 \in \{m_1 + 1, \dots, m\}$ and $\beta = \beta^0 - n_1 - m_1$ if $\beta^0 \in \{m + n_1 + 1, \dots, n\}$.

Then there is at least one solution $y = y(x)$ of problem (10), (2) *such that for its components the inequalities*

$$|y_i(x) - A_i| < \delta_i(x), \quad i = 1, \dots, n, \tag{21}$$

hold on the interval I.

Proof. First we prove that the set Ω^0 described by (16) (where the functions n_i $(i=1,\dots,l)$, p_j $(j=1,\dots,q)$ are defined by formulas (14), (15)) satisfies the property $\Omega^0 = \Omega^0_{a,b}$ and generates some (n,p)-subset with respect to system (10). The property $\Omega^0 = \Omega^0_{a,b}$ is a consequence of formulas (11) and (16). Indeed, if $l = \{(x,y) : x \in I,\ y = y(x)\}$ is a continuous curve for which the relation $(x, y(x)) \subset \overline{\Omega}^0$ holds on I, then from (11) and (16) it follows that

$$\lim_{x\to a+} y_i(x) = A_i,\ \ i \in \{1,\dots,m\},\quad \lim_{x\to b-} y_k(x) = A_k\ \ k \in \{m+1,\dots,n\}.$$

Further we will compute the derivative of the function n_α, $\alpha \in \{1,\dots,l\}$, along the trajectories of system (10) on the set N_α. In view of (17) and (19) we obtain

$$\frac{dn_\alpha(x,y)}{dx} = 2(y_\alpha - A_\alpha)y'_\alpha - 2\delta_\alpha\delta'_\alpha = 2(y_\alpha - A_\alpha)\Bigg[\sum_{j=1, j\neq\alpha}^{n} a_{\alpha j}(y_j - A_j) + \\ + \sum_{j=1}^{n} a_{\alpha j}A_j + g_\alpha + a_{\alpha\alpha}(y_\alpha - A_\alpha)\Bigg] - 2\delta_\alpha\delta'_\alpha < \\ < 2\delta_\alpha\Bigg[a_{\alpha\alpha}\delta_\alpha - \delta'_\alpha + \sum_{j=1, j\neq\alpha}^{n} |a_{\alpha j}|\delta_j + \sum_{j=1}^{n} |a_{\alpha j}A_j| + |g_\alpha|\Bigg] < 0.$$

By analogy we can compute that in view of (18) and (20) for the derivative of the function p_β, $\beta \in \{1,\dots,q\}$, along the trajectories of system (10) the inequality $dp_\beta/dx > 0$ holds on the set P_β. Inequalities (6) and (7) hold and, by Definition 2, the set Ω^0 is an (n,p)-subset with respect to system (10).

Let $\{x_i\}$ be some decreasing sequence of numbers with $x_i \in I$ and $\lim_{i\to\infty} x_i = a$. For each fixed i we denote $S_i = (\Omega^0 \cup \Omega^0_e) \cap \{(x_i, y) : y \in \mathbb{R}^n\}$, where

$$\Omega^0_e = \Omega^0_{se} = \bigcup_{\beta=1}^{q} P_\beta \setminus \bigcup_{\alpha=1}^{l} N_\alpha$$

(see the proof of Theorem 3). The set $S_i \cap \Omega^0_e$ is a retract of the set Ω^0_e because the continuous mapping

$$\Pi:\ (x,y) \in \Omega^0_e \mapsto (x_i, y^0) \in S_i \cap \Omega^0_e,$$

with

$$y^0_j = A_j - \delta_j(x_i) + \big(y_j - A_j + \delta_j(x)\big)\frac{\delta_j(x_i)}{\delta_j(x)}\ ,\quad j = 1,\dots,n,$$

is identical on $S_i \cap \Omega_e^0$. On the other hand, the set $S_i \cap \Omega_e^0$ is not a retract of the set S_i. This follows from the fact that the set $\widetilde{S}_i \subset S_i$, where

$$\widetilde{S}_i = \Big\{(x,y) \in S_i,\ x = x_i,\ y_j = C_j \in (A_j - \delta_j(x_i), A_j + \delta_j(x_i)),\\ C_j = const,\ j = 1,\dots,m_1;\ m+1,\dots,m+n_1\Big\}$$

with the property that $\widetilde{S}_i \cap \Omega_e^0 \subset S_i \cap \Omega_e^0$, is not a retract of the set $\widetilde{S}_i \cap \Omega_e^0$ as the boundary of the sphere is not its retract ([18]). Consequently, all the assumptions of Theorem 3 are fulfilled and therefore Theorem 4 is valid. We obtain inequalities (21) from inequalities (8) and (9) or from (16). □

Example 1. Let problem (10), (2) be of the form

$$\begin{aligned} y_1' &= -4x^{-2}y_1 + x^5(x-1)^{-1}y_2 + \cos y_2,\\ y_2' &= (x-1)^4 x^{-1} y_1 + 4(x-1)^{-2}y_2 + \cos y_1,\\ y_1(0+) &= y_2(1-) = 0. \end{aligned}$$

Then all the assumptions of Theorem 4 are fulfilled if we put $n = 2$, $a_{11}(x) = -4x^{-2}$, $a_{12}(x) = x^5(x-1)^{-1}$, $a_{21}(x) = (x-1)^4x^{-1}$, $a_{22}(x) = 4(x-1)^{-2}$, $g_1(x,y) = \cos y_2$, $g_2(x,y) = \cos y_1$, $m_1 = 1$, $n_1 = 0$, $m = 1$, $a = 0$, $b = 1$, $A_1 = A_2 = 0$, $\delta_1(x) = x$, $\delta_2(x) = 1 - x$. Consequently, problem (10), (2) has at least one solution $y = y(x)$ such that $|y_1(x)| < x$, $|y_2(x)| < 1 - x$ on $(0,1)$.

Remark 1. [6] contains some theorems on the existence and uniqueness of solutions of singular Cauchy–Nicoletti problems for systems of ordinary differential equations. We note that these theorems are independent of the above-proved results. For example, if we apply Theorem 4.1 from [6, Chapter II, §4, pp. 37–38] to Example 1 then, in addition, the inequality

$$\big(-4x^{-2}y_1 + x^5(x-1)^{-1}y_2 + \cos y_2\big)\,\mathrm{sign}\, y_1 \le -a(x)|y_1| + g(x,|y_1|,|y_2|)$$

must be valid on a set $\{(x,y) : 0 < x < 1,\ y \in \mathbb{R}^2\}$, where $a(x) \ge 0$, $a(x) \in L(0+,1-)$ on $(0,1)$, and

$$\sup\Big\{|g(x,|y_1|,|y_2|)| :\ |y_1| + |y_2| \le \rho\Big\} \in L(0,1) \tag{22}$$

for each $\rho \in (0,+\infty)$. In our case $a(x) \equiv -4x^{-2}$, $g(x,|y_1|,|y_2|) \equiv x^5(x-1)^{-1}|y_2| + \cos|y_2|$ and, consequently, relation (22) does not hold.

(**B**) Let system (23) be of the form

$$y_1' = f(x)y_1 + F(x,y_1,y_2), \quad y_2' = y_1, \tag{23}$$

where $f \in C(I,\mathbb{R})$, $F \in C(\Omega,\mathbb{R}^2) \cap L_{loc}(\Omega)$. For system (23) we consider problem (2) if $a = 0$, $b = T$, $0 < T = \text{const}$, $m = 1$, $A_1 = 0$, $A_2 = -\alpha$, $0 \le \alpha = \text{const}$.

Theorem 5. *Let there exist a positive function $h \in C^1(I_1,\mathbb{R}^+)$, $I_1 = (0,T)$ and a negative function $\omega \in C^1(I_1,\mathbb{R}^-)$ such that $h(0+) = 0$, $\omega(x) < -\alpha$ on I_1, $\omega(T-) = -\alpha$, $h(x) < \omega'(x)$ on I_1 and on the set*

$$\mathcal{D} = \big\{(x,y_2) : \ x \in I_1,\ \omega(x) < y_2 < -\alpha\big\}$$

the following inequalities hold:

$$f(x)h(x) - h'(x) + F(x,h(x),y_2) < 0 < F(x,0,y_2). \tag{24}$$

Then there is at least one solution $y = y(x)$ of problem (23), (25) *where*

$$y_1(0+) = 0, \quad y_2(T-) = -\alpha \tag{25}$$

such that the inequalities $0 < y_1(x) < h(x)$, $\omega(x) < y_2(x) < -\alpha$ hold on I_1.

Proof. Let $n_1 \equiv y_1(y_1 - h(x))$ and $p_1 \equiv (y_2 - \omega(x))(y_2 + \alpha)$. Then the set Ω^0 defined by (5) satisfies the condition $\Omega^0 = \Omega^0_{0,T}$. Compute the derivatives along the trajectories of system (23). We obtain

$$\frac{dn_1(x,y)}{dx} = \big[f(x)y_1 + F(x,y_1,y_2)\big](y_1 - h(x)) + \\ + y_1\big[f(x)y_1 + F(x,y_1,y_2) - h'(x)\big].$$

For the value y_1 we have $y_1 = h(x)$ or $y_1 = 0$ on the set $N_1(x,y)$. Then from (24) it follows that $dn_1(x,y)/dx < 0$. Analogously, $dp_1(x,y)/dx > 0$ on the set $P_1(x,y)$. Consequently, the set Ω^0 is an (n,p)-subset.

The property that for some decreasing sequence of numbers $\{x_i\}$ with $x_i \in I_1$ and $\lim_{i\to\infty} x_i = 0$ there is a set S_i with the properties described in Theorem 3 can be verified in a similar fashion as in the corresponding part of the proof of Theorem 4. Now all the assumptions of Theorem 3 are fulfilled and therefore Theorem 5 holds. □

Example 2. In system (23) let us put $f(x) = -Lx^{-m}$, where $0 < L = \text{const}$ and $0 < m = \text{const}$. Let $h(x) = \varepsilon x^p$ where $\varepsilon T^p < \alpha$, $0 < \varepsilon = \text{const}$, p is an even positive number, $\omega(x) = [-\alpha - (x-T)^p]\exp(T-x)$, $F(x,0,y_2) > 0$, and $F(x,h(x),y_2) < \varepsilon x^p(Lx^{-m} + px^{-1})$ on $\mathcal{D}$. Then all the assumptions of Theorem 5 are valid and its conclusion is true.

Remark 2. Some classes of singular problems were studied in [14], [15]. For example, in [15] the problem

$$y_1' = -(N-1)y_1/x + F_1(y_2,x), \quad y_2' = y_1, \tag{26}$$

$$y_1(0+) = 0, \quad y_2(T-) = -\alpha \le 0, \tag{27}$$

where $2 \leq N$, N is an integer, $x \in I_1$ and $F_1 \in C^1(\mathbb{R}^- \times I_1, \mathbb{R}^+)$, is considered in connection with the study of increasing negative radial solutions of semilinear elliptic equations. In particular, this work contains the following result:

Let $0 \leq d \leq l \leq NT^{-1}$, $0 < K < sT^{-1}$, and $0 < F_1(y_2, x) < (N - lx)K \exp(-lx)$ hold for some constants d, l, K, and s if $\psi(x) \equiv -\alpha - s(T - x)\exp(-dx) < y_2 < -\alpha$ and $x \in I_1$. Then problem (26), (27) has at least one solution $y = y(x)$ which satisfies the inequalities $0 < y_1(x) < \varphi(x) \equiv Kx\exp(-lx)$ and $\psi(x) < y_2(x) < -\alpha$ on I_1.

We note that problem (23), (25) is more general that the one given above. If we put $f(x) = -(N-1)x^{-1}$ and $F(x, y_1, y_2) \equiv F_1(y_2, x)$ then from Theorem 5 (if $h \equiv \varphi$ and $\omega \equiv \psi$) it follows that there is at least one solution of problem (26), (27) with the mentioned properties. Moreover, as Example 2 shows, we may obtain more precise estimations of this solution if the functions h and ω are chosen in a proper way.

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