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ON THE SOLUTION OF THE DIRICHLET PROBLEM FOR AN ELLIPTIC EQUATION DEGENERATING ON THE BOUNDARY

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ABSTRACT. The Dirichlet problem for the equation

$$y\Delta W + \alpha W_y = -F$$

is studied in the semi-circle $x^2 + y^2 < 1$, $y > 0$. The restrictions on F are established under which the problem is uniquely solvable in the definite generalized sense.

Let $K = \{(x,y) : (x,y) \in \mathbb{R}^2,\ x^2 + y^2 < 1,\ y > 0\}$, and let ∂K be the boundary of K. Denote $\Delta W = W_{xx} + W_{yy}$ and consider the equations

$$y\Delta W + \alpha W_y = -F(x,y), \quad (x,y) \in K, \tag{1}$$

$$y\Delta W + \alpha W_y = 0, \quad (x,y) \in K. \tag{2}$$

The classical Dirichlet problem for these equations is formulated as follows: Find in the domain K a regular solution $u = u(x,y)$ of the given equation which is continuous in $\overline{K} = K \cup \partial K$ and satisfies the boundary condition $u(x,y) = f(x,y)$ for $(x,y) \in \partial K$, where $f(x,y)$ is the known continuous function on ∂K. Bitsadze [1] investigated the existence and uniqueness of the solution of the Dirichlet problem for equations of more general type than (1), while the solutions of boundary value problems for equations of more general type were studied by Keldysh [2]. In [3] Nikol'sky and Lizorkin considered some boundary value problems for degenerating elliptic equations. Weinstein [4], [5] constructed fundamental solutions for equation (2). As is well known (see, e.g., [6]), using fundamental solutions one can construct a Green function of the Dirichlet problem for equation

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(1) which has the form

$$
\begin{gathered}
G(x,y;\xi,\eta;\gamma)y^{1-\alpha}=[Z_0(x,y;\xi,\eta;\gamma)-\\
-\rho^{-\gamma}Z_0(x,y;\xi/\rho^2,\eta/\rho^2;\gamma)]y^{1-\alpha}=[Z_0-Z_1]y^{1-\alpha},\\
Z_0=\frac{1}{2\pi}\int_0^\pi \sin^{\gamma-1}\theta[r^2+4y\eta\sin^2\theta/2]^{-\gamma/2}d\theta,\\
\zeta,z\in K \text{ with } \zeta=\xi+i\eta,\ z=x+iy,\ \rho=|\xi|,\ r=|z-\zeta|,\ \gamma=2-\alpha.
\end{gathered}
\tag{3}
$$

A unique solution for the classical Dirichlet problem of equation (1) can be represented by the following

Theorem A. *Let $0<\alpha<1$, and let the function F satisfy the conditions*

$$
\begin{gathered}
|F(x_1,y_1)-F(x_2,y_2)|\le c\big\{|x_1-x_2|^\delta+|y_1-y_2|^\delta\big\},\\
0<\delta\le 1,\quad (x_k,y_k)\in K,\quad k=1,2.
\end{gathered}
$$

Moreover, $F(x,y)=0$ when $0\le y\le\omega$, where $\omega>0$ depends on F. Then for any function f continuous on ∂K there exists a unique function $W=W(x,y)$ which is continuous on $\overline{K}$, $W(x,y)=f(x,y)$, $(x,y)\in\partial K$, and satisfies equation (1) *in K. For all $(x,y)\in K$ we have* [7]

$$
\begin{aligned}
W(x,y)=y^{1-\alpha}&\int_K F(\xi,\eta)G(x,y;\xi,\eta;2-\alpha)\,d\xi\,d\eta+\\
&+(1-\alpha)y^{1-\alpha}\int_{-1}^1 f(t,0)G(x,y;t,0;2-\alpha)\,dt-\\
&-y^{1-\alpha}\int_0^\pi f(\varphi)\sin\varphi\,\frac{\partial G(x,y;\cos\varphi,\sin\varphi;2-\alpha)}{\partial n}\,d\varphi=\\
&=J_1(F,x,y)+J_2(f,x,y)+J_3(f,x,y),
\end{aligned}
\tag{4}
$$

where $\frac{\partial G}{\partial n}$ is the derivative with respect to the outer normal to ∂K and $f[\varphi]=f(\cos\varphi,\sin\varphi)$, $0\le\varphi\le\pi$.

The aim of this paper is to show that expression (4) is a solution of the Dirichlet problem of equation (1) for a wider class of given functions F and f. In the sequel we shall use the following propositions.

Proposition 1. *Let $\gamma<0$. Then for all $(x,y)\in K$, $(\xi,\eta)\in K$ we have*

$$
Z_0(x,y;\xi,\eta;\gamma)\ge C(y\cdot\eta)^{-\gamma/2}\int_0^{\pi/4}\theta^{-1}d\theta=\infty,
$$

where $C\in(0,\infty)$ is a constant independent of x,y,ξ,η.

Proposition 2. *For each $(a,b)\in(0,1]\times(0,1]$ and a fixed $p\in(0,\infty)$ the inequality $\ell\le\frac{(a+b)^p}{a^p+b^p}\le L$ holds with $\ell=\min[1,2^{p-1}]$, $L=\max[1,2^{p-1}]$.*

Proposition 3. *For all* $(x,y) \in K$, $(\xi,\eta) \in K$, $0 < \alpha < 2$, *there exists a constant* $C \in (0,\infty)$ *such that*

$$|G(x,y;\xi,\eta;2-\alpha)| \le \frac{C}{r^{2-\alpha}}.$$

Proposition 4. *If* $z = x+iy \in K$, $\zeta = \xi + i\eta \in K$, $\zeta = \rho e^{i\varphi}$, $\zeta^* = \xi^* + i\eta^* = \frac{1}{\rho}e^{i\varphi}$, *then* $|z-\zeta| \le |z-\zeta^*|$,

$$\lim |\zeta|\,|z-\zeta^*| = 1, \quad \lim |\zeta|^2 \cdot y \cdot \eta^* = 0 \quad \textit{as} \quad |\zeta| \to 0.$$

Below we assume that F is summable on K, $F(x,y) = 0$ if $(x,y) \notin \overline{K}$, and for any point $(x,y) \in \overline{K}$ we have

$$\int_0^{2\pi} |F(x+\rho\cos\varphi, y+\rho\sin\varphi)|\,d\varphi \le C[1 + |\ln\rho^{-1}|] = C\ln^+\rho, \qquad \text{(L)}$$

where $C \in (0,\infty)$ is a constant depending on F only.

Theorem 1. *Let* $0 < \alpha < 2$, *and let* F *satisfy the condition* (L). *Then for all* $(x,y) \in \overline{K}$ *there exists, in the Lebesgue sense, an integral*

$$\int_E |F|\,|G(x,y;\xi,\eta;2-\alpha)|\,d\xi\,d\eta, \quad E \subseteq K,$$

which is absolutely continuous with respect to the measurable set E *uniformly for all* $(x,y) \in \overline{K}$. *If, however,* $y = 0$, $0 < \alpha < 1$, *then for an arbitrary summable function* $F(x,y)$ *we have* $J_1(F,x,0) = 0$, $-1 < x \le 1$.

Theorem 2. *Let* $0 < \alpha < 1$ *and let* F *satisfy the condition* (L). *Then for all* $(x_0,y_0) \in \partial K$, $x_0 \ne \pm 1$ *we have*

$$\lim y^{1-\alpha} \int_K F(\xi,\eta)G(x,y;\xi,\eta;2-\alpha)\,d\xi\,d\eta = 0 \ \textit{as}\ x \to x_0,\ y \to y_0. \qquad (5)$$

Thus for the function F satisfying on K condition (L) there exists an expression $J_1(F,x,y)$ for all $(x,y) \in K$, and at every point $(x_0,y_0) \in \partial K$, $x_0 \ne \pm 1$, equality (5) is fulfilled. As for the function $G(x,y;\xi,\eta;\gamma)$, we are aware of the following inequalities ([4], [5]).

Proposition 5. *Let us introduce the functions* h *and* N *defined by the equalities*

$$G(x,y;\xi,\eta;\gamma) = \frac{1}{2\pi}(y\cdot\eta)^{-\gamma/2}\ln r^{-1} + N(z,\zeta,\gamma) = h(y,\eta)\ln r^{-1} + N(z,\zeta,\gamma)$$

with $z,\zeta \in K$. *For all* $0 < \omega \le y$, $\eta \ge \omega$, $0 < r \le \omega$ *we have*

$$\begin{aligned} &|N| \le C_\omega, \quad |N_y| \le C_\omega, \quad |N_x| \le C_\omega r|\ln r^{-1}|, \\ &|N_\xi| \le C_\omega r|\ln r^{-1}|, \quad |N_\eta| \le C_\omega r|\ln r^{-1}|, \end{aligned} \qquad (6)$$

where $C_\omega \in (0,\infty)$ *is a constant depending on* ω.

To study boundary properties of the integral

$$J_3(f,x,y) = -y^{1-\alpha}\int_0^\pi f[\varphi]\,\frac{\partial G(x,y;\cos\varphi,\sin\varphi;2-\alpha)}{\partial n}\,\sin\varphi\,d\varphi$$

let us introduce an angular neighborhood of the point $e^{i\varphi_0}\in\Gamma=\partial K\setminus(-1,1)$, $0<\varphi_0<\pi$. We say that the point z belongs to $\Delta_1(\varphi_0,\delta)$ if

$$|\arg z-\varphi_0|<\delta(1-|z|),\quad \delta>0, \tag{7}$$

where δ is an arbitrarily fixed positive number. From (7) we obtain that for $z\in\Delta_1(\varphi_0,\delta)$,

(a) $|z-e^{i\varphi_0}|\le C_\delta(1-|z|)$ with a constant $C_\delta>0$; (8)

(b) there exist constants $m>0$, $n>0$ depending on δ such that for all $\psi\in(\varphi_0-\delta,\varphi_0+\delta)$

$$m|z-e^{i\psi}|\le|z-e^{i\varphi_0}|\le n|z-e^{i\psi}|; \tag{9}$$

(c) $m|x-\cos\psi|\le|x-\cos\varphi_0|\le n|y-\sin\psi|,\ \ \psi\in(\varphi_0-\delta,\varphi_0+\delta)$. (10)

Let f be a function which is summable on $[0,\pi]$ and assume that

$$\widetilde{f}(\varphi_0)=\sup\Big[\Big|\Big(\int\limits_0^{\varphi_0-\varepsilon}+\int\limits_{\varphi_0+\varepsilon}^{\pi}\Big)\frac{f[\varphi]}{\varphi-\varphi_0}\,d\varphi\Big|,\ \varepsilon>0\Big],$$

$$M(f,\varphi_0)=\sup\Big[\Big|\frac{1}{h+k}\int\limits_{\varphi_0-h}^{\varphi_0+k}|f[\varphi]|\,d\varphi\Big|,\ h>0,\ k>0\Big],$$

$$\overline{J}_3(f,\varphi_0)=\sup\big[|J_3(f,z)|,\ z\in\Delta_1(\varphi_0,\delta)\big].$$

Suppose that the arbitrarily fixed numbers introduced in (7) and in Proposition 5 are equal, i.e., $\delta=\omega=\lambda>0$. Introduce the set $I(\lambda)=(\lambda,\frac{\pi}{2}-\lambda)\cup(\frac{\pi}{2}+\lambda,\pi-\lambda)$, $|I(\lambda)|=\pi-4\lambda>0$, $\lambda<\frac{\pi}{8}$. Let $\varphi_0\in I(2\lambda)$, in particular, $\varphi_0\in(2\lambda,\frac{\pi}{2}-2\lambda)$. Let $0<\alpha<1$, and let f be summable on $[0,\pi]$. Using Proposition 5 and relations (7)–(10), we can estimate the expression

$$\Big|\int\limits_{\varphi_0-\lambda}^{\varphi_0+\lambda} y^{1-\alpha}f[\varphi]\,\frac{\partial}{\partial n}(h\cdot\ln r^{-1})\sin\varphi\,d\varphi\Big|\le$$

$$\le C_\lambda\Big[\int\limits_{\varphi_0-\lambda}^{\varphi_0+\lambda}|f[\varphi]|\,|\ln r^{-1}|\,d\varphi+\Big|\int\limits_{\varphi_0-\lambda}^{\varphi_0+\lambda} f\cdot h\,\frac{\partial\ln r^{-1}}{\partial n}\,d\varphi\Big|\Big].$$

For $z \in \Delta_1(\varphi_0, \lambda)$, $z = |z|e^{i\psi} = x + iy$, $r = |z - e^{i\varphi}|$ we have

$$\frac{\partial h \ln r^{-1}}{\partial n} = \frac{1}{\varphi - \varphi_0} + R(x, y, \varphi) \ln |\varphi - \varphi_0|,$$

where $R(x, y, \varphi)$ is the bounded measurable function of x, y, φ, $z = x + iy \in \Delta_1(\varphi_0, \lambda)$. Thus from the latter relations we get

$$\Big| \int\limits_{\varphi_0-\lambda}^{\varphi_0+\lambda} y^{1-\alpha} f[\varphi] \frac{\partial}{\partial n} \left(h \cdot \ln r^{-1}\right) \sin\varphi \, d\varphi \Big| \le$$

$$\le C_\lambda \Big[\Big| \int\limits_{\varphi_0-\lambda}^{\varphi_0+\lambda} \frac{f_1[\varphi]}{\varphi - \varphi_0} \, d\varphi \Big| + M(f, \varphi_0) \Big] \le$$

$$\le C_\lambda [\widetilde{f}_1(\varphi_0) + M(f, \varphi_0)], \quad |f_1[\varphi]| \le |f[\varphi]|, \quad \varphi \in (\varphi_0 - \lambda, \varphi_0 + \lambda), \tag{11}$$

where $C_\lambda \in (0, \infty)$ is a constant depending on λ only. Further, using inequalities (6)–(10), we can easily get the following inequalities:

$$\begin{aligned}
&\Big| \int\limits_{0}^{\varphi_0-\lambda} y^{1-\alpha} f[\varphi] \frac{\partial}{\partial n} \left(h \cdot \ln r^{-1}\right) \sin\varphi \, d\varphi \Big| \le C_\lambda M(f, \varphi_0), \\
&\Big| \int\limits_{\varphi_0+\lambda}^{\pi} y^{1-\alpha} f[\varphi] \frac{\partial}{\partial n} \left(h \cdot \ln r^{-1}\right) \sin\varphi \, d\varphi \Big| \le C_\lambda M(f, \varphi_0), \\
&\Big| \int\limits_{0}^{\pi} y^{1-\alpha} f[\varphi] \frac{\partial N}{\partial n} \sin\varphi \, d\varphi \Big| \le C_\lambda M(f, \varphi_0), \quad z = |z|e^{i\psi}, \quad r = |z - e^{i\varphi}|.
\end{aligned} \tag{12}$$

Analogous inequalities are also valid for the rest of the summands.

Let f be summable on $[-1, 1]$, and let $\Delta_1(x_0, \lambda)$ be an angular neighborhood of the point $x_0 \in [-1, 1]$, i.e., $z = x + iy \in \Delta_1(x_0, \lambda)$ if

$$|x - x_0|/y \le \lambda. \tag{13}$$

Using the previous arguments and all the inequalities provided by (13), we obtain the following inequality: let f be summable on $[-1, 1]$, $0 < \alpha < 1$, $-1 < x_0 < 1$. Then

$$|\overline{J}_2(f, x_0)| \le C_\lambda M(f, x_0), \tag{14}$$

where $C_\lambda \in (0, \infty)$ is a constant depending on $\lambda > 0$ only, while

$$\overline{J}_2(f, x_0) = \sup \big[|J_2(f, z)|, \ \ z \in \Delta_1(x_0, \lambda) \big].$$

On the basis of (14), (12) and (11) we obtain

Proposition 6. *Let $0<\alpha<1$. Let f be summable on $\partial K=\Gamma\cup(-1,1)$ and let $\lambda>0$ be an arbitrarily fixed number, $\Gamma_\lambda=\{e^{i\varphi},\ \varphi\in I(\lambda)\}$. Then for any point $t\in\Gamma_\lambda\cup(-1,1)$ we have*

$$\overline{J}_3(f,t)+\overline{J}_2(f,t)\le C_\lambda[M(f,t)+\widetilde{f}_1(t)],\quad |f_1(t)|\le|f(t)|. \tag{15}$$

Theorem 3. *Let $0<\alpha<1$, and let f be summable on ∂K. Then for almost all $t\in\Gamma\cup(-1,1)$ we have*

$$\lim[J_2(f,z)+J_3(f,z)]=f(t),\quad z\to t,\quad z\in\Delta_1(t,\lambda), \tag{16}$$

where $\lambda>0$ is an arbitrarily fixed number.

Proof. By virtue of Theorem A and Theorem 2 for all $t_0=(x_0,y_0)\in\partial K$, $x_0\ne\pm1$ and any function g continuous on ∂K we have

$$\lim[J_2(g,z)+J_3(g,z)]=g(t_0),\quad z\to t_0,\quad z\in\Delta_1(t_0,\lambda). \tag{17}$$

Let f be a function summable on $\Gamma\cup(-1,1)$ and suppose that (16) is not fulfilled. Then there exist sufficiently small $\lambda_0>0$ and $\varepsilon_0>0$ and a set E, where either $|E\cap\Gamma_{\lambda_0}|\ge16\varepsilon_0$ or $|(-1,1)\cap E|\ge16\varepsilon_0$, such that for every point $t_0\in E$ there exists a sequence $z_n\to t_0$, $z_n\in\Delta_1(t_0,\lambda_0)$, $n\ge1$, and

$$\big|[J_2(f,z_n)+J_3(f,z_n)]-f(t_0)\big|\ge3\varepsilon_0,\quad n\ge1. \tag{18}$$

On the other hand, for $\varepsilon_0>0$ and for the function f there exists a function g continuous on $\Gamma\cup(-1,1)$ such that

$$\int_{\partial K}|f(t)-g(t)|\,dt\le\frac{\varepsilon_0^2}{C_{\lambda_0}},\quad |f(t)-g(t)|<\varepsilon_0,\ \ t\in E. \tag{19}$$

Next, on the basis of the well-known Kolmogoroff inequality ([8], [9]) we have

$$\begin{gathered}\big|\{t\in\Gamma\cup(-1,1);\ \widetilde{f}_1(t)+M(f,t)>\mu\}\big|\le\\ \le\frac{4}{\mu}\int_{\partial K}(|f_1(t)|+|f(t)|)\,dt=\frac{8}{\mu}\int_{\partial K}|f|\,dt,\quad |f_1|\le|f|.\end{gathered} \tag{20}$$

Using relations (15), (17)–(20), we get

$$\begin{aligned}&16\varepsilon_0\le|E|\le|\{t_0\in E;\ |J_2(f,z_n)+J_3(f,z_n)-f(t_0)|\ge3\varepsilon_0\}|\le\\ &\le|\{t\in E;\ |J_2(f-g,z_n)+J_3(f-g,z_n)|>\varepsilon_0\}|+|\{t\in E;\ |f-g|>\varepsilon_0\}|+\\ &+|\{t\in E;\ |J_2(g,z_n)+J_3(g,z_n)-g(t_0)|>\varepsilon_0\}|<8\varepsilon_0.\end{aligned}$$

This contradiction proves the theorem. □

For our further investigation let us consider a maximal function for the function of two variables (see [8]).

Let f be the function summable on K. Assume that for $(x_0,y_0)\in K$

$$M_1(f,x_0,y_0)=\sup\left\{\frac{1}{|\sigma|}\int_\sigma |f|\,dx\,dy : |\sigma|>0\right\},$$
$$\sigma=\{(x-x_0)^2+(y-y_0)^2\le\delta^2\}.$$

Using (6) and the relation $\Delta[u,v]=v\cdot\Delta u+2[u_x\cdot v_x+u_y\cdot v_y]+u\cdot\Delta v$, we can prove the following

Proposition 7. *Let $0<\alpha<1$, and let the function Ψ satisfy the condition* (L). *Then for all*

$$(x_0,y_0)\in K_\delta=\{(x,y)\in K;\ x^2+y^2\le(1-\delta)^2,\ y\ge\delta\},$$

where $\delta>0$ is an arbitrarily fixed number, we have

$$\begin{aligned}&|\Delta[J_1(\Psi,x_0,y_0)]|\le C_\delta M_1(\Psi,x_0,y_0),\\ &\left|\frac{\partial}{\partial y}J_1(\Psi,x_0,y_0)\right|\le C_\delta M_1(\Psi,x_0,y_0).\end{aligned}\tag{21}$$

Here the constant $C_\delta\in(0,\infty)$ depends on δ only.

According to Kolmogoroff [9] the inequality

$$\big|\{(x,y)\in K;\ M_1(\Psi,x,y)>\mu\}\big|\le\frac{\theta}{\mu}\int_K|\Psi|\,dx\,dy\tag{22}$$

holds, where the constant $\theta\in(0,\infty)$ does not depend on μ and Ψ.

Proposition 8. *Let $0<\alpha<1$, and let f be summable on ∂K. Then for all $(x,y)\in K_\delta$ we have*

$$\big|\Delta[J_2(f,x,y)+J_3(f,x,y)]\big|\le C_\delta M_1(f,x,y),\tag{23}$$

$$\left|\frac{\partial}{\partial y}[J_2(f,x,y)+J_3(f,x,y)]\right|\le C_\delta M_1(f,x,y).\tag{24}$$

Repeating now the arguments from the proof of Theorem 3 and taking into account relations (24), (23), (22), (21) and (20), we obtain the main result.

Theorem 4. *Let $0<\alpha<1$. Let F satisfy the condition* (L), *and let f be a function summable on ∂K. Then we have*

$$\begin{aligned}&y\Delta[J_1(F,x,y)+J_2(f,x,y)+J_3(f,x,y)]+\\ &+\alpha\frac{\partial}{\partial y}[J_1(F,x,y)+J_2(f,x,y)+J_3(f,x,y)]=-F(x,y)\end{aligned}$$

for almost all $(x,y)\in K$ *and*

$$\lim W(x,y)=\lim[J_1+J_2+J_3]=f(x_0,y_0) \text{ as } z=x+iy\to x_0+iy_0,$$
$$z\in\Delta_1(\varphi_0,\lambda),\quad e^{i\varphi_0}=x_0+iy_0,$$

for almost all $(x_0,y_0)\in\partial K$, *where* $\lambda>0$ *is an arbitrarily fixed number.*

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