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LYAPUNOV TYPE INTEGRAL INEQUALITIES FOR CERTAIN DIFFERENTIAL EQUATIONS

B. G. PACHPATTE

ABSTRACT. In the present paper we establish Lyapunov type integral inequalities related to the zeros of solutions of certain second-order differential equations by using elementary analysis. We also present some immediate applications of our results to study the asymptotic behavior of solutions of the corresponding differential equations.

1. INTRODUCTION

In a celebrated paper of 1893 Russian mathematician A. M. Lyapunov [1] proved the following remarkable inequality.

If y is a nontrivial solution of

$$y'' + q(t)y = 0,$$

on an interval containing the points a and b ($a < b$) such that $y(a) = y(b) = 0$, then

$$4 < (b-a)\int_a^b |q(s)|\,ds.$$

Since the appearance of Lyapunov's fundamental paper [1], various proofs and generalizations or improvements have appeared in the literature (see [2]–[10] and the references cited therein). The object of this paper is to prove similar Lyapunov type inequalities for differential equations of the forms

$$\big(r(t)|y'|^{\alpha-1}y'\big)' + p(t)y' + q(t)y + f(t,y) = 0, \tag{A}$$

$$\big(r(t)|y|^{\beta}|y'|^{\gamma-2}y'\big)' + p(t)y' + q(t)y + f(t,y) = 0. \tag{B}$$

In equations (A), (B), throughout we assume that $t \in I = [t_0,\infty)$, $t_0 \geq 0$, and I contains the points a and b ($a < b$), $\alpha \geq 1$, $\beta \geq 0$, $\gamma \geq 2$ are real constants and $\gamma > \beta$, the functions $r, p, q : I \to R = (-\infty,\infty)$ are

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continuous, r and p are continuously differentiable and $r(t) > 0$, the function $f : I \times R \to R$ is continuous and satisfies the condition $|f(t,y)| \le w(t,|y|)$, where the function $w : I \times R_+ \to R_+ = [0,\infty)$ is continuous and satisfies $w(t,u) \le w(t,v)$ for $0 \le u \le v$.

There is vast literature devoted to the study of special variants of equations (A) and (B) from different viewpoints. Our interest in such problems was motivated by the interesting results recently established by various investigators [11]–[16] for equations like (A) and (B). Concerning the existence of solutions of equations of the type (A) and (B), we refer to [11], [12], [15], [16]. The Lyapunov type integral inequalities that we propose here relate points a and b in I, at which solutions of (A) and (B) have zeros, and can be used as handy tools in the study of the qualitative nature of solutions of equations (A) and (B). Here we give some of such applications to convey the importance of our results to the literature.

2. Main Results

In this section we establish our main results on Lyapunov type integral inequalities related to differential equations (A) and (B).

Theorem 1. *Let y be a solution of equation* (A) *with $y(a) = y(b) = 0$, and $y(t) \ne 0$ for $t \in (a,b)$. Let $|y|$ be maximized at a point $c \in (a,b)$. Then*

$$\begin{aligned}1 \le 1/2^{\alpha+1}&\left(\int_a^b x^{-(1/\alpha)}(s)\,ds\right)^{\alpha} \times \\ &\times \left(1/M^{\alpha-1}\int_a^b |q(s)-p'(s)/2|\,ds + 1/M^{\alpha}\int_a^b w(s,M)\,ds\right), \qquad (1)\end{aligned}$$

where $M = \max\{|y(t)| : a \le t \le b\}$.

Proof. By assumption, we have

$$M = |y(c)| = \left|\int_a^c y'(s)\,ds\right| = \left|-\int_c^b y'(s)\,ds\right|. \qquad (2)$$

From (2) we observe that

$$\begin{aligned}2M &\le \int_a^b |y'(s)|\,ds = \\ &= \int_a^b r^{-(1/(\alpha+1))}(s) r^{1/(\alpha+1)}(s)|y'(s)|\,ds. \qquad (3)\end{aligned}$$

Now raising both sides of (3) into the $(\alpha+1)$st power and using the Hölder inequality on the right side of the resulting inequality with indices $(\alpha+1)/\alpha$,

$\alpha + 1$, performing integration by parts and using the fact that $y(t)$ is a solution of equation (A) such that $y(a) = y(b) = 0$, we observe that

$$\begin{aligned}
(2M)^{\alpha+1} &\le \left(\int_a^b r^{-(1/\alpha)}(s)\,ds\right)^{\alpha}\left(\int_a^b r(s)|y'(s)|^{\alpha+1}ds\right) = \\
&= \left(\int_a^b r^{-(1/\alpha)}(s)\,ds\right)^{\alpha}\left(\int_a^b \big(r(s)|y'(s)|^{\alpha-1}y'(s)\big)y'(s)\,ds\right) = \\
&= \left(\int_a^b r^{-(1/\alpha)}(s)\,ds\right)^{\alpha}\left(-\int_a^b \big(r(s)|y'(s)|^{\alpha-1}y'(s)\big)'y(s)\,ds\right) = \\
&= \left(\int_a^b r^{-(1/\alpha)}(s)\,ds\right)^{\alpha}\left(\int_a^b y(s)\big[p(s)y'(s) + \right. \\
&\qquad\qquad \left. + q(s)y(s) + f(s, y(s))\big]\,ds\right) = \\
&= \left(\int_a^b r^{-(1/\alpha)}(s)\,ds\right)^{\alpha}\left(\int_a^b \big(q(s) - p'(s)/2\big)y^2(s)\,ds + \right. \\
&\qquad\qquad \left. + \int_a^b y(s)f(s, y(s))\,ds\right) \le \\
&\le \left(\int_a^b r^{-(1/\alpha)}(s)\,ds\right)^{\alpha}\left(\int_a^b |q(s) - p'(s)/2|\,|y(s)|^2 ds + \right. \\
&\qquad\qquad \left. + \int_a^b |y(s)|\,|f(s, y(s))|\,ds\right) \le \\
&\le \left(\int_a^b r^{-(1/\alpha)}(s)\,ds\right)^{\alpha}\left(\int_a^b M^2|q(s) - p'(s)/2|\,ds + \right. \\
&\qquad\qquad \left. + \int_a^b Mw(s, M)\,ds\right). \qquad (4)
\end{aligned}$$

Now dividing both sides of (4) by $(2M)^{\alpha+1}$, we get the desired inequality (1). □

Theorem 2. *Let y be a solution of equation* (B) *with $y(a) = y(b) = 0$, and $y(t) \ne 0$ for $t \in (a, b)$. Let $|y(t)|$ be maximized at a point $c \in (a, b)$. Then*

$$\begin{aligned}
1 \le &\left(\int_a^b r^{-(1/(\gamma-1))}(s)\,ds\right)^{\gamma-1} \times \\
&\times \left(1/M^{\beta+\gamma-2}\int_a^b |q(s) - p'(s)/2|\,ds + \right. \\
&\left. + 1/M^{\beta+\gamma-1}\int_a^b w(s, M)\,ds\right), \qquad (5)
\end{aligned}$$

where $M = \max\{y(t) : a \le t \le b\}$.

Proof. By assumption, we have

$$M^2 = 2\int_a^c y(s)y'(s)\,ds = -2\int_c^b y(s)y'(s)\,ds. \tag{6}$$

From (6) we observe that

$$\begin{aligned} M^2 &\le \int_a^b |y(s)|\,|y'(s)|\,ds = \\ &= \int_a^b \big(r^{-(1/\gamma)}(s)|y(s)|^{1-(\beta/\gamma)}\big)\big(r^{1/\gamma}(s)|y(s)|^{\beta/\gamma}|y'(s)|\big)\,ds. \end{aligned} \tag{7}$$

By taking the γth power of both sides of (7) and using the Hölder inequality on the right side of the resulting inequality with indices $\gamma/(\gamma-1)$, performing integration by parts, and using the fact that $y(t)$ is a solution of equation (B) such that $y(a) = y(b) = 0$, we observe that

$$\begin{aligned} M^{2\gamma} &\le \left(\int_a^b r^{-(1/(\gamma-1))}(s)|y(s)|^{(\gamma-\beta)/(\gamma-1)}ds\right)^{\gamma-1} \times \\ &\times \left(\int_a^b r(s)|y(s)|^\beta|y'(s)|^\gamma ds\right) = \\ &= \left(\int_a^b r^{-(1/(\gamma-1))}(s)|y(s)|^{(\gamma-\beta)/(\gamma-1)}ds\right)^{\gamma-1} \times \\ &\times \left(\int_a^b \big(r(s)|y(s)|^\beta|y'(s)|^{\gamma-2}y'(s)\big)y'(s)\,ds\right) = \\ &= \left(\int_a^b r^{-(1/(\gamma-1))}(s)|y(s)|^{(\gamma-\beta)/(\gamma-1)}ds\right)^{\gamma-1} \times \\ &\times \left(-\int_a^b \big(r(s)|y(s)|^\beta|y'(s)|^{\gamma-2}y'(s)\big)y(s)\,ds\right) = \\ &= \left(\int_a^b r^{-(1/(\gamma-1))}(s)|y(s)|^{(\gamma-\beta)/(\gamma-1)}ds\right)^{\gamma-1} \times \\ &\times \left(\int_a^b y(s)\big[p(s)y'(s) + q(s)y(s) + f(s, y(s))\big]\,ds\right) = \\ &= \left(\int_a^b r^{-(1/(\gamma-1))}(s)|y(s)|^{(\gamma-\beta)/(\gamma-1)}ds\right)^{\gamma-1} \times \\ &\times \left(\int_a^b \big(q(s) - p'(s)/2\big)y^2(s) + \int_a^b y(s)f(s, y(s))\,ds\right) \le \end{aligned}$$

$$\leq \left(\int_a^b r^{-(1/(\gamma-1))}(s)|y(s)|^{(\gamma-\beta)/(\gamma-1)}ds\right)^{\gamma-1} \times$$
$$\times \left(\int_a^b |q(s)-p'(s)/2|\,|y(s)|^2ds + \int_a^b |y(s)|\,|f(s,y(s))|\,ds\right) \leq$$
$$\leq M^{\gamma-\beta}\left(\int_a^b r^{-(1/(\gamma-1))}(s)ds\right)^{\gamma-1} \times$$
$$\times \left(\int_a^b M^2|q(s)-p'(s)/2|\,ds + \int_a^b Mw(x,M)\,ds\right). \tag{8}$$

Now dividing both sides of (8) by $M^{2\gamma}$, we get the required inequality in (5). □

Remark 1. We note that in the special case with $r(t) \equiv 1$, the obtained inequalities (1) and (5) reduce respectively to the following inequalities:

$$1 \leq 1/2^{\alpha+1}(b-a)^{\alpha}\left(1/M^{\alpha-1}\int_a^b |q(s)-p'(s)/2|\,ds + \right.$$
$$\left. + 1/M^{\alpha}\int_a^b w(s,M)\,ds\right), \tag{9}$$
$$1 \leq (b-a)^{\gamma-1}\left(1/M^{\beta+\gamma-2}\int_a^b |q(s)-p'(s)/2|\,ds + \right.$$
$$\left. + 1/M^{\beta+\gamma-1}\int_a^b w(s,M)\,ds\right). \tag{10}$$

Inequalities (9) and (10) obviously yield the lower bounds of the distance between the consecutive zeros of the nontrivial solutions of equations (A), (B) with $r(t) \equiv 1$. For similar results see [7], [8], [10].

3. Some Applications

In this section we apply our results to the Lyapunov type inequalities given by Theorems 1 and 2 to study the asymptotic behavior of the oscillatory solutions of equations (A) and (B).

Theorem 3. *Assume that*

$$\int^{\infty} r^{-(1/\alpha)}(s)\,ds < \infty, \tag{11}$$
$$\int^{\infty} w(s,L)\,ds < \infty, \tag{12}$$

for any constant $L > 0$, and there exist continuous functions $h, g : I \to R_+$ such that

$$|q(t) - p'(t)/2| \leq h(t)v^{\alpha-1}, \tag{13}$$
$$w(t, v) \leq g(t)v^{\alpha}, \tag{14}$$

for all large v, and

$$\int^{\infty} h(s)\, ds < \infty, \tag{15}$$
$$\int^{\infty} g(s)\, ds < \infty. \tag{16}$$

Then every oscillatory solution of equation (A) *converges to zero as $t \to \infty$.*

Proof. Let y be an oscillatory solution of equation (A) on the interval I. Because of (11), (15), and (16), we can choose $T \geq t_0$ large enough so that for every $t \geq T$,

$$\begin{aligned} &\int_T^{\infty} r^{-(1/\alpha)}(s)\, ds < 1, \\ &\int_T^{\infty} h(s)\, ds < 1, \\ &\int_T^{\infty} g(s)\, ds < 1. \end{aligned} \tag{17}$$

First we shall show that every oscillatory solution y of equation (A) is bounded on I. Suppose on the contrary that $\lim_{t\to\infty} \sup |y(t)| = \infty$. Because of (12) we can choose L large enough so that (14) holds for all $v \geq L$. Indeed, since y is oscillatory, there exists an interval (t_1, t_2) such that $t_1 > T$, $y(t_1) = y(t_2) = 0$, $|y(t)| > 0$ on (t_1, t_2), and $M = \max\{|y(t)| : t_1 \leq t \leq t_2\} = \max\{|y(t)| : t \leq t_2\} > L$. Choose c in (t_1, t_2) such that $M = |y(c)|$. Clearly, inequality (1) in Theorem 1 is true on the interval (t_1, t_2) and we have

$$\begin{aligned} 1 \leq 1/2^{\alpha+1} &\left(\int_{t_1}^{t_2} r^{-(1/\alpha)}(s)\, ds \right)^{\alpha} \times \\ &\times \left(1/M^{\alpha-1} \int_{t_1}^{t_2} |q(s) - p'(s)/2|\, ds + \right. \\ &\left. + 1/M^{\alpha} \int_{t_1}^{t_2} w(s, M)\, ds \right). \end{aligned} \tag{18}$$

From (18), (13), (14), and (17) we observe that

$$\begin{aligned}
1 &\le 1/2^{\alpha+1}\left(\int_{t_1}^{t_2} r^{-(1/\alpha)}(s)\,ds\right)^{\alpha}\left(\int_{t_1}^{t_2} h(s)\,ds + \int_{t_1}^{t_2} g(s)\,ds\right) \le \\
&\le 1/2^{\alpha+1}\left(\int_{t_1}^{\infty} r^{-(1/\alpha)}(s)\,ds\right)^{\alpha}\left(\int_{t_1}^{\infty} h(s)\,ds + \int_{t_1}^{\infty} g(s)\,ds\right) < \\
&\le 1/2^{\alpha+1}(1)(1+1) = 1/2^{\alpha}.
\end{aligned}$$

This contradiction shows that the solution y of equation (A) is bounded, say, $|y(t)| \le N$.

Next we shall prove that $y(t) \to 0$ as $t \to +\infty$. If $y(t) \not\to 0$ as $t \to +\infty$, then there exists a constant $d > 0$ such that

$$2d \le \lim_{t\to+\infty} \sup |y(t)| \le N. \tag{19}$$

Choose $T_0 \ge T$ such that

$$\begin{aligned}
&\int_{T_0}^{\infty} r^{-(1/\alpha)}(s)\,ds < 1, \\
&\int_{T_0}^{\infty} |q(s) - p'(s)/2|\,ds < d^{\alpha-1}, \\
&\int_{T_0}^{\infty} w(s,N)\,ds < d^{\alpha}.
\end{aligned} \tag{20}$$

Because of (19) there exists $t_2 > t_1 > T_0$ such that $y(t_1) = y(t_2)$ and

$$M_0 = \max\left\{|y(t)| : t_1 \le t \le t_2\right\} > d. \tag{21}$$

From the conclusion of Theorem 1 we have

$$\begin{aligned}
1 &\le 1/2^{\alpha+1}\left(\int_{t_1}^{t_2} r^{-(1/\alpha)}(s)\,ds\right)^{\alpha} \times \\
&\times \left(1/M_0^{\alpha-1}\int_{t_1}^{t_2} |q(s) - p'(s)/2|\,ds + \right. \\
&\left. + 1/M_0^{\alpha}\int_{t_1}^{t_2} w(s,N)\,ds\right).
\end{aligned} \tag{22}$$

From (20), (21), and (22) we have

$$
\begin{aligned}
1 &\le 1/2^{\alpha+1}\left(\int_{t_1}^{\infty} r^{-(1/\alpha)}(s)\,ds\right)^{\alpha} \times \\
&\times \left(1/M_0^{\alpha-1}\int_{t_1}^{\infty} |q(s)-p'(s)|\,ds + 1/M_0^{\alpha}\int_{t_1}^{\infty} w(s,N)\,ds\right) < \\
&< 1/2^{\alpha+1}(1)\big(1/M_0^{\alpha-1}(d^{\alpha-1}) + 1/M_0^{\alpha}(d^{\alpha})\big) < \\
&\le 1/2^{\alpha+1}(1)(1+1) = 1/2^{\alpha}.
\end{aligned}
$$

This contradiction proves the theorem. □

Theorem 4. *Assume that*

$$\int^{\infty} r^{-(1/(\alpha-1))}(s)\,ds < \infty, \tag{23}$$

$$\int^{\infty} w(s,L)\,ds < \infty, \tag{24}$$

for any constant $L > 0$, and there exist continuous functions $h_0, g_0 : I \to R_+$ such that

$$|q(t) - p'(t)/2| \le h(t)v^{\beta+\gamma-2}, \tag{25}$$

$$w(t,v) \le g(t)v^{\beta+\gamma-1}, \tag{26}$$

for all large v, and

$$\int^{\infty} h_0(s)\,ds < \infty, \tag{27}$$

$$\int^{\infty} g_0(s)\,ds < \infty. \tag{28}$$

Then every oscillatory solution of equation (B) *converges to zero as $t \to \infty$.*

The proof of this theorem can be completed by the arguments as in the proof of Theorem 3 and by applying Theorem 2. Here we omit the details.

Remark 2. We note that the inequalities established by Theorems 1 and 2 can be very easily extended to the following differential equations:

$$y\big(r(t)|y'|^{\alpha-1}y'\big)' + p(t)y' + q(t)y + f(t,y) = 0, \tag{C}$$

$$y\big(r(t)|y|^{\beta}|y'|^{\gamma-2}y'\big)' + p(t)y' + q(t)y + f(t,y) = 0, \tag{D}$$

where $r, p, q, f, \alpha, \beta, \gamma$ are as given in equations (A), (B). For the study of equations of this type see [7], [14]. Proceeding along the same lines as in the proofs of Theorems 1 and 2 corresponding to equations (C), (D), the obtained inequalities (1), (5) take the forms

$$1 \leq 1/2^{\alpha+1}\left(\int_a^b r^{-(1/\alpha)}(s)\,ds\right)^{\alpha} \times$$
$$\times\left(1/M^{\alpha}\int_a^b |q(s)-p'(s)|\,ds + 1/M^{\alpha+1}\int_a^b w(s,M)\,ds\right), \tag{29}$$
$$1 \leq \left(\int_a^b r^{-(1/(\gamma-1))}(s)\,ds\right)^{\alpha} \times$$
$$\times\left(1/M^{\beta+\gamma-1}\int_a^b |q(s)-p'(s)|\,ds + 1/M^{\beta+\gamma}\int_a^b w(s,M)\,ds\right). \tag{30}$$

Here it is to be noted that the obtained inequalities (29) and (30) can be used to study the asymptotic behavior of the solutions of equations (C) and (D) by arguments as used in the proofs of Theorems 3 and 4 with suitable changes. We also note that our results in Theorems 1–4 can be extended to differential equations (A), (B), (C), (D), where the function f involved therein depends on the functional arguments and is of the form $f(t, y(t), y(\Phi(t)))$, by making use of the same hypothesis on Φ and f as in [13] with suitable modifications.

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Author's address:
Department of Mathematics
Marathwada University
Aurangabad 431 004
(Maharashtra) India