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OBSTRUCTIONS TO THE SECTION PROBLEM IN A FIBRATION WITH A WEAK FORMAL BASE

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ABSTRACT. For a class of Serre fibrations $F \longrightarrow E \xrightarrow{\xi} X$ with a weak formal base X (or with a degenerated A_∞-algebra structure on the integral cohomology $H^*(X)$), obstructions are defined by means of spherical twisting cochains of ξ. In particular, for a given section $s^n : X^n \longrightarrow E$ on n-skeleton of X, the problem of avoiding the $(n+1)$th obstruction $o(s^n) \in H^{n+1}(X; \pi_n(F))$ to the existence of a section on X^{n+1} reduces to solving a system of linear equations with respect to cohomology elements of the groups $H^i(X; \pi_i(F))$, $i < n$. Homotopy classification theorems for sections as well as for weak formal maps are given, too.

1. INTRODUCTION

The paper continues the study of the obstruction theory to the section problem in a Serre fibration which we began in [1], [2]. We consider a Serre fibration $F \longrightarrow E \xrightarrow{\xi} X$ of path connected spaces, where X is a polyhedron and weak formal (weak $\mathbb{Z}$-formal in the terminology of [1]), i.e., an A_∞-algebra structure in the sense of Stasheff on the integral cohomology $H^*(X)$ [3] is degenerate. We also assume that $\pi_1(X)$ acts trivially on $\pi_*(F)$ and $H_*(F)$. Here we extend the class of spaces playing the role of a fibre F by replacing the split injectivity of the Hurewicz homomorphism

$$u : \pi_i(F) \longrightarrow H_i(F)$$

by condition (A) at the cost of condition (B) below.

Namely, for the spaces X and F, we consider the following condition:

(A) The short sequence

$$0 \longrightarrow H^i(X; \pi_{i-1}(F)) \xrightarrow{u^*} H^i(X; H_{i-1}(F))$$

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is exact, where u^* is induced by the Hurewicz homomorphism u in the coefficients.

To state condition (B) first consider the differential (tri)-graded algebra (dga)

$$(\mathcal{H}_H, d), \quad \mathcal{H}_H^{i,j,t} = H^i\big(X; \mathrm{Hom}^{j,t}(RH_*(F), RH_*(F))\big),$$

with total degree $n = i - j - t$, where

$$\rho : (R_{\geq 0}H_q(F), \partial^R) \to H_q(F), \quad \partial^R : R_iH_q(F) \to R_{i-1}H_q(F),$$

is a fixed free group resolution of $H_q(F)$ (for more details see Section 2). An element $h \in \mathcal{H}_H$, $h = \{h^{i,j,t}\}$, is referred to as twisting, if it is of total degree 1, $i - j \geq 2$, and $d(h) = -hh$. Denote

$$(\tilde{C}_n^k, \partial^{R*}) = \Big(\prod_{j\geq 0} H^{k+j+n}(X; R_jH_n(F)), \partial^{R*}\Big)$$

and

$$(\tilde{L}_n^k, \partial^{R*}) = \Big(\prod_{j\geq 0} H^{k+j+n}(X; R_j\pi_n(F)), \partial^{R*}\Big).$$

For convenience, we can consider $\tilde{L}_*^*$ as a subcomplex of $\tilde{C}_*^*$, $i : \tilde{L}_*^* \subset \tilde{C}_*^*$, by putting $\pi_0(F) = \mathbb{Z}$ and appropriately choosing a resolution $RH_*(F)$.

We have the homomorphism

$$d_h : \tilde{C}_*^k \to \tilde{C}_*^{k+1}$$

defined by $d_h = \partial^R + h\cup\_$.

For any twisting element $h \in \mathcal{H}_H$, $h = \{h^{i,j,t}\}$, with $d_{h^{i,j,t}}$ preserving $\tilde{L}_*^*, i < m$, we consider the following condition:

(B) For any elements $a^{(n)} = (a^1, \ldots, a^n)$, $b^{(n-1)} = (b^1, \ldots, b^{n-1})$, $a^j \in \tilde{C}_j^0$, $b^j \in \tilde{L}_j^0$, $n < m - 1$, with $\partial^{R*}(a^j) = \partial^{R*}(b^j) = 0$, $[d_h(a^{(n-1)} - b^{(n-1)})]^{j+1} = 0 \in H^{j+1}(X; \pi_j(F))$, $j \leq n$, $(d_h(a^{(n)}))^{n+2} \in \tilde{L}_{n+1}^1$, there exists $b^n \in \tilde{L}_n^0$ such that $\partial^{R*}(b^n) = 0$ and $[d_h(a^{(n)} - b^{(n)})]^{n+2} = 0 \in H^{n+2}(X; \pi_{n+1}(F))$.

For example, (B) is easily satisfied in the following cases:

(B1) $u^* : H^i(X; \pi_i(F)) \to H^i(X; H_i(F))$, $i > 0$, is an epimorphism.
(B2) $H^*(X)$ has the trivial multiplication.
(B3) There is a homomorphism $\tilde{\beta} : \tilde{C}_*^q \to \tilde{L}_*^q$ with $\tilde{\beta} \circ i = id$ and $\tilde{\beta} \circ d_h = d_h \circ \tilde{\beta}$ for $q = -1, 0, 1$.
(B4) $H^{2i}(X; H_{2j}(F)) = 0$ and $H^{2i}(X; \pi_{2i-1}(F)) = 0$, $i, j > 0$.
(B5) $h = 0$.

To a fibration ξ with a weak formal base X we assign a twisting element $h \in \mathcal{H}_H$ (induced, in fact, by Brown's twisting cochain [4]) which is well defined modulo the action of the multiplicative group of the algebra $\mathcal{H}_H$. Thus the factorization of all twisting elements by this action yields the set $D_H(X; H_*(F))$ so that we have the class $d_H(\xi)$ of h in this set. In fact, there is a bijection

$$\lambda : D(X; H_*(F)) \approx D_H(X; H_*(F))$$

and $d_H(\xi) = \lambda(d(\xi))$, where $D(X; H_*(F))$ and $d(\xi)$ are defined in [5] (see also [2], Section 2).

We denote by h_0 a component of a twisting element h in the subcomplex $H^*(X; RH_*(F)) = H^*(X; Hom(RH_0(F), RH_*(F)))$, and we will refer to it as the *transgressive* one. We say that $d_H(\xi)$ is *transgressively trivial* (in *perturbation degrees* $\leq n$), if there is $h \in d_H(\xi)$, $h = \{h^{i,j,t}\}$ with $h_0 = 0$ ($h_0^{i,j,t} = 0, i - j \leq n$). Next, we say that the fibration ξ satisfies conditions (A) and (B), if the base X and fibre F satisfy (A) and there is some $h \in d_H(\xi)$ satisfying (B).

We have the following main theorem:

Theorem 1.1. *Let $F \to E \xrightarrow{\xi} X$ be a Serre fibration with X weak formal and satisfying (A) and (B). Then ξ has a section if and only if $d_H(\xi)$ is transgressively trivial.*

This theorem is the key point for our obstruction theory. Namely, let $s^n : X^n \to E$ be a section on n-skeleton of X, and let $o(s^n) \in H^{n+1}(X; \pi_n(F))$ be the obstruction element for extending s^n on X^{n+1}. Using Proposition 2.5 below and the bijection λ, choose a twisting element $g = \{g^{i,j,t}\}$ with $g_0^{i,j,t} = 0$, $i - j \leq n$ and $g^{i,j,t} \in H^i(X; \tilde{\mathrm{H}}\mathrm{om}^{j,t}(RH_*(F), RH_*(F)))$, $i \leq n$, where $\tilde{\mathrm{H}}\mathrm{om}$ consists of those homomorphisms of Hom which preserve $R\pi_*(F)$. Let now ξ satisfy (A), where $H^{i+1}(X; \pi_i(F))$ is identified with its image. Then define the subgroup $I^{n+1}(\xi) \subset H^{n+1}(X; \pi_n(F))$ as follows:

$$I^{n+1}(\xi) = \Big\{ \big[(d_g(y^1 + \cdots + y^n))\big]^{n+1} \in H^{n+1}(X; \pi_n(F)) | \\ (d_g(y^1 + \cdots + y^n))^j = 0,\ j < n, y^j \in \tilde{L}_j^0 \Big\}.$$

The group $I^{n+1}(\xi)$ has a filtration

$$0 = I_0^{n+1}(\xi) \subset \cdots \subset I_{n-1}^{n+1}(\xi) = I^{n+1}(\xi),$$

where $I_j^{n+1}(\xi)$ is obtained from the definition of $I^{n+1}(\xi)$ by putting $y^i = 0$ for $i \leq n - 1 - j$. Further, let $O^{n+1}(\xi)$ be the set consisting of $o(s^n)$ corresponding to all sections on X^n. We find that $O^{n+1}(\xi)$ also has a filtration

$$0 = O_0^{n+1}(\xi) \subset \cdots \subset O_{n-1}^{n+1}(\xi) = O^{n+1}(\xi),$$

where $O_j^{n+1}(\xi)$ is defined by those sections which coincide with each other on X^{n-1-j}.

We have

Theorem 1.2. *Let ξ be as in Theorem 1.1 and having a section $s^n : X^n \to E$. Then*

$$O_j^{n+1}(\xi) = o(s^n) + I_j^{n+1}(\xi), \quad j \leq n-1.$$

Consequently,

$$O^{n+1}(\xi) = o(s^n) + I^{n+1}(\xi).$$

Now $O_j^{n+1}(\xi)$ can be regarded as a single element in the quotient group $H^{n+1}(X;\pi_n(F))/I_j^{n+1}(\xi)$.

Thus we obtain

Corollary 1.3. *There is a section ${s'}^{n+1} : X^{n+1} \to E$ with ${s'}^{n+1}|_{X^m} = s^n|_{X^m}, m < n$, if and only if*

$$O_{n-1-m}^{n+1}(\xi) = 0,$$

i.e., there exist elements $y^i \in \tilde{L}_i^0, i = 1, \ldots, n-1$, so that the following system of equalities holds:

$$\partial^{R*}(y^1) = 0, \partial^{R*}(y^2) + g^2 y^1 = 0, \ldots, \left[\sum_{j=1}^{n-1} g^{n+1-j} y^j\right] = o(s^n).$$

The problem of the vanishing of the $(n+1)$th obstruction element for $n \geq 2$ reduces to solving a system of linear equations with respect to elements of the groups $H^{*+i}(X; R_*\pi_i(F))$, $i \leq n-1$.

The definition of the group $I^n(\xi)$ suggests a further formalization of twisting cochains by replacing homology groups by homotopy ones of the fibre so that we could develop an obstruction theory (see Section 3). In fact, the twisting element g determining $I^n(\xi)$ is restricted to a spherical twisting element $\nu \in H^*(X^{n+1}; \mathrm{Hom}(R\pi_*(F), R\pi_*(F)))$ which defines the element $do^{n+1}(\xi)$ in the set

$$D_H^{n+1}(X;\pi_*(F)) = D_H(X^{n+1};\pi_*(F)).$$

We refer to $D_H^{n+1}(X;\pi_*(F))$ as the $(n+1)$th obstruction functor and to $do^{n+1}(\xi)$ as the $(n+1)$th obstruction element of ξ, being motivated by [6] (see also [7]), where an attempt is made to define such functors and elements without any assumption on the existence of a section.

The role of spherical twisting elements is emphasized for the homotopy classification of sections, too. Namely, for a spherical twisting element $\tilde{\nu} \in do^{n+1}(\xi)$, we can form the complex

$$(\tilde{L}_*^*, d_{\tilde{\nu}}), \quad d_{\tilde{\nu}} = \partial^{R*} + \tilde{\nu} \cup_- .$$

Then we obtain the following generalization of Theorem 2.10 of [2]:

Theorem 1.4. *Let ξ with a weak formal base have a section $s^n : X^n \to E$. If ξ satisfies (B3), then there is a bijection*

$$[X^{n-1}, E]_s \approx H^0(\tilde{L}_{(n-1)}, d_{\tilde{\nu}}),$$

where $[\ ,\ ]_s$ denotes the set of homotopy classes of sections.

2. Weak Formal Spaces, Maps and the Functor D

First we recall some facts about weak $\mathbb{Z}$-formal spaces and maps [1] which for simplicity we will call weak formal ones.

A space X is called weak formal, if the cohomology algebra $H^*(X)$ and the singular cochain algebra $C^*(X)$ are weak homotopy equivalent, i.e., there are a differential graded algebra (dga) A and maps of dga's

$$H^*(X) \xleftarrow{\rho} A \xrightarrow{k} C^*(X)$$

inducing an isomorphism in cohomology.

It is not hard to show that this is equivalent to the fact that an A_∞-algebra structure on $H^*(X)$ ([3], [8]) is degenerated.

Moreover, a map $f : X \to Y$ is called weak formal, if there exists the following (derivation) homotopy commutative diagram of dga's

$$\begin{array}{ccc}
C^*(X) & \xleftarrow{C(f)} & C^*(Y) \\
\uparrow k & & \uparrow k \\
A_X & \longleftarrow & A_Y \\
\downarrow \rho & & \downarrow \rho \\
H^*(X) & \xleftarrow{H(f)} & H^*(Y).
\end{array}$$

For example, any suspension, a space X with $H^*(X)$ polynomial or with $H^i(X) = 0$ for $i < n$ and $i > 3n - 2$ are weak formal spaces. A suspension map [1], a map $X \to Y$, where X, Y are weak formal and $H^i(X) = 0$ for $i > 2n - 2$ and $H^i(Y) = 0$ for $i < n$, are weak formal maps.

Moreover, we have

Proposition 2.1. *Any map $X \to Y$ is weak formal provided X is weak formal and $H^*(Y)$ is polynomial.*

Proof. It is analogous to that of Theorem 5.6 [1]. □

Now we recall the definition of the set $D(A)$ for a dga (A,d) (cf. [5]): Suppose $A=\{A^{i,j}\}$ is bigraded with $d:A^{i,j}\to A^{i+1,j}$ and total degree $n=i-j$. By definition, $D(A)=M(A)/G(A)$, where

$$M(A)=\Big\{a\in A^1|d(a)=-aa,\ a=a^{2,1}+a^{3,2}+\cdots\Big\},$$
$$G(A)=\{p\in A^0|p=1+p^{1,1}+p^{2,2}+\cdots\Big\},$$

and the action $G(A)\times M(A)\to M(A)$ is given by the formula

$$p*a=pap^{-1}+d(p)p^{-1}.$$

In other words, two elements $a,a'\in M(A)$ are on the same orbit if there is $p\in G(A)$, $p=1+p'$, with $a'-a=p'a-a'p'+d(p')$.

We have the following two theorems (cf. [1], [5]):

Theorem 2.2. *If two dga maps $f,g:A\to B$ are homotopic, then $D(f)=D(g):D(A)\to D(B)$.*

Thus D becomes the functor on the category of dga's and (derivation) homotopy classes of dga maps to the category of pointed sets.

Another useful property of D is the following comparison theorem:

Theorem 2.3. *If $f:A\to B$ is a cohomology isomorphism, then $D(f):D(A)\to D(B)$ is a bijection.*

Now let H_* be a graded group and

$$\rho:\big(R_{\geq 0}H_q,\partial^R\big)\to H_q,\quad \partial^R:R_iH_q\to R_{i-1}H_q,$$

its free group resolution. For a space X, consider the dga

$$(\mathcal{H},\nabla)=\big(C^*(X;Hom(RH_*,RH_*)),\nabla=d^C+\partial^{R*}\big)$$

which is bigraded via $\mathcal{H}^{r,t}=\prod_{r=i-j}C^i(X;Hom^{j,t}(RH_*,RH_*))$, where $Hom(RH_*,RH_*)$ is the standard bigraded complex (algebra): $f^{s,t}:R_jH_q\to R_{j+s}H_{q+t}$ if $f^{s,t}\in Hom^{s,t}(R_*H_*,R_*H_*)$. We refer to i as the *base topological degree*, to j as the *fibre resolution degree*, to t as the *fibre weight*, and to $n=i-j-t$ as the *total degree*. Moreover, we refer to $r=i-j$ as the *perturbation* degree, which will be exploited by the induction arguments below (in particular, when $RH_*=H_*$, the perturbation degree coincides with the base topological one). Thus we have

$$\mathcal{H}=\{\mathcal{H}^n\},\quad \mathcal{H}^n=\prod_{n=r-t}\mathcal{H}^{r,t},$$
$$\nabla:\mathcal{H}^{r,t}\to\mathcal{H}^{r+1,t}.$$

A twisting cochain h is an element of $\mathcal{H}$ of total degree 1 and at least of perturbation degree 2 satisfying the condition $\nabla h = -hh$, i.e., h has the form

$$h = h^2 + \cdots + h^r + \cdots, \quad h^r \in \mathcal{H}^{r,r-1}.$$

For the perturbation degree the condition that h is twisting reads as

$$\nabla(h^2) = 0, \quad \nabla(h^3) = -h^2h^2, \quad \nabla(h^4) = -h^2h^3 - h^3h^2, \ldots .$$

Note that a single superscript for an element of $\mathcal{H}$ will always denote the perturbation degree. On the other hand, it is obvious that the dga $\mathcal{H}_H = (H^*(X; \mathrm{Hom}(RH_*, RH_*)), \partial^{R*})$ inherits all the gradings from $\mathcal{H}$.

Define

$$D(X; H_*) = D(\mathcal{H})$$

and

$$D_H(X; H_*) = D(\mathcal{H}_H).$$

Then we have

Proposition 2.4. *There is an isomorphism of functors*

$$\lambda : D(X; H_*) \approx D_H(X; H_*)$$

on the category of weak formal spaces and weak formal maps.

Proof. It follows from Theorems 2.3 and 2.2. □

Now let $F \longrightarrow E \xrightarrow{\xi} X$ be a Serre fibration. Then there is a twisting cochain $h \in \mathcal{H}$, $H_* = H_*(F)$, which defines a twisted differential on the tensor product $C_*(X) \otimes RH_*(F)$ to obtain the Hirsch complex of ξ [9], [4], [10], [5]. The class of h in $D(X; H_*)$ is denoted by $d(\xi)$ and is referred to as (homological) predifferential.

The image of $d(\xi)$ under λ will be denoted by $d_H(\xi)$.

By a slight modification of the proof of Proposition 2.4 we obtain [2]

Proposition 2.5. *Let ξ have a section $s^n : X^n \longrightarrow E$. Then there is a twisting cochain $h \in d(\xi)$ with $h_0^r = 0$, $r \leq n$, $\rho{*}(h_0^{n+1}) = u^*(c(s^n))$, and $h^r(\sigma) : RH_*(F) \longrightarrow RH_*(F)$, $r \leq n$, preserves the subgroup $R\pi_*(F) \subset RH_*(F)$ for each simplex $\sigma \in X^n$.*

3. Spherical Twisting Cochains and Obstruction Functors $D^n(X;\pi_*)$

The ordinary theory of twisting cochains and, in particular, the set $D(X;H_*)$ were available for the study of the homology theory of a fibration. However, for the needs of the obstruction theory to the section problem in a fibration we defined, in [2], a spherical twisting cochain by a further investigation of the connection between a twisting cochain and the obstruction cocycle for extending a section [11]. Motivated by [7],[6], this can be formalized to obtain the so-called obstruction functors $D^n(X;\pi_*)$ and obstruction elements of a fibration in the manner as follows: First we simply define

$$D^n(X;\pi_*) = D(X^n;\pi_*).$$

But difficulties arise when we want to assign to a fibration ξ an element in $D^n(X;\pi_*)$ similarly to $d(\xi)$ in the case $\pi_* = \pi_*(F)$ (assuming $\pi_0 = \mathbb{Z}$). The reason is that homotopy groups are not realizable as homologies of a functorial chain complex of a space. Instead, for a fibration ξ with a section $s^{n-1} : X^{n-1} \to E$ we use Proposition 2.5 to obtain a twisting cochain $h \in d(\xi|_{X^n})$ so that $h(\sigma)$, $\sigma \in X^{n-1}$, preserves $R\pi_*(F) \subset RH_*(F)$, $h_0^r = 0$, $r \leq n-1$, and $\rho^*(h_0^n) = u^*(c(s^{n-1}))$. Restrict $h^{(n-1)} + h_0^n$ to $C^*(X^n; \mathrm{Hom}(R\pi_*(F), R\pi_*(F)))$ to obtain a spherical twisting cochain ν of ξ. The class of ν in $D^n(X;\pi_*(F))$ is called the *obstruction element* of ξ and is denoted by $dc^n(\xi)$. We will refer to $D^n(X;\pi_*(F))$ as the nth *obstruction functor* (cf. [7], [6], where an attempt is made to define the global obstruction functor without any assumption on the existence of a section).

By Proposition 2.4 we also have a bijection

$$\lambda_\pi : D^n(X;\pi_*) \approx D^n_H(X;\pi_*).$$

Define $do^n(\xi) = \lambda_\pi(dc^n(\xi))$

Now we get a criterion for the existence of a section in terms of the obstruction element:

Theorem 3.1. *Let ξ be as in Corollary 1.3. Then there exists a section on X^{n+1} if and only if $do^{n+1}(\xi)$ is transgressively trivial.*

Proof. It is easy to see that the conditions of the theorem are equivalent to $O^{n+1}(\xi) = 0$ and the proof follows from Corollary 1.3. $\square$

4. The Proofs of Theorems 1.1 and 1.2

The proof of Theorem 1.1 goes along the lines of that of Theorem 2.8 [2]. Given a section of ξ, the existence of a twisting cochain $h \in d(\xi)$ with $h_0 = 0$ follows from Proposition 2.5. To prove the converse we need to recall some previous constructions.

We have that ξ defines a colocal system of singular chain complexes over the base X: To each simplex $\sigma \in X$ is assigned the complex

$$(C_*(F_\sigma), \gamma_\sigma), \quad F_\sigma = \xi^{-1}(\sigma),$$

and to a pair $\tau \subset \sigma$ the induced chain map

$$C_*(F_\tau) \to C_*(F_\sigma).$$

Then $\sigma \to Hom(RH_*(F), C_*(F_\sigma))$ also forms a colocal system over X. Define $\mathcal{K}$ canonically as the simplicial cochain complex of X with coefficients in the latter colocal system:

$$\mathcal{K} = \{\mathcal{K}^{i,j,t}\}, \quad \mathcal{K}^{i,j,t} = C^i\big(X; Hom^{j,t}(RH_*(F), C_*(F_\sigma))\big)$$

(C_* is regarded as bigraded via $C_{0,*} = C_*$, $C_{j>0,*} = 0$). Hence $\mathcal{K}$ becomes a bicomplex via

$$\mathcal{K}^{r,t} = \prod_{r=i-j} \mathcal{K}^{i,j,t},$$

$$\delta : \mathcal{K}^{r,t} \to \mathcal{K}^{r+1,t}, \quad \delta = d^C + \partial^R,$$

$$\gamma : \mathcal{K}^{r,t} \to \mathcal{K}^{r,t-1}, \quad \gamma = \{\gamma_\sigma\}.$$

For convenience we refer to the gradings of $\mathcal{K}$ as to those of $\mathcal{H}$. Next we have a natural differential graded pairing (defined by the $\cup$-product and by the composition of homomorphisms in coefficients)

$$(\mathcal{K}, \delta + \gamma) \otimes (\mathcal{H}, \nabla) \to (\mathcal{K}, \delta + \gamma),$$

and, since $\gamma(kh) = \gamma(k)h$, an induced differential graded pairing

$$(\mathcal{K}_\gamma, \delta_\gamma) \otimes (\mathcal{H}, \nabla) \to (\mathcal{K}_\gamma, \delta_\gamma)$$

where

$$(\mathcal{K}_\gamma, \delta_\gamma) = (H(\mathcal{K}, \gamma), \delta_\gamma) = C^*\big(X; Hom(RH_*(F), H_*(F))\big).$$

Note that there is an epimorphism

$$\rho^* : (\mathcal{H}, \nabla) \to (\mathcal{K}_\gamma, \delta_\gamma)$$

induced by the resolution map ρ above. Clearly, it induces an isomorphism in cohomology.

Now consider the equation

$$(\delta + \gamma)(k) = kh \tag{1}$$

with respect to a pair (k, h),

$$k = k^0 + \cdots + k^r + \cdots, \quad k^r \in \mathcal{K}^{r,r},$$

$$h = h^2 + \cdots + h^r + \cdots, \quad h^r \in \mathcal{H}^{r,r-1}.$$

We also have the initial conditions

$$\nabla(h) = -hh,$$
$$\gamma(k^0) = 0, \quad [k^0]_\gamma = e \in \mathcal{K}_\gamma^{0,0}, \quad e = \rho^*(1), \quad 1 \in \mathcal{H}.$$

A twisting cochain $h \in d(\xi)$ is just a solution of this equation. On the other hand, a given section, s, of ξ can be regarded as an element k_0 of $\mathcal{K}_0 = C^*(X; C_*(F_\sigma)) \subset \mathcal{K}$ as follows: $k_0^r(\sigma^r) = s|_{\sigma^r}$, for an r-simplex $\sigma^r \in X$, $r \geq 0$, so that we can write $k_0 = s$.

Let (k, h) be any solution of the equation with $h_0 = 0$. We must find some solution (k', h) of the equation with k_0' defined by a section of ξ. Take a section $s^0 : X^0 \to E$ with $s^0|_{\sigma^0} = k_0^0(\sigma^0)$. Since F is path connected, we have an extension $s^1 : X^1 \to E$ of s^0. But we choose s^1 so that $[k_0^1 - s^1]_\gamma = 0$ (assuming $u : \pi_1(F) \approx H_1(F)$). Then we have $o(s^1) = 0$. So there is a section $s^2 : X^2 \to E$. Again we will have $o(s^2) = [h_0^3] = 0$. Hence we obtain a section ${s'}^3 : X^3 \to E$. Now we have $o({s'}^3) = [h^2 a^2]$, where some cocycle $a^2 \in C^{*+2}(X; R_* H_2(F))$. By the conditions of the theorem (using the bijection λ) there is a cocycle $c^2 \in C^{*+2}(X; R_*\pi_2(F))$ with $[h^2 a^2] = [h^2 c^2]$. Now in a standard manner change ${s'}^3$ on X^2 by the cochain $\rho^*(c^2) \in C^2(X; \pi_2(F))$ to obtain a section $s^3 : X^3 \to E$. Then $o(s^3) = 0$. So a section on X^4 exists.

Suppose that we have constructed by induction a section ${s'}^n : X^n \to E$ with

$$o({s'}^n) = \left[h^2 a^{n-1} + h^3(a^{n-2} - c^{n-2}) + \cdots + h^{n-1}(a^2 - c^2)\right],$$

where some cocycles $a^i \in C^{*+i}(X; R_* H_i(F))$ and $c^i \in C^{*+i}(X; R_*\pi_i(F))$. By the conditions of the Theorem (using λ) there is a cocycle $c^{n-1} \in C^{*+n-1}(X; R_*\pi_{n-1}(F))$ with

$$[h^2 c^{n-1}] = \left[h^2 a^{n-1} + h^3(a^{n-2} - c^{n-2}) + \cdots + h^{n-1}(a^2 - c^2)\right].$$

Now change ${s'}^n$ on X^{n-1} by the cochain $\rho^*(c^{n-1}) \in C^{n-1}(X; \pi_{n-1}(F))$ to obtain a section $s^n : X^n \to E$. Then $o(s^n) = 0$. So a section ${s'}^{n+1} : X^n \to E$ exists, and

$$o({s'}^{n+1}) = \left[h^2 a^n + h^3(a^{n-1} - c^{n-1}) + \cdots + h^n(a^2 - c^2)\right],$$

where some cocycle $a^n \in C^{*+n}(X; R_* H_n(F))$. Thus we have constructed by induction a global section of ξ. □

Proof of Theorem 1.2. Let $\bar{s}^n : X^n \to E$ be another section of ξ coinciding with s^n on X^j. Consider the fibration ξ' over $X \times I$ induced by the projection $X \times I \to X$ and equation (1) for it. (For convenience, we regard the standard cellular decomposition of the cylinder $X \times I$.) In the initial conditions we fix the solution (k, h) on $X \times 0$ with $h \in d(\xi), k_0^n = s^n$ and $(\bar{k}, \bar{h})$ on $X^n \times 1$ with $\bar{h} \in d(\xi), \bar{k}_0^n = {s'}^n$. (In particular, we have

$[\bar{h}_0^{n+1}] = o(\bar{s}^n)$.) Let $h' \in d(\xi')$ be any twisting cochain satisfying these conditions. Consider $c^{(n-1)} \in C^0_{(n-1)}$, $C^k_n = \prod_{j\geq 0} C^{k+j+n}(X; R_j H_n(F))$, defined by

$$c^{(n-1)}(\sigma) = {h'}_0^{(n)}(\sigma \times I).$$

We can easily choose h' so that $c^i = 0$, $i \leq n-1-j$. The fact that h' is twisting implies that $d_h(c^{(n-1)}) = 0$ in $C^1_{(n-1)}$ and

$$\big[{h'}_0^{n+1}|_{X\times 1}\big] = \big[{h'}_0^{n+1}|_{X\times 0}\big] + \big[h^2 c^{n-1} + \cdots + h^n c^1\big],$$

i.e.,

$$o(\bar{s}^n) = o(s^n) + \big[h^2 c^{n-1} + \cdots + h^n c^1\big].$$

Now using the bijection λ and condition (B) we can replace h and c^i above by the twisting element g and some elements $y^i \in \tilde{L}^0_i$, respectively.

Conversely, let some $v \in I^{n+1}_{n-j-1}(\xi)$ and $y^i \in \tilde{L}^0_i$, $n-j \leq i \leq n-1$, be elements determining v. Then the proof of Theorem 1.1 shows that y^i define a section ${s'}^n : X^n \to E$ such that ${s'}^n|_{X^{n-1-j}} = s^n|_{X^{n-1-j}}$ and

$$o({s'}^n) = o(s^n) + v. \qquad \square$$

5. Homotopy Classification of Sections

We will prove Theorem 1.4. The proof is similar to that of Theorem 2.10 [2].

Proof of Theorem 1.4. Observe that we have a bijection

$$\alpha : H^*(\tilde{L}_{(n-1)}, d_{\tilde{\nu}}) \approx H^*(L_{(n-1)}, d_\nu),$$

where $L^k_n = \prod_{j\geq 0} C^{k+j+n}(X; R_j \pi_n(F))$, $\nu \in \lambda_\pi^{-1}(do^{n+1}(\xi))$. So it is sufficient to show that there is a bijection

$$\big[X^{n-1}, E\big]_s \approx H^0(L_{(n-1)}, d_\nu).$$

First define a map

$$\psi : \big[X^{n-1}, E\big]_s \to H^0(L_{(n-1)}, d_\nu)$$

as follows: Let $\bar{s}^n : X^n \to E$ be another section of ξ. Consider the fibration ξ' over $X \times I$ induced from ξ by the projection $X \times I \to X$. We consider equation (1) for ξ' with the initial conditions, where we fix a given solution (k, h) on $X \times 0$, $h \in d(\xi)$, while $\bar{k}^n_0 = \bar{s}^n$ on $X \times 1$. Let (k', h') be any solution satisfying these conditions, where we require by the choice of k' that

$${k'}_0^{i+1}(\sigma^i \times I) = \chi(f_0^{i+1})(\sigma^i \times I)$$

in which $f^{i+1} = \delta({k'}^i) - {k'}^{i-1}{h'}^2 - \cdots - {k'}^1 {h'}^i$ and χ is some fixed homomorphism

$$\chi : C^{i+1}\big(X \times I; ZC_i(F_\sigma)\big) \to C^{i+1}\big(X \times I; C_{i+1}(F_\sigma)\big)$$

defined by $\gamma(\chi(c^{i+1}))(\sigma^{i+1}) = c^{i+1} - k^0[c^{i+1}]_\gamma$, and ZC denotes the cycles of C. Let $a^j(\sigma) = {h'}_0^{j+1}(\sigma \times I)$. Put $b^j = \beta(a^j)$ where $\beta : C_*^* \to L_*^*$ is induced by $\tilde{\beta}$.

It is easy to see the fact that h' is twisting and condition (B3) implies

$$d_\nu(b^{(n-1)}) = 0$$

in $L_{(n-1)}$. Now show that the assignment $\bar{s}^n \to b_{\bar{s}^n}^{(n-1)}$ does not depend on the homotopy class of $\bar{s^n}$.

Let $\bar{t}^n$ be a section homotopic to $\bar{s}^n$. Then we consider the fibration ξ'' over $X \times I \times I$ induced from ξ by the projection $X \times I \times I \to X$. Again consider equation (1) for ξ'' with the initial conditions, where we fix the solution $(\bar{s}^n, \bar{h})$ for ξ' on $X \times I \times 0$, the solution $(\bar{t}^n, \bar{h})$ for ξ' on $X \times 0 \times I$, a homotopy between $\bar{s}^n$ and $\bar{t}^n$ on $X \times I \times 1$, and the constant homotopy for $\bar{s}^n$ on $X \times 1 \times I$. Let $h'' \in d(\xi'')$ be any twisting cochain satisfying these conditions. Put

$${\theta'}^j(\sigma) = {h''}_0^{j+2}(\sigma \times I \times I), \quad 0 \le j \le n-1,$$

and $\theta^j = \beta({\theta'}^j)$. Then we have

$$d_\nu(\theta^{(n-1)}) = b_{\bar{s}^n}^{(n-1)} - b_{\bar{t}^n}^{(n-1)}.$$

Define the map ψ by $\psi([\bar{s}^n]) = b_{\bar{s}^n}^{(n-1)}$.

Conversely, we assign to a d_ν-cocycle, $a^{(n-1)} \in L_{(n-1)}^0$, a section of ξ on X^n as follows. We have that the argument of the proof of Theorem 1.2 defines a section, ${s'}^n : X^n \to E$ (up to homotopy, since we again use the fixed homomorphism χ above) such that if we fix s^n and ${s'}^n$ respectively on $X \times 0$ and $X \times 1$ in the initial conditions of (1) for ξ', then there will be a twisting cochain $h' \in d(\xi')$ with ${h'}_0^{(n)}(\sigma \times I) = {a'}^{(n-1)}(\sigma)$, $\beta({a'}^{(n-1)}) = a^{(n-1)}$.

Let $a^{(n-1)}$ and $b^{(n-1)}$ be two d_ν-cocycles and let ${s'}^n$ and ${t'}^n$ be the corresponding sections. If $a^{(n-1)} - b^{(n-1)}$ is a d_ν-boundary, then by considering the fibration ξ'' over $X \times I \times I$ we will get that ${s'}^n$ is homotopic to ${t'}^n$. Therefore the map

$$H^0(L_{(n-1)}, d_\nu) \to [X^{n-1}, E]_s$$

is defined, which is obviously the converse of ψ. $\square$

6. Applications

We will give some applications of the obstruction theory to the homotopy classification of maps which immediately follow from the statements above. For the application to the Lusternik-Schnirelmann category see [12]. We hope to consider other applications elsewhere.

We have

Theorem 6.1. *Let $f, g : X \longrightarrow Y$ be two weak formal maps. Let X, ΩY satisfy conditions (A), $(B1)$ and $H^*(Y \times Y) = H^*(Y) \otimes H^*(Y)$. Then f is homotopic to g if and only if*

$$H(f) = H(g) : H^*(Y) \longrightarrow H^*(X).$$

Proof. Let $\Omega Y \to E \xrightarrow{\xi} X$ be the fibration induced from the path fibration $\Omega Y \longrightarrow Y^I \xrightarrow{\zeta} Y \times Y$ by the composition $(f \times g) \circ \Delta : X \longrightarrow Y \times Y$, where $\Delta : X \longrightarrow X \times X$ is the diagonal map. Then by Theorem 1.1 it is sufficient to show that $d(\xi)$ is transgressively trivial. Indeed, we have $d(\xi) = D(f) \circ D(\Delta) d(\zeta)$. But $D(\Delta)(d(\zeta))$ corresponds to the free loop fibration on Y; therefore, it is transgressively trivial by Proposition 2.5, and so is $d(\xi)$. $\square$

Now this theorem and Proposition 2.1 imply the following:

Corollary 6.2. *Let X, ΩY satisfy conditions (A) and $(B1)$, let X be weak formal and $H^*(Y)$ be polynomial. Then two maps $f, g : X \longrightarrow Y$ are homotopic if and only if*

$$H(f) = H(g) : H^*(Y) \longrightarrow H^*(X).$$

Theorem 6.3. *Let a fibration ξ have the trivial twisting cochain $0 \in d(\xi)$ (e.g., $H_*(E) \approx H_*(X) \otimes H_*(F)$ and $H_*(F)$ has no torsion) and $u^* : H^i(X; \pi_j(F)) \longrightarrow H^i(X; H_j(F))$ be injective for $j = i - 1$ and split injective for $j = i$, $i > 0$. Then*

$$[X, E]_s \approx \prod_i H^i(X; \pi_i(F)).$$

Proof. It is similar to that of Theorem 1.4. $\square$

Corollary 6.4. *Let for spaces X, Y, $u^* : H^i(X; \pi_j(Y)) \longrightarrow H^i(X; H_j(Y))$ be injective for $j = i - 1$ and split injective for $j = i$, $i > 0$. Then*

$$[X, Y] \approx \prod_i H^i(X; \pi_i(Y)).$$

Note that in Theorem 6.3 the space X is not needed to be weak formal.

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