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# ON DIFFERENTIAL BASES FORMED OF INTERVALS

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*In memory of young mathematician*
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ABSTRACT. Translation invariant subbases of the differential basis $B_2$ (formed of all intervals), which differentiates the same class of all non-negative functions as $B_2$ does, are described. A possibility for extending the results obtained to bases of more general type is discussed.

## 1. DEFINITIONS AND NOTATION

A mapping $B$ defined on $\mathbb{R}^n$ is said to be a differential basis in $\mathbb{R}^n$ if, for every $x \in \mathbb{R}^n$, $B(x)$ is a family of open bounded sets containing the point $x$ such that there exists a sequence $\{R_k\} \subset B(x)$, $\operatorname{diam} R_k \to 0$ $(k \to \infty)$.

For $f \in L_{loc}(\mathbb{R}^n)$ the numbers

$$\overline{D}_B\Big(\int f, x\Big) = \overline{\lim_{\operatorname{diam} R \to 0, R \in B(x)}} \frac{1}{|R|} \int_R f$$

and

$$\underline{D}_B\Big(\int f, x\Big) = \underline{\lim_{\operatorname{diam} R \to 0, R \in B(x)}} \frac{1}{|R|} \int_R f$$

are said to be respectively the upper and the lower derivative of the integral of $f$ at the point $x$. If the upper and the lower derivative coincide, then their common value is called the derivative of the integral of $f$ at the point $x$, and we denote it by $D_B\big(\int f, x\big)$. They say that the basis $B$ differentiates the integral of $f$ if $D_B\big(\int f, x\big) = f(x)$ for almost all $x$. The set of those functions $f \in L_{loc}(\mathbb{R}^n)$, $f \geq 0$, whose integrals are differentiable with respect to the basis $B$ will be denoted by $F_B^+$. Under $M_B$ we mean the maximal operator

$$M_B(f)(x) = \sup_{R \in B(x)} \frac{1}{|R|} \int_R |f| \quad (f \in L_{loc}(\mathbb{R}^n),\ x \in \mathbb{R}^n),$$

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corresponding to the basis $B$. It will be assumed here that $\overline{B} = \bigcup\limits_{x \in \mathbb{R}^n} B(x)$.

$B$ is said to be a subbasis of $B'$ (writing $B \subset B'$), if $B(x) \subset B'(x)$ $(x \in \mathbb{R}^n)$. The basis $B$ is said to be translation invariant or the $TI$-basis, if $B(x) = \{x + R : R \in B(O)\}$ $(x \in \mathbb{R}^n)$ (here $O$ is the origin in $\mathbb{R}^n$). Let us have the bases $B$ and $B'$. We shall say that the family $\overline{B}$ is locally regular with respect to the family $\overline{B'}$ (writing $\overline{B} \in LR(\overline{B'})$), if there exist $\delta > 0$ and $c > 0$ such that for any $R \in \overline{B}$, $\operatorname{diam} R < \delta$, there is $R' \in \overline{B}'$ such that $R \subset R'$ and $|R'| < c|R|$.

We shall agree that $I^n = [0,1]^n$ and $f \in L(I^n)$, if $f \in L(\mathbb{R}^n)$ and $\operatorname{supp} f \subset I^n$.

## 2. $TI$-Bases Formed of Intervals

Let $B_2$ be the basis in $\mathbb{R}^2$ for which $B_2(x)$ $(x \in \mathbb{R}^2)$ consists of all two-dimensional intervals containing the point $x$.

The theorem below characterizes $B \subset B_2$, $TI$-bases for which $F_B^+ = F_{B_2}^+$.

**Theorem 1.** *Let $B \subset B_2$ be a $TI$-basis. Then the following conditions are equivalent:*

$(a)$ $F_B^+ = F_{B_2}^+$;

$(b)$ *for every $f \in L(\mathbb{R}^2)$, $f \geq 0$, a.e. on $\mathbb{R}^2$*

$$\overline{D}_B\Big(\int f, x\Big) = \overline{D}_{B_2}\Big(\int f, x\Big) \ \ \text{and} \ \ \underline{D}_B\Big(\int f, x\Big) = \underline{D}_{B_2}\Big(\int f, x\Big);$$

$(c)$ *$\overline{B}_2$ is locally regular with respect to $\overline{B}$.*

The implication $(b) \Rightarrow (a)$ is evident. Therefore to prove Theorem 1 it suffices to show that $(a) \Rightarrow (c)$ and $(c) \Rightarrow (b)$.

The implication $(a) \Rightarrow (c)$ follows from the following assertion.

**Theorem 2.** *Let $B \subset B_2$ be a $TI$-basis. If $\overline{B}_2$ is not locally regular with respect to $\overline{B}$, then there exists a function $f \in L(I^2)$, $f \geq 0$, such that*

$$\overline{D}_{B_2}\Big(\int f, x\Big) = \infty \ \ \text{a.e. on} \ \ I^2,$$

$$D_B\Big(\int f, x\Big) = f(x) \ \ \text{a.e. on} \ \ I^2.$$

Before proving Theorem 2 we shall give several lemmas.

For the interval $I$ we denote by $\alpha I$ $(\alpha > 0)$ the interval $H(I)$, where $H$ is the homothety with the coefficient $\alpha$ whose center is the center of $I$.

**Lemma 1.** *Let $I \in \overline{B}_2$ and $h > 1$. Then $\{M_{B_2}(h\chi_I) > 1\} \subset (2h+1)I$ and $|\{M_{B_2}(h\chi_I) > 1\}| \geq h(\ln h)|I|$.*

The validity of this lemma can be shown by a direct checking.

Projections of $I$ onto the $ox^1$- and $ox^2$-axes will be denoted by $\mathrm{pr}_1 I$ and $\mathrm{pr}_2 I$, respectively. For $I \in \overline{B}_2$ and $h > 0$ we have

$$\begin{gathered} R_1(I,h) = (2h+1)\,\mathrm{pr}_1 I \times 3\,\mathrm{pr}_2 I, \\ R_2(I,h) = 3\,\mathrm{pr}_1 I \times (2h+1)\,\mathrm{pr}_2 I, \\ R(I,h) = R_1(I,h) \cup R_2(I,h). \end{gathered}$$

It is clear that $|R(I,h)| \le 18h|I|$.

**Lemma 2.** *Let $B \subset B_2$ be a $TI$-basis; $h > 1$. Suppose that for $I \in \overline{B}_2$ there is no $J \in \overline{B}$, $J \supset I$, such that $|J| \le h|I|$. Then $\{M_B(h\chi_I) > 1\} \subset R(I,h)$.*

*Proof.* Let $J \in \overline{B}$ and $(1/|J|)\int_J h\chi_I > 1$. From the condition of the lemma we can easily find that either

$$|pr_1 J|_1 < |\,\mathrm{pr}_1 I|_1 \ \text{ or } \ |\,\mathrm{pr}_2 J|_1 < |\,\mathrm{pr}_2 I|_1, \tag{1}$$

where $|\cdot|_1$ is the Lebesgue measure on $\mathbb{R}$.

By Lemma 1, $J \subset (2h+1)I$, which by virtue of (1) implies that either $J \subset R_1(I,h)$ or $J \subset R_2(I,h)$, so that we have $J \subset R(I,h)$. It follows from the latter inclusion that $\{M_B(h\chi_I) > 1\} \subset R(I,h)$. $\square$

It can be easily seen that the following two lemmas are valid.

**Lemma 3.** *Let $I, J \in \overline{B}_2$; $h > 1$. If either $|\,\mathrm{pr}_1 J|_1 \ge h|\,\mathrm{pr}_1 J|_1$ or $|\,\mathrm{pr}_2 J|_1 \ge h|\,\mathrm{pr}_2 J|_1$, then $h|J \cap I| \le |I \cap R(I,h)|$.*

**Lemma 4.** *Let $I, J \in \overline{B}_2$. If $J \cap I \ne \varnothing$ and $J \backslash R(I,h) \ne \varnothing$, then either $|\,\mathrm{pr}_1 J|_1 \ge |\,\mathrm{pr}_1 I|_1$, or $|\,\mathrm{pr}_2 J|_1 \ge |\,\mathrm{pr}_2 I|_1$.*

The following lemma is also valid.

**Lemma 5.** *Let $B \subset B_2$; $h > 1$, and let $I_1, \dots, I_k \in \overline{B}_2$ be the equal intervals. If*

$$\begin{gathered} \{M_B(h\chi_{I_m}) > 1\} \subset R(I_m,h) \quad (m = \overline{1,k}), \\ R(I_m,h) \cap R(I_{m'},h) = \varnothing \quad (m \ne m'), \end{gathered}$$

*then*

$$\Big\{ M_B\Big(\sum_{m=1}^{k} h\chi_{I_m}\Big) > 1 \Big\} \subset \bigcup_{m=1}^{k} R(I_m,h). \tag{2}$$

*Proof.* Inequality (2) is equivalent to the following inequality: if $x \notin \bigcup\limits_{m=1}^{k} R(I_m, h)$ and $I \in B(x)$, then

$$\frac{1}{|I|} \int_I \sum_{m=1}^{k} h\chi_{I_m} \leq 1. \tag{3}$$

Let $x \notin \bigcup\limits_{m=1}^{k} R(I_m, h)$ and $I \in B(x)$. Then we may have three cases. Let us consider each of them separately.

(i) $I$ intersects none of the intervals $I_m$ $(m = \overline{1,k})$; in this case the validity of (3) is evident.

(ii) $I$ intersects only one interval $I_m$ $(m = \overline{1,k})$.

Let $I_n$ be the interval for which $I \cap I_n \neq \varnothing$. $x \notin R(I_m, h)$, and therefore by the condition of the lemma,

$$\frac{1}{|I|} \int_I h\chi_{I_n} \leq M_B(h\chi_{I_n})(x) \leq 1,$$

from which, taking into account the equality $I \cap I_m = \varnothing$ $(m \neq n)$, we obtain (3).

(iii) $I$ intersects more than one interval $I_m$ $(m = \overline{1,k})$.

It follows from the equality $R(I_m, h) \cap R(I_{m'}, h) = \varnothing$ $(m \neq m')$ that

$$(2h+1)I_m \cap I_{m'} = \varnothing \tag{4}$$

Denote $P = \{m \in [1,k] : I \cap I_m \neq \varnothing\}$. By (4) we have

$$I \cap I_m \neq \varnothing, \quad I \backslash (2h+1)I_m \neq \varnothing \quad (m \in P),$$

which gives either

$$|\operatorname{pr}_1 I|_1 \geq h |\operatorname{pr}_1 I_m|_1 \ \text{ or } \ |\operatorname{pr}_2 I|_1 \geq h |\operatorname{pr}_2 I_m|_1 \ \ (m \in P).$$

Now according to Lemma 3 we can conclude that

$$h|I \cap I_m| \leq |I \cap R(I_m, h)| \ \ (m \in P).$$

From the obtained inequality, taking into consideration the pairwise non-intersection of $R(I_m, h)$ $(m = \overline{1,k})$, we write

$$\int_I \sum_{m=1}^{k} h\chi_{I_m} = \sum_{m \in P} \int_I h\chi_{I_m} = \sum_{m \in P} h|I \cap I_m| \leq \sum_{m \in P} |I \cap R(I_m, h)| \leq |I|.$$

Thus inequality (3) and Lemma 5 are proved. □

**Lemma 6.** *Let $B \subset B_2$, $h > 1$, and for every $m \in [1,k]$ let $\{I_{m,q}\}_{q=1}^{q_m} \subset \overline{B}_2$ be a family of equal intervals. If*

$$\begin{gathered}\{M_B(h\chi_{I_{m,q}}) > 1\} \subset R_{m,q} \\ (R_{m,q} = R(I_{m,q},h); m = \overline{1,k}) \;\; q = \overline{1,q_m}),\end{gathered} \tag{5}$$

$$R_{m,q} \cap R_{m',q'} = \varnothing \quad (m,q) \neq (m,q'), \tag{6}$$

$$\begin{aligned}&|\operatorname{pr}_1 I_{m,1}|_1 \geq h|\operatorname{pr}_1 I_{m+1,1}|_1, \\ &|\operatorname{pr}_2 I_{m,1}|_1 \geq h|\operatorname{pr}_2 I_{m+1,1}|_1,\end{aligned} \quad (m = \overline{1,k-1}) \tag{7}$$

*then*

$$\Big\{M_B\Big(\sum_{m=1}^{k}\sum_{q=1}^{q_m} h\chi_{I_{m,q}}\Big) > 2\Big\} \subset \bigcup_{m=1}^{k}\bigcup_{q=1}^{q_m} R_{m,q}.$$

*Proof.* The inclusion we have to prove is equivalent to the following inequality: if $x \notin \bigcup\limits_{m=1}^{k}\bigcup\limits_{q=1}^{q_m} R_{m,q}$ and $I \in B(x)$, then

$$\frac{1}{|I|}\int_I \sum_{m=1}^{k}\sum_{q=1}^{q_m} h\chi_{I_{m,q}} \leq 2. \tag{8}$$

Assume $x \notin \bigcup\limits_{m=1}^{k}\bigcup\limits_{q=1}^{q_m} R_{m,q}$ and $I \in B(x)$. It is clear that (8) is fulfilled for $I \cap I_{m,q} = \varnothing$ $(m = \overline{1,k},\ q = \overline{1,q_m})$. Denote otherwise $n = \min\{m \in [1,k] : \exists I_{m,q}(q = \overline{1,q_m}), I \cap I_{m,q} \neq \varnothing\}$ and consider first the case with $1 < n < k$.
By virtue of Lemma 5 (see (5) and (6)), we write

$$M_B\Big(\sum_{q=1}^{q_n} h\chi_{I_{n,q}}\Big)(x) \leq 1,$$

which implies

$$\int_I \sum_{q=1}^{q_n} h\chi_{I_{n,q}} = \sum_{q=1}^{q_n} h|I \cap I_{n,q}| \leq |I|. \tag{9}$$

For some $q \in [1,q_n]$ $I \cap I_{n,q} \neq \varnothing$, it is clear that $I \backslash R_{n,q} \neq \varnothing$. Therefore, according to Lemma 4, we have either

$$|\operatorname{pr}_1 I|_1 \geq |\operatorname{pr}_1 I_{n,q}|_1 \;\text{ or }\; |\operatorname{pr}_2 I|_1 \geq |\operatorname{pr}_2 I_{n,q}|_1.$$

Taking into account that the intervals $\{I_{m,q}\}_{q=1}^{q_m}$ $(m = \overline{1,k})$ are equal and using (7), we can write that either

$$|\operatorname{pr}_1 I|_1 \geq h|\operatorname{pr}_1 I_{m,q}|_1 \quad (n < m \leq k,\ q = \overline{1,q_m})$$

or

$$|\operatorname{pr}_2 I|_1 \geq h|\operatorname{pr}_2 I_{m,q}|_1 \quad (n < m \leq k,\ q = \overline{1, q_m}),$$

whence by Lemma 3 we get

$$h|I \cap I_{m,q}| \leq |I \cap R_{m,q}| \quad (n < m \leq k;\ q = \overline{1, q_m}). \tag{10}$$

Clearly, $I \cap I_{m,q} = \varnothing$ $(n \leq m < n;\ q = \overline{1, q_m})$, so that by (6), (9), (10) and $1 \leq m < n$ we have

$$\int_I \sum_{m=1}^{k} \sum_{q=1}^{q_m} h\chi_{I_{m,q}} = \sum_{m=1}^{k} \sum_{q=1}^{q_m} h|I \cap I_{m,q}| = \sum_{m=1}^{n-1} \sum_{q=1}^{q_m} h|I \cap I_{m,q}| +$$
$$+ \sum_{q=1}^{q_n} h|I \cap I_{n,q}| + \sum_{m=n+1}^{k} \sum_{q=1}^{q_m} h|I \cap I_{m,q}| =$$
$$= A_1 + A_2 + A_3 \leq 0 + |I| + \sum_{m=n+1}^{k} \sum_{q=1}^{q_m} |I \cap R_{m,q}| \leq |I| + |I| = 2|I|.$$

Inequality (8) can be proved in a simpler way for $n = 1$ or $n = k$, since in these cases we do not have the terms $A_1$ and $A_3$. $\square$

For the basis $B$ let us define the operator

$$M_B^*(f)(x) = \sup_{R \in B(x)} \frac{1}{|R|} \Big| \int_R f \Big| \quad (f \in L_{loc}(\mathbb{R}^n), x \in \mathbb{R}^n).$$

**Lemma 7.** *Let the basis $B$ differentiate the integrals of the functions $f_k \in L(\mathbb{R}^n)$ $(k \in \mathbb{N})$, $\sum_{k=1}^{\infty} \|f_k\|_1 < \infty$. If*

$$\sum_{k=1}^{\infty} |\{M_B^*(f_k) > \lambda_k\}|_e < \infty,$$

*where $\lambda_k > 0$ $(k \in \mathbb{N})$, $\sum_{k=1}^{\infty} \lambda_k < \infty$ and $|\cdot|_e$ is an outer measure, then $B$ differentiates the integral of the function $\sum_{k=1}^{\infty} f_k$.*

*Note that since $M_B^*(f) \leq M_B(f)$ $(f \in L(\mathbb{R}^n))$, the conclusion of Lemma 7 will be the more so valid when the inequality $\sum_{k=1}^{\infty} |\{M_B(f_k) > \lambda_k\}|_e < \infty$ is fulfilled.*

*Proof.* Let us note that:

(i) $\sum_{k=1}^{\infty} \|f_k\|_1 < \infty$; therefore the set $A_0 = \{x \in \mathbb{R}^n : \sum_{k=1}^{\infty} |f_k(x)| < \infty\}$ is of measure 0;

(ii) $B$ differentiates $\int f_k$ $(k \in \mathbb{N})$, i.e., the sets $A_k = \{x \in \mathbb{R}^n : D_B(\int f, x) = f(x)\}$ $(k \in \mathbb{N})$ have a complete measure on $\mathbb{R}^n$.

(iii) $\sum_{k=1}^{\infty} |\{M_B^*(f_k) > \lambda_k\}|_e < \infty$; therefore the set $\overline{\lim}_{k\to\infty}\{M_B^*(f_k) > \lambda_k\}$ is of measure 0.

It follows from (i)–(iii) that the set $A = \bigcup_{k=1}^{\infty} A_k \setminus \big(\overline{\lim}_{k\to\infty}\{M_B^*(f_k) > \lambda_k\} \bigcup A_0\big)$ is of complete measure. Let us show that for every $x \in A$ we have $D_B\Big(\int \sum_{k=1}^{\infty} f_k, x\Big) = \sum_{k=1}^{\infty} f_k(x)$, which will prove the lemma.

Let $x \in A$ and $\varepsilon > 0$. From the inclusion $x \in A$ we can conclude that:

1) there is $k_0 \in \mathbb{N}$ ($k_0 = k(x,\varepsilon)$) such that $x \notin \bigcap_{k=k_0}^{\infty} \{M_B^*(f_k) > \lambda_k\}$, $\sum_{k=k_0}^{\infty} \lambda_k < \varepsilon/3$ and $\sum_{k=k_0}^{\infty} |f_k(x)| < \varepsilon/3$;

2) there is $\delta > 0$ such that for every $R \in B(x)$, $\operatorname{diam} R < \delta$

$$\left| \frac{1}{|R|} \int_R \sum_{k=1}^{k_0-1} f_k - \sum_{k=1}^{k_0-1} f_k(x) \right| < \varepsilon/3.$$

According to 1) and 2) we can write that for every $R \in B(x)$, $\operatorname{diam} R < \delta$,

$$\left| \frac{1}{|R|} \int_R \sum_{k=1}^{\infty} f_k - \sum_{k=1}^{\infty} f_k(x) \right| \le \left| \frac{1}{|R|} \int_R \sum_{k=1}^{k_0-1} f_k - \sum_{k=1}^{k_0-1} f_k(x) \right| +$$
$$+ \sum_{k=k_0}^{\infty} \frac{1}{|R|} \left| \int_R f_k \right| + \sum_{k=k_0}^{\infty} |f_k(x)| < \varepsilon/3 + \sum_{k=k_0}^{\infty} M_B^*(f_k)(x) +$$
$$+\varepsilon/3 \le 2\varepsilon/3 + \sum_{k=k_0}^{\infty} \lambda_k < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that the equality $D_B\Big(\int \sum_{k=1}^{\infty} f_k, x\Big) = \sum_{k=1}^{\infty} f_k(x)$ is valid. $\square$

*Proof of Theorem* 2. Let $k \ge 3$. Choose $h_k$ such that $\frac{1}{k} \ln \frac{h_k}{k} \ge 18 \cdot 2^{2k}$.

Let $\alpha_k = \frac{1}{k} \ln \frac{h_k}{k} / 2^{2(k+2)} h_k$. Obviously, $0 < \alpha_k < 1$.

For $I \in \overline{B}_2$ and $h > 0$ denote $Q^0(I,h) = (2h+1)I$. Let $2^{-m} < |\operatorname{pr}_1 Q^0(I,h)|_1 \le 2^{-m+1}$ and $2^{-m'} < |\operatorname{pr}_2 Q^0(I,h)|_1 \le 2^{-m'+1}$, where $m, m' \in \mathbb{N}$. Denote by $Q(I,h)$ the interval concentric with $Q^0(I,h)$, and $|\operatorname{pr}_1 Q(I,h)|_1 = 2^{-m+1}$, $|\operatorname{pr}_2 Q(I,h)|_1 = 2^{-m'+1}$.

For the basis $B$ denote by $M_B^{(r)}$ $(r > 0)$ the operator

$$M_B^{(r)}(f)(x) = \sup_{R \in B(x), \operatorname{diam} R < r} \frac{1}{|R|} \int_R |f| \quad (f \in L(\mathbb{R}^n),\ x \in \mathbb{R}^n).$$

$\overline{B}_2 \notin LR(\overline{B})$, since there is $I \in \overline{B}_2$ such that:

1) there is no $J \in \overline{B}$, $J \supset I$ for which $|J| < 2^k h_k |I|$;

2) $\operatorname{diam} Q(I, 2^k h_k) < 1/k$.

Divide $I^2$ into intervals equal to $Q(I, 2^k h_k)$ and denote them by $Q_{1,q}$ $(1 \le q \le q_1)$. Take $I_{1,q}$ $(1 \le q \le q_1)$ equal to $T_q(I)$, where $T_q$ is a shift translating $Q(I, 2^k h_k)$ to $Q_{1,q}$.

Let the families $\{I_{1,q}\}_{q=1}^{q_1} \dots \{I_{m,q}\}_{q=1}^{q_m}$ consisting of equal intervals be already constructed. Consider the sets

$$A_m^1 = \bigcup_{j=1}^{m} \bigcup_{q=1}^{q_j} \{M_{B_2}^{(1/k)}(h_k, \chi_{I_{j,q}}) > k\},$$

$$A_m^2 = \bigcup_{j=1}^{m} \bigcup_{q=1}^{q_j} R_{j,q} \quad (R_{j,q} = R(I_{j,q}, 2^k h_k)).$$

If $|A_m^1| > 1 - \dfrac{1}{k}$, then we stop the construction. If $|A_m^1| \le 1 - \dfrac{1}{k}$, then we shall construct the family $\{I_{m+1,q}\}_{q=1}^{q_{m+1}}$ as follows:

Consider the set $A_m = I^2 = I^2 \backslash (A_m^1 \cup A_m^2)$ which, obviously, can be represented as $G_1 \cup G_2$, where $G_1$ is open and $G_2$ consists of a finite number of smooth closed lines. It is clear that there is $\delta \in (0, 1/k)$ such that if we divide $I^2$ into equal intervals $\{J_j\}$ with diameters less than $\delta$, then

$$\Big| A_m \cap \bigcup_{J_j \subset A_m} J_j \Big| \ge \Big(1 - \frac{\alpha_k}{4}\Big)|A_m|.$$

$\overline{B}_2 \notin LR(\overline{B})$, since there is $I \in \overline{B}_2$ such that:

1) there is no $J \in \overline{B}$, $J \supset I$ for which $|J| \le 2^k h_k |I|$;

2) $\operatorname{diam} Q(I, 2^k h_k) < \delta$;

3) $|\operatorname{pr}_1 I_{m,1}|_1 \ge 2^k h_k |\operatorname{pr}_1 I|_1$ and $|\operatorname{pr}_2 I_{m,1}|_1 \ge 2^k h_k |\operatorname{pr}_2 I|_1$.

Divide $I^2$ into intervals equal to $Q(I, 2^k h_k)$. Denote the intervals included in $A_m$ by $Q_{m+1,q}$ $(1 \le q \le q_{m+1})$. Take $I_{m+1,q}$ $(1 \le q \le q_{m+1})$ equal to $T_q(I)$, where $T_q$ is a shift translating $Q(I, 2^k h_k)$ to $Q_{m+1,q}$.

By our construction we obtain

$$\Big| A_m \cap \bigcup_{q=1}^{q_{m+1}} Q_{m+1,q} \Big| \ge \Big(1 - \frac{\alpha_k}{4}\Big)|A_m|. \tag{11}$$

By Lemma 1,- for $q = \overline{1, q_{m+1}}$

$$\{M_{B_2}(h_k \chi_{I_{m+1,q}}) > k\} \subset \Big(2\frac{h_k}{k} - 1\Big) I_{m+1,q} \subset Q_{m+1,q}, \tag{12}$$

$$\big|\{M_{B_2}(h_k \chi_{I_{m+1,q}}) > k\}\big| > \frac{h_k}{k} \ln \frac{h_k}{k} |I_{m+1,q}|. \tag{13}$$

Since $\operatorname{diam} Q_{m+1,q} < 1/k$ $(q = \overline{1, q_{m+1}})$, because of (12) we have

$$\{M_{B_2}(h_k \chi_{I_{m+1,q}}) > k\} = \{M_{B_2}^{(1/k)}(h_k \chi_{I_{m+1,q}}) > k\} \quad (q = \overline{1, q_{m+1}}). \tag{14}$$

It can be easily seen that (see (13), (14))

$$\left|\{M_{B_2}^{(1/k)}(h_k\chi_{I_{m+1,q}}) > k\}\right| \ge \alpha_k |Q_{m+1,q}| \quad (q = \overline{1, q_{m+1}}),$$

which by (11) readily implies

$$\Big|\bigcup_{q=1}^{q_{m+1}} \{M_{B_2}^{(1/k)}(h_k\chi_{I_{m+1,q}}) > k\}\Big| \ge \alpha_k \Big|\bigcup_{q=1}^{q_{m+1}} Q_{m+1,q}\Big| \ge$$
$$\ge \alpha_k\Big(1 - \frac{\alpha_k}{4}\Big)|A_m| > \frac{\alpha_k}{2}|A_m|. \tag{15}$$

According to the construction and Lemma 2 we have for $q = \overline{1, q_{m+1}}$

$$\{M_B(h_k\chi_{I_{m+1,q}}) > 1/2^k\} \subset R_{m+1,q} \;\; (R_{m+1,q} = R(I_{m+1,q}, 2^k h_k)).$$

On account of our choice of $h_k$, because of (13) and (14) we write

$$\left|\{M_{B_2}^{(1/k)}(h_k\chi_{I_{m+1,q}}) > k\}\right| > 2^k |R_{m+1,q}| \quad (q = \overline{1, q_{m+1}}),$$

whence

$$\Big|\bigcup_{q=1}^{q_{m+1}} \{M_{B_2}^{(1/k)}(h_k\chi_{I_{m+1,q}}) > k\}\Big| > 2^k \Big|\bigcup_{q=1}^{q_{m+1}} R_{m+1,q}\Big|. \tag{16}$$

Let us show that for sufficiently large $mk$ the construction ceases, i.e., we shall have the inequality

$$\Big|\bigcup_{m=1}^{m_k}\bigcup_{q=1}^{q_m} \{M_{B_2}^{(1/k)}(h_k\chi_{I_{m+1,q}}) > k\}\Big| > 1 - \frac{1}{k}. \tag{17}$$

Assume the contrary, i.e., $|A_m^1| \le 1 - \dfrac{1}{k}$ $(m \in \mathbb{N})$. Introduce the notation

$$\left.\begin{aligned} A_m^3 &= \bigcup_{q=1}^{q_m} \{M_{B_2}^{(1/k)}(h_k\chi_{I_{m,q}}) > k\}, \\ A_m^4 &= \bigcup_{q=1}^{q_m} R_{m,q}. \end{aligned}\right. \qquad (m \in \mathbb{N})$$

Clearly, (15) and (16) will be valid for all $m \in \mathbb{N}$, and therefore we write

$$|A_{m+1}^3| > \frac{\alpha_k}{2}|A_m| = \frac{\alpha_k}{2}\Big(1 - \sum_{j=1}^{m} |A_j^3 \cup A_j^4|\Big) \ge$$
$$\ge \frac{\alpha_k}{2}\Big(1 - \sum_{j=1}^{m}\Big(1 + \frac{1}{2^k}\Big)|A_j^3|\Big) \ge \frac{\alpha_k}{2}\Big(1 - \frac{1}{2^k} - \sum_{j=1}^{m}|A_j^3|\Big),$$

which by the equality $|A^1_m| = \sum_{j=1}^m |A^3_m|$ implies that $|A^1_m| > 1 - \dfrac{1}{2^{k-1}} > 1 - \dfrac{1}{k}$ for sufficiently large $m$. From this contradiction we conclude that (17) is valid.

Consider the function $f_k = \sum_{m=1}^{m_k} \sum_{q=1}^{q_m} h_k \chi_{I_{m,q}}$. Clearly, $f_k \in L(I^2)$ and $f_k \geq 0$. From (17) we get

$$\big|\{M^{(1/k)}_{B_2}(f_k) > k\}\big| > 1 - \frac{1}{k}. \tag{18}$$

From the construction we can easily see that $2^k h_k$ and the families $\{I_{1,q}\}_{q=1}^{q_1}, \dots, \{I_{m_k,q}\}_{q=1}^{q_{m_k}}$ satisfy all the conditions of Lemma 6. Therefore by Lemma 6 we write

$$\begin{aligned} &\{M_B(f_k) > 1/2^{k-1}\} = \{M_B(2^k f_k) > 2\} = \\ &= \Big\{M_B\Big(\sum_{m=1}^{m_k}\sum_{q=1}^{q_m} 2^k h_k \chi_{I_{m,q}}\Big) > 2\Big\} \subset \bigcup_{m=1}^{m_k}\bigcup_{q=1}^{q_m} R_{m,q}. \end{aligned} \tag{19}$$

Obviously, (16) is fulfilled for all $m \in [1, m_k - 1]$, so that taking the construction into account, we have

$$2^k \Big|\bigcup_{m=1}^{m_k}\bigcup_{q=1}^{q_m} R_{m,q}\Big| < \Big|\bigcup_{m=1}^{m_k}\bigcup_{q=1}^{q_m} \{M^{(1/k)}_{B_2}(h_k \chi_{I_{m,q}}) > k\}\Big| \leq 1. \tag{20}$$

Due to (19) and (20) we write

$$\Big|\{M_B(f_k) > 1/2^{k-1}\}\Big| < \frac{1}{2^k}. \tag{21}$$

It can be easily seen that

$$\frac{h_k}{k} \ln \frac{h_k}{k} \Big|\bigcup_{m=1}^{m_k}\bigcup_{q=1}^{q_m} I_{m,q}\Big| < \Big|\bigcup_{m=1}^{m_k}\bigcup_{q=1}^{q_m} \{M^{(1/k)}_{B_2}(h_k \chi_{I_{m,q}}) > k\}\Big| \leq 1,$$

Hence, due to our choice of $h_k$, we obtain

$$\|f_k\|_1 = h_k \Big|\bigcup_{m=1}^{m_k}\bigcup_{q=1}^{q_m} I_{m,q}\Big| < 1 \Big/ \frac{1}{k} \ln \frac{h_k}{k} < \frac{1}{2^k}. \tag{22}$$

Consider the function $f = \sum_{k=3}^{\infty} f_k$. Clearly, $f \in L(I^2)$ (see (22)) and $f \geq 0$.

It is obvious that if $x \in \overline{\lim\limits_{k\to\infty}}\{M^{(1/k)}_{B_2}(f_k) > k\}$, then $\overline{D}_{B_2}\Big(\int \sum_{k=3}^{\infty} f_k, x\Big) = \infty$. Because of (18) $\overline{\lim\limits_{k\to\infty}}\{M^{(1/k)}_{B_2}(f_k) > k\}$ has a complete measure on $I^2$. Therefore the latter equality holds almost everywhere on $I^2$.

It is clear that $B$ differentiates $\int f_k$ ($k \in \mathbb{N}$). Moreover, owing to (21) we get $\sum_{k=3}^{\infty} \left|\{M_B(f_k) > 1/2^{k-1}\}\right| < \sum_{k=3}^{\infty} \frac{1}{2^k} < \infty$, which by Lemma 7 implies that $B$ differentiates $\int f$. $\square$

After making some technical changes in the proof of Theorem 2 we can obtain the following generalization.

**Theorem 3.** *Let $B \subset B_2$ be a $TI$-basis. If $\overline{B}_2$ is not locally regular with respect to $\overline{B}$ then for any function $f \in L\backslash L\ln^+ L(I^2)$, $f \geq 0$, there is a Lebesgue measure-preserving and invertible mapping $\omega : \mathbb{R}^2 \to \mathbb{R}^2$, $\{x : \omega(x) \neq x\} \subset I^2$ such that*

$$\overline{D}_{B_2}\Big(\int f \circ \omega, x\Big) = \infty \ \ a.e. \ on \ I^2,$$

$$D_B\Big(\int f \circ \omega, x\Big) = (f \circ \omega)(x) \ \ a.e. \ on \ I^2.$$

To prove the implication $(c) \Rightarrow (b)$ let us show the validity of

**Lemma 8.** *Let $B \subset B_2$ be a $TI$-basis. If $\overline{B}_2$ is locally regular with respect to $\overline{B}$, then for every $f \in L(\mathbb{R}^2)$*

$$\left|\{M_{B_2}^{(r)}(f) > \lambda\}\right| \leq 25\left|\left\{M_B^{(cr)}(f) > \frac{\lambda}{4c}\right\}\right| \ \ (\lambda > 0, 0 < r < \delta),$$

*where $\delta$ and $c$ are the constants from the definition of local regularity of $\overline{B}_2$ with respect to $\overline{B}$.*

*Proof.* Note first that if $I$ and $I'$ ($I \subset I'$) are the one-dimensional intervals and $I^*$ is either the left or the right half of the interval $I$, then for the point $y \in I'$ the set $E = \{z \in \mathbb{R} : (I' + z - y) \supset I^*\}$ possesses the following properties:

1) $E$ is an interval;

2) $E \cap I' \neq \varnothing$;

3) $|E|_1 = |I'|_1 - \dfrac{1}{2}|I|_1 \geq \dfrac{1}{2}|I'|_1$.

It follows from these properties that

4) $5E \supset I'$

is likewise valid.

Let $f \in L(\mathbb{R}^2)$. Denote by $P$ the set of all two-dimensional intervals $E \in \overline{B}_2$ such that $E \subset \left\{M_B^{(cr)}(f) > \dfrac{\lambda}{4c}\right\}$ and show the inclusion

$$\{M_{B_2}^{(r)}(f) > \lambda\} \subset \bigcup_{E \in P} 5E \ \ (\lambda > 0, \ 0 < r < \delta). \tag{23}$$

Let $x \in \{M_{B_2}^{(r)}(f) > \lambda\}$ $(\lambda > 0,\ 0 < r < \delta)$. Then there is $I \in B_2(x)$ such that $\operatorname{diam} I < r$ and

$$\frac{1}{|I|}\int_I |f| > \lambda. \tag{24}$$

Since $\overline{B}_2 \in LR(\overline{B})$, there is an interval $I'$ from $\overline{B}$ such that $I \subset I'$ and $|I'| \le c|I|$. Then it is obvious that $\operatorname{diam} I' \le c \operatorname{diam} I < cr$. Let $y \in I'$ be the point for which $I' \in B(y)$. Divide the interval $I$ into four equal intervals. Because of (24) the integral mean will be greater than $\lambda$ at least on one of these intervals. Denote this interval by $I^*$. Thus

$$\frac{1}{|I^*|}\int_{I^*} |f| > \lambda. \tag{25}$$

Consider the set $E = \{z \in \mathbb{R}^2 : (I' + z - y) \supset I^*\}$. By virtue of the above discussion we can easily note that

1) $E \in \overline{B}_2$;
2) $E \cap I' \neq \varnothing$;
3) $|E| \ge \frac{1}{4}|I'|$;
4) $5E \supset I'$.

Since $x \in I \subset I'$, then to prove (23) it suffices to show that $E \in P$. Let $z \in E$. Then $(I' + z - y) \supset I^*$. Since $B$ is the $TI$-basis and $I' \in B(y)$, we have $(I' + z - y) \in B(z)$. Moreover, owing to (25), we have

$$\frac{1}{|I' + z - y|}\int_{I'+z-y} |f| \ge \frac{1}{|I'|}\int_{I^*} |f| \ge \frac{1}{c|I|}\int_{I^*} |f| = \frac{1}{c4|I^*|}\int_{I^*} |f| > \frac{\lambda}{4c},$$

from which by the inequality $\operatorname{diam}(I' + z - y) < cr$, we conclude that $z \in \left\{M_B^{(cr)}(f) > \frac{\lambda}{4c}\right\}$. Hence $E \in P$. Thus inclusion (23) is proved.

Now, using (23) and Lemma 1 from [2], we will have

$$\left|\{M_{B_2}^{(r)}(f) > \lambda\}\right| \le \Big|\bigcup_{E\in P} 5E\Big| \le 25\Big|\bigcup_{E\in P} E\Big| \le 25\left|\left\{M_B^{(cr)}(f) > \frac{\lambda}{4c}\right\}\right|$$

for $\lambda > 0$, $0 < r < \delta$. □

The basis $B$ is said to be regular if there is a constant $c < \infty$ such that for every $R \in \overline{B}$ there exists a cubic interval $Q$ with the following properties: $Q \supset R$, $|Q| \le c|R|$.

According to the well-known Lebesgue theorem on differentiability of integrals, the regular basis $B$ differentiates $L$, i.e., $D_B\left(\int f, x\right) = f(x)$ a.e. on $\mathbb{R}^n$ for every $f \in L(\mathbb{R}^n)$.

We shall say that the basis $B$ possesses property $(E)$ if for every $f \in L(\mathbb{R}^n)$, $\underline{D}_B\left(\int f, x\right) \le f(x) \le \overline{D}_B\left(\int f, x\right)$ a.e. on $\mathbb{R}^n$.

Obviously, the basis $B$, containing a regular subbasis, possesses the property $(E)$.

It directly follows from the relation $\overline{B}_2 \in LR(\overline{B})$ that $B$ contains a regular subbasis, and hence possesses property $(E)$. Now, to prove the implication $(b) \Rightarrow (c)$, it suffices to consider the following assertion.

**Lemma 9.** *Let $B$ be a subbasis of $B_2$ with property $(E)$. Suppose that there exist positive constants $c_1, c_2, c_3$ and $\delta$ such that for every $f \in L(\mathbb{R}^2)$,*

$$\left|\{M_{B_2}^{(r)}(f) > \lambda\}\right| \le c_1 \left|\left\{M_B^{(c_2 r)}(f) > \frac{\lambda}{c_3}\right\}\right| \quad (\lambda > 0,\ 0 < r < \delta).$$

*Then for every $f \in L(\mathbb{R}^2)$, $f \ge 0$,*

$$\underline{D}_B\Big(\int f, x\Big) = \underline{D}_{B_2}\Big(\int f, x\Big) \ \text{ and } \ \overline{D}_B\Big(\int f, x\Big) = \overline{D}_{B_2}\Big(\int f, x\Big),$$

*a.e. on $\mathbb{R}^2$.*

The proof of Lemma 9 is based on the well-known Besicovitch theorem on possible values of upper and lower derivative numbers (see [1], Ch.IV, §3) and is carried out analogously to the proof of Theorem 1 from [2].

## 3. On a Possibility of Extending Theorem 1 to More General Bases

It should be noted that beyond the scope of $TI$-bases Theorem 1 becomes invalid. Moreover, even for bases which are similar enough to $TI$-bases the local regularity of $\overline{B}_2$ with respect to $\overline{B}$ is insufficient for the equality $F_B^+ = F_{B_2}^+$ to be fulfilled.

A basis $B$ will be called a $TI^*$-basis if for every $x \in \mathbb{R}^n$, $R \in B(x)$ and $y \in \mathbb{R}^n$ there is a translation $T$ such that $T(R) \in B(y)$.

The following theorem is valid.

**Theorem 4.** *There is a $TI^*$-basis $B \subset B_2$ with the properties:*
1) *$\overline{B}_2$ is locally regular with respect to $\overline{B}$, and $B(O) = B_2(O)$;*
2) *there is a function $f \in L(I^2)$, $f \ge 0$ such that*

$$\overline{D}_{B_2}\Big(\int f, x\Big) = \infty \ \text{ a.e. on } I^2,$$
$$D_B\Big(\int f, x\Big) = f(x) \ \text{ a.e. on } I^2.$$

*Proof.* Consider the sequences $\alpha_k \uparrow \infty$, $\alpha_k > 0$ and $h_k \uparrow \infty$, $h_k > 10$ with the properties

$$\sum_{k=1}^{\infty} \frac{\alpha_k}{h_k} < \infty, \ \sum_{k=1}^{\infty} \frac{\ln h_k}{h_k} = \infty. \tag{26}$$

Let $q_k = 2^{m_k}$, where $m_k \in \mathbb{N}$ $(k \in \mathbb{N})$ and $m_k \uparrow \infty$. For every $k$, we divide $I^2$ into $q_k^2$ equal square interval and denote them by $I^2_{k,q}$. The length of square intervals sides will be denoted by $\Delta_k$. For every $I^2_{k,q}$ let us consider the square interval $I_{k,q}$ concentric with $I^2_{k,q}$ having the side length $\delta_k = \Delta_k/(2h_k+1)$. We shall assume $q_k \uparrow \infty$ so that $\delta_k > \Delta_{k+1}$ $(k \in \mathbb{N})$.

Let $f_k = \sup\{\alpha_k h_k \chi_{I_{k,q}} : q = \overline{1, q_k^2}\}$ $(k \in \mathbb{N})$ and $f = \sum_{k=1}^{\infty} f_k$. Clearly, $f \geq 0$. It can be easily checked that

$$\|f_k\|_1 = \alpha_k h_k \Big|\bigcup_{q=1}^{q_k^2} I_{k,q}\Big| = \alpha_k h_k \sum_{q=1}^{q_k^2} \frac{1}{(2h_k+1)^2} |I^2_{k,q}| < \frac{\alpha_k}{h_k},$$

which according to (26) implies

$$\|f\|_1 \leq \sum_{k=1}^{\infty} \|f_k\|_1 < \sum_{k=1}^{\infty} \frac{\alpha_k}{h_k} < \infty,$$

i.e., $f \in L(I^2)$.

By Lemma 1 for $k \in \mathbb{N}$, $q = \overline{1, q_k^2}$ we have

$$\{M_{B_2}(\alpha_k h_k \chi_{I_{k,q}}) > \alpha_k\} \subset (2h_k+1) I_{k,q} = I^2_{k,q}, \tag{27}$$

and

$$\big|\{M_{B_2}(\alpha_k h_k \chi_{I_{k,q}}) > \alpha_k\}\big| \geq h_k \ln h_k |I_{k,q}|.$$

Thus we write

$$\begin{aligned} \Big|\bigcup_{q=1}^{q_k^2} \{M_{B_2}(\alpha_k h_k \chi_{I_{k,q}}) > \alpha_k\}\Big| &= \sum_{q=1}^{q_k^2} \big|\{M_{B_2}(\alpha_k h_k \chi_{I_{k,q}}) > \alpha_k\}\big| \geq \\ &\geq \sum_{q=1}^{q_k^2} h_k \ln h_k |I_{k,q}| = \sum_{q=1}^{q_k^2} h_k \ln h_k \frac{|I^2_{k,q}|}{(2h_k+1)^2} > \frac{\ln h_k}{9h_k} \end{aligned} \tag{28}$$

for $k \in \mathbb{N}$.

For every $k \in \mathbb{N}$ let us consider those intervals $I^2_{k+1,q}$ $(q = \overline{1, q_{k+1}^2})$ which are contained in $\bigcup_{q=1}^{q_k^2} \{M_{B_2}(\alpha_k h_k \chi_{I_{k,q}}) > \alpha_k\}$. Denote their union by $O_k$. It can be easily seen that if $\{q_k\}$ tends rapidly enough to $\infty$, then

$$|O_k| \geq \frac{1}{2} \Big|\bigcup_{q=1}^{q_k^2} \{M_{B_2}(\alpha_k h_k \chi_{I_{k,q}}) > \alpha_k\}\Big| \quad (k \in \mathbb{N}). \tag{29}$$

Using the property of dyadic intervals, we can see that $\{O_k\}$ is the sequence of independent sets, i.e., for every $n \geq 2$ and for arbitrary pairwise different natural numbers $k_1, k_2 \dots, k_n$,

$$\Big|\bigcap_{m=1}^{n} O_{k_m}\Big| = \prod_{m=1}^{n} |O_{k_m}|.$$

Owing to (26), (28), and (29), $\sum_{k=1}^{\infty} |O_k| > \sum_{k=1}^{\infty} \dfrac{1}{2}\dfrac{\ln h_k}{9h_k} = \infty$. Using now Borel–Kantelly's lemma, we get

$$\overline{\lim_{k\to\infty}} O_k, \text{ has a complete measure in } I^2. \tag{30}$$

From (27) we have for $k \in \mathbb{N}$, $q = \overline{1, q_k^2}$

$$\{M_{B_2}(\alpha_k h_k \chi_{I_{k,q}}) > \alpha_k\} = \{M_{B_2}^{(\Delta_k)}(\alpha_k h_k \chi_{I_{k,q}}) > \alpha_k\},$$

which for $k \in \mathbb{N}$ implies

$$O_k \subset \bigcup_{q=1}^{q_k^2} \{M_{B_2}(\alpha_k h_k \chi_{I_{k,q}}) > \alpha_k\} =$$

$$= \bigcup_{q=1}^{q_k^2} \{M_{B_2}^{(\Delta_k)}(\alpha_k h_k \chi_{I_{k,q}}) > \alpha_k\} \subset \{M_{B_2}^{(\Delta_k)}(f_k) > \alpha_k\}.$$

Because of (30), from the latter inclusion we obtain

$$\overline{\lim_{k\to\infty}} \{M_{B_2}^{(\Delta_k)}(f_k) > \alpha_k\} \text{ has a complete measure in } I^2. \tag{31}$$

Since $\alpha_k \uparrow \infty$, $\Delta_k \downarrow 0$ $(k \to \infty)$, we can easily see that if $x \in \overline{\lim_{k\to\infty}} \{M_{B_2}^{(\Delta_k)}(f_k) > \alpha_k\}$, then $\overline{D}_{B_2}\big(\int f, x\big) = \infty$. By virtue of (31) we conclude that the latter equality holds a.e. on $I^2$.

Assume $k$ to be fixed. For every $I_{k,q}$ $(q = \overline{1, q_k^2})$ let us consider the intervals $J_{k,q}^1$ $(\mathrm{pr}_1\, J_{k,q}^1 = 3\,\mathrm{pr}_1\, I_{k,q},\ \mathrm{pr}_2\, J_{k,q}^1 = (0,1))$ and $J_{k,q}^2$ $(\mathrm{pr}_1\, J_{k,q}^2 = (0,1),\ \mathrm{pr}_2\, J_{k,q}^2 = 3\,\mathrm{pr}_2\, I_{k,q})$. Denote $G_k^i = \bigcup_{q=1}^{q_k^2} J_{k,q}^i$ $(i = \overline{1,2})$, $G_k = G_k^1 \cup G_k^2$. It can be easily verified (see (26)) that $\sum_{k=1}^{\infty} |G_k| < \infty$ which implies that $\overline{\lim_{k\to\infty}} G_k$ has zero measure.

Determine now an unknown basis $B$. If $x \in \Big(\overline{\lim_{k\to\infty}} G_k\Big) \cup \mathbb{Q}^2 \cup (\mathbb{R}^2 \backslash I^2)$ ($\mathbb{Q}$ is a set of rational numbers), then we put $B(x) = B_2(x)$. For $x \in I^2 \backslash \Big(\overline{\lim_{k\to\infty}} G_k \cup \mathbb{Q}^2\Big)$ choose $B(x)$ such that if $d_I$ is the length of the lesser side of the interval $I \ni x$, and $d_I \notin \bigcup\limits_{k \geq k(x)} [\delta_k, \Delta_k/2]$ (for $x \notin \overline{\lim\limits_{k\to\infty}} G_k$, $k(x)$ is the number with the property $x \notin G_k$ for $k \geq k(x)$), then $I \in B(x)$, while if $d_I \in [\delta_k, \Delta_k/2]$ for some $k \geq k(x)$, then $I \in B(x)$ iff either $I \cap G_k^1 = \varnothing$ or $I \cap G_k^2 = \varnothing$.

It can be easily verified that for every $I \in \overline{B}_2$ and $x \in \mathbb{R}^2$ there is a translation $T, T(I) \in B(x)$; this implies that $B$ is the $TI^*$-basis. The local regularity of $\overline{B}_2$ with respect to $\overline{B}$ is also easily checked. By proving the inclusion $f \in F_B^+$ the proof of the theorem will be completed.

Denote $A_k = \{x \in \mathbb{R}^n : D_B(\int f_k, x) = f_k(x)\}$ $(k \in \mathbb{N})$. Clearly, $A_k$ $(k \in \mathbb{N})$ has a complete measure on $\mathbb{R}^2$. Denote $A = \Big(I^2 \cap \bigcap\limits_{k=1}^{\infty} A_k\Big) \setminus \Big(\overline{\lim\limits_{k\to\infty}} G_k \cup \mathbb{Q}^2\Big)$. Since $f(x) = 0$ for $x \notin I^2$, $B$ differentiates $\int f$ on $\mathbb{R}^2 \setminus I^2$; $A$ is the set of a complete measure on $I^2$. Therefore to prove the inclusion $f \in F_B^+$, it suffices to show the differentiability of $\int f$ with respect to $B$ on the set $A$.

Let $x \in A$ and $\varepsilon > 0$. Find $\delta > 0$ for which $\Big|(1/|I|)\int_I f - f(x)\Big| < \varepsilon$ when $I \in B(x)$, $\operatorname{diam} I < \delta$. Consider $k'(x) \geq k(x)$ for which

$$\sum_{k \geq k'(x)} 16\|f_k\|_1 < \varepsilon/2. \tag{32}$$

It is clear that $\sum_{k=1}^{\infty} f_k(x) = \sum_{k=1}^{k'(x)-1} f_k(x)$, and $B$ differentiates $\int \sum_{k=1}^{k'(x)-1} f_k$ at the point $x$. This implies that there is $\delta \in (0, \Delta_{k'(x)}/2)$ such that

$$\Big|\frac{1}{|I|}\int_I \sum_{k=1}^{k'(x)-1} f_k - \sum_{k=1}^{k'(x)-1} f_k(x)\Big| < \varepsilon/2, \ \text{ for } \ I \in B(x), \ \operatorname{diam} I < \delta. \tag{33}$$

Consider an arbitrary interval $I \in B(x)$, $\operatorname{diam} I < \delta$. For $k_I \in \mathbb{N}$ let $\Delta_{K_I+1}/2 \leq d_I \leq \Delta_{k_I}/2$. Clearly, $k_I \geq k'(x)$. By virtue of the inequality $\Delta_{k+1} < \delta_k$ $(k \in \mathbb{N})$ and the condition $x \notin G_k$ $(k \geq k(x))$, we have $I \cap \operatorname{supp} f_k = \varnothing$ for $k'(x) \leq k \leq k_I - 1$, which gives

$$\int_I f_k = 0 \ \text{ for } \ k'(x) \leq k \leq k_I - 1. \tag{34}$$

Analogously, $\int_I f_{k_I} = 0$ if $d_I < \delta_{k_I}$, while if $\delta_{k_I} \leq d_I \leq \Delta_{k_I}/2$, then $\int_I f_{k_I} = 0$ because $I \cap \operatorname{supp} f_k = \varnothing$ by construction of the basis $B$. Thus

$$\int_I f_{k_I} = 0. \tag{35}$$

Let is show that

$$|I \cap \operatorname{supp} f_k| \leq 16|I|\,|\operatorname{supp} f_k| \ \text{ for } \ k \geq k_I + 1. \tag{36}$$

To this end it is sufficient to consider the following easily verifiable facts:

1) $d_I \geq \Delta_k/2$ for $k \geq k_I + 1$;

2) let $P_k$ be a set of vertices of the squares $I^2_{k,q}$ $(q = \overline{1, q_k^2})$. If the vertices of the interval $J \subset I^2$ belong to the set $P_k$, then $|J \cap \operatorname{supp} f_k| = |I| |\operatorname{supp} f_k|$;

3) for every interval $J \subset I^2$, $d_J \geq \Delta_k/2$ there is an interval $J' \supset J$, $|J'| \leq 16|J|$ with vertices at the points of the set $P_k$.

By (36) we can write that for $k \geq k_I + 1$,

$$\int_I f_k = \alpha_k h_k |I \cap \operatorname{supp} f_k| \leq 16 \alpha_k h_k |I| \, |\operatorname{supp} f_k| = 16|I| \, \|f_k\|_1,$$

which, owing to (32) and the inequality $k_I \geq k'(x)$, implies

$$\frac{1}{|I|} \int_I \sum_{k \geq k_I + 1} f_k = \frac{1}{|I|} \sum_{k \geq k_I + 1} \int_I f_k \leq \frac{1}{|I|} \sum_{k \geq k_I + 1} 16|I| \, \|f_k\|_1 < \varepsilon/2. \quad (37)$$

By virtue of relations (33)–(35) and (37) (it should also be noted that $f_k(x) = 0$ for $k \geq k'(x)$), we have

$$\left| \frac{1}{|I|} \int_I \sum_{k=1}^{\infty} f_k - \sum_{k=1}^{\infty} f_k(x) \right| \leq \left| \frac{1}{|I|} \int_I \sum_{k=1}^{k'(x)-1} f_k - \sum_{k=1}^{k'(x)-1} f_k(x) \right| +$$

$$+ \frac{1}{|I|} \int_I \sum_{k=k'(x)}^{k_I} f_k + \frac{1}{|I|} \int_I \sum_{k \geq k_I + 1} f_k + \sum_{k \geq k'(x)} f_k(x) < \frac{\varepsilon}{2} + 0 + \frac{\varepsilon}{2} + 0 = \varepsilon.$$

Due to the arbitrariness of $x \in A$ and $\varepsilon > 0$, we conclude that $D_B\left(\int f, x\right) = f(x)$ for every $x \in A$. $\square$

## 4. Remarks

(1) Let us consider one application of the obtained results. Let $B_Z$ be the basis in $\mathbb{R}^2$ for which $B_Z(x)$ $(x \in \mathbb{R}^2)$ consists of all intervals $I \ni x$, $D_I^2 \leq d_I \leq D_I \leq 1$, where $D_I$ and $d_I$ are the lengths, respectively, of the greater and of the lesser side of $I$. This basis was introduced by Zygmund and it was he who initiated (see [1], Ch. VI, §4) the study of the differential properties of $B_Z$.

Morion showed (see [1], Appendix IV) that for the integral classes $B_Z$ behaves like $B_2$, i.e., $B_Z$ does not differentiate a wider integral class than $L \ln^+ L$.

A question arises: does there exist in general a function whose integral does not differentiate $B_2$ and differentiates $B_Z$? From the above proven theorems the answer is positive.

$B_Z$ is the $TI$-basis. The fact that $\overline{B}_2 \notin LR(\overline{B}_z)$ is easily verified. By Theorem 1 this implies that the strict inclusion $F^+_{B_2} \subset F^+_{B_Z}$ holds. Moreover, Theorem 3 implies that by perturbing the values $\omega$ of any function $f \in L\backslash L\ln^+ L(I^2)$, $f \geq 0$, one can get the function $f \circ \omega$ with the following properties: $\overline{D}_{B_2}\big(\int f \circ \omega, x\big) = \infty$ a.e. on $I^2$, and $B_Z$ differentiates $\int f \circ \omega$.

Thus, for the integral classes the bases $B_2$ and $B_Z$ behave similarly, while for individual nonnegative functions the basis $B_Z$ behaves better than the basis $B_2$.

(2) The bases $B$ and $B'$ are said to be positive equivalent ($B \overset{+}{\Leftrightarrow} B'$) if for every $f \in L(\mathbb{R}^n)$, $f \geq 0$, $\overline{D}_B\big(\int f, x\big) = \overline{D}_{B'}\big(\int f, x\big)$ and $\underline{D}_B\big(\int f, x\big) = \underline{D}_{B'}\big(\int f, x\big)$ a.e. on $\mathbb{R}^n$ (i.e., condition (b) in Theorem 1 means that the bases $B$ and $B'$ are positive equivalent).

$B$ is said to be a Busemann–Feller type basis ($BF$-basis) if for any $R \in \overline{B}$ and $x \in R$ we have $R \in B(x)$.

We shall say that $B$ exactly differentiates $\varphi(L)$ (writing $B \in D(\varphi(L))$ if $B$ differentiates $\varphi(L)$ and does not differentiate a wider integral class than $\varphi(L)$.

For integral classes the behavior of $B \subset B_2$, $BF$, $TI$-bases was studied by Stokoloc in [3], where he introduced the property $(S)$ and proved that if $B$ possesses the property $(S)$, then $B \in D(L\ln^+ L)$, and if $B$ does not possess this property, then $B \in D(L)$.

One can easily see that ignoring the $BF$ property, this result remains valid. Thus, for the integral classes $B \subset B_2$, the $TI$-basis behaves like $B_2$ or $B_1$ ($B_1$ is the basis formed of square intervals). The $B_Z$-basis illustrates that an analogous fact does not hold for nonnegative individual functions and, generally speaking, if we combine Stokoloc's assertion and Theorem 1, then we shall have:

1) if $\overline{B}_2 \in LR(\overline{B})$, then $B \overset{+}{\Leftrightarrow} B_2$;

2) if $\overline{B}_2 \notin LR(\overline{B})$ and $B$ possesses the property $(S)$, then $B \in D(L\ln^+ L)$ and $F^+_{B_2} \subset F^+_B$ (strictly);

3) if $B$ does not possess the property $(S)$, then $B \in D(L)$.

(3) Let $B \subset B_2$ be a $TI$-basis. Consider the intervals $I \in \overline{B}$ of the type $I = (0, x^1) \times (0, x^2)$. Denote the set of points $(x^1, x^2)$ by $A_B$. The set $A_B$ indicates how rich the family $\overline{B}$ is.

One can easily prove the following criterion of local regularity of $\overline{B}_2$ with respect to $\overline{B}$ : $(\overline{B}_2 \in LR(\overline{B})) \Leftrightarrow (\exists m, k_0 \in \mathbb{N} : A_B \cap ([1/m^k, 1/m^{k-1}) \times [1/m^{k'}, 1/m^{k'-1})) \neq \varnothing$ for $k, k' > k_0)$.

(4) The basis $B$ is said to be invariant ($HI$-basis) with respect to homotheties if for every $x \in \mathbb{R}^n$ and every homothety $H$ centered at $x$ we have $B(x) = \{H(R) : R \in B(x)\}$.

If $\overline{B}$ is locally regular with respect to $\overline{B}'$ and, moreover, if $\delta$ is equal to $\infty$, then $\overline{B}$ is called regular with respect to $\overline{B}'$.

For the basis $B \subset B_2$ define the sets: $R_{1,B} = \{r > 1 : \exists I \in \overline{B}, |\operatorname{pr}_1 I|_1 = r|\operatorname{pr}_2 I|_1\}$, $R_{2,B} = \{r > 1 : \exists I \in \overline{B}, |\operatorname{pr}_2 I|_1 = r|\operatorname{pr}_1 I|_1\}$.

For the basis $B \subset B_2$ which is simultaneously the $TI$- and $HI$ basis one can easily verify that $(\overline{B}_2 \in LR(\overline{B})) \Leftrightarrow (\overline{B}_2$ is regular with respect to $B) \equiv\leftrightarrow (\exists m \in \mathbb{N} : R_{i,B} \cap [m^k, m^{m+1}) \neq \varnothing,\ k \in \mathbb{N},\ i = \overline{1,2})$.

(5) Let $B_2$ $(B_2 = B_2(\mathbb{R}^n))$ be the basis in $\mathbb{R}^n$ for which $B_2(x)$ $(x \in \mathbb{R}^n)$ consists of all $n$-dimensional intervals containing the point $x$.

Theorems 1–4 are also valid for $B \subset B_2(\mathbb{R}^n)$ $(n \geq 3)$, $TI$-bases. Proofs for the $n$-dimensional case are similar to those for the two-dimensional case.

(6) Let $\delta_k^1 \downarrow 0, \dots, \delta_k^n \downarrow 0$ $(k \to \infty)$. Denote $\Delta_{k,m}^i = [(m-1)\delta_k^i, m\delta_k^i)$ $(i = \overline{1,n}; k, m \in \mathbb{N})$. Let $B$ be the basis in $\mathbb{R}^n$ for which $B(x) = \{\prod_{i=1}^n \Delta_{k^i,m^i}^i : \prod_{i=1}^n \Delta_{k^i,m^i}^i \ni x; k^i, m^i \in \mathbb{N})$. Such bases are sometimes called nets. We call them $N$-bases.

Let $B$ be the $N$-basis in $\mathbb{R}^n$ $(n \geq 2)$. Denote by $B_T$ the least $TI$-basis containing $B$. For $B$ the following analogue of Theorem 1 is valid: the following conditions are equivalent: (i) $F_B^+ = F_{B_2}^+$; (ii) $B \overset{+}{\Leftrightarrow} B_2$; (iii) $\overline{B}_2$ is locally regular with respect to $\overline{B}_T$.

Obviously, (ii)$\Rightarrow$(i), the implication (i)$\Rightarrow$(iii) follows directly from Theorem 1. Thus it remains only to show that (iii)$\Rightarrow$(i). To this end, note that using Lemma 1 of [2] one can easily obtain the inequality

$$|\{M_{B_T}^{(r)}(f) > \lambda\}| \leq 3^n \left|\left\{M_B^{(r)}(f) > \frac{\lambda}{2^n}\right\}\right| \quad (f \in L(\mathbb{R}^n); \lambda, r > 0).$$

From this and Lemma 5 we get the upper bound of the distribution function of $M_{B_2}^{(r)}(f)$ by means of the distribution function of $M_B^{(cr)}(f)$. It easily follows from the relation $\overline{B}_2 \in LR(\overline{B})$ that $B$ contains a regular subbasis. Hence $B$ possesses the property $(E)$. Next, we obtain the implication (iii)$\Rightarrow$(ii) from Lemma 6.

Note that for $\delta_k^i = 1/2^k$ $(i = \overline{1,n}; k \in \mathbb{N})$ the relation $B \overset{+}{\Leftrightarrow} B_2$ was proved earlier in [2].

(7) The basis $B$ constructed in Theorem 4 is not a $BF$-basis, which is not a casual fact. In particular, the following assertion is valid: let $B \subset B_2$ $(B_2 = B_2(\mathbb{R}^n))$ be a $BF$-basis and let $\overline{B}_2 \in LR(\overline{B})$. Then $B \overset{+}{\Leftrightarrow} B_2$.

To prove the above assertion we have to consider the following facts which easily follow from the Busemann-Feller property of the basis $B$ and from the relation $\overline{B}_2 \in LR(\overline{B}$: (i) $B$ contains a regular subbasis; (ii) for every $f \in L(\mathbb{R}^n)$,

$$|\{M_{B_2}^{(r)}(f) > \lambda\}| \leq \left|\left\{M_B^{(cr)}(f) > \frac{\lambda}{c}\right\}\right| \quad (\lambda > 0,\ 0 < r < \delta),$$

where $c$ and $\delta$ are the constants from the definition of local regularity of $\overline{B}_2$ with respect to $\overline{B}$.

From (i) we have that $B$ possesses the property $(E)$. Now, taking into account (ii) and using Lemma 6, we can conclude that the relation $B \overset{+}{\Leftrightarrow} B_2$ is valid.

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