

## ON DIFFERENTIAL BASES FORMED OF INTERVALS

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*In memory of young mathematician  
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ABSTRACT. Translation invariant subbases of the differential basis  $B_2$  (formed of all intervals), which differentiates the same class of all non-negative functions as  $B_2$  does, are described. A possibility for extending the results obtained to bases of more general type is discussed.

### 1. DEFINITIONS AND NOTATION

A mapping  $B$  defined on  $\mathbb{R}^n$  is said to be a differential basis in  $\mathbb{R}^n$  if, for every  $x \in \mathbb{R}^n$ ,  $B(x)$  is a family of open bounded sets containing the point  $x$  such that there exists a sequence  $\{R_k\} \subset B(x)$ ,  $\text{diam } R_k \rightarrow 0$  ( $k \rightarrow \infty$ ).

For  $f \in L_{loc}(\mathbb{R}^n)$  the numbers

$$\overline{D}_B \left( \int f, x \right) = \overline{\lim}_{\text{diam } R \rightarrow 0, R \in B(x)} \frac{1}{|R|} \int_R f$$

and

$$\underline{D}_B \left( \int f, x \right) = \underline{\lim}_{\text{diam } R \rightarrow 0, R \in B(x)} \frac{1}{|R|} \int_R f$$

are said to be respectively the upper and the lower derivative of the integral of  $f$  at the point  $x$ . If the upper and the lower derivative coincide, then their common value is called the derivative of the integral of  $f$  at the point  $x$ , and we denote it by  $D_B \left( \int f, x \right)$ . They say that the basis  $B$  differentiates the integral of  $f$  if  $D_B \left( \int f, x \right) = f(x)$  for almost all  $x$ . The set of those functions  $f \in L_{loc}(\mathbb{R}^n)$ ,  $f \geq 0$ , whose integrals are differentiable with respect to the basis  $B$  will be denoted by  $F_B^+$ . Under  $M_B$  we mean the maximal operator

$$M_B(f)(x) = \sup_{R \in B(x)} \frac{1}{|R|} \int_R |f| \quad (f \in L_{loc}(\mathbb{R}^n), x \in \mathbb{R}^n),$$

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corresponding to the basis  $B$ . It will be assumed here that  $\overline{B} = \bigcup_{x \in \mathbb{R}^n} B(x)$ .

$B$  is said to be a subbasis of  $B'$  (writing  $B \subset B'$ ), if  $B(x) \subset B'(x)$  ( $x \in \mathbb{R}^n$ ). The basis  $B$  is said to be translation invariant or the  $TI$ -basis, if  $B(x) = \{x + R : R \in B(O)\}$  ( $x \in \mathbb{R}^n$ ) (here  $O$  is the origin in  $\mathbb{R}^n$ ). Let us have the bases  $B$  and  $B'$ . We shall say that the family  $\overline{B}$  is locally regular with respect to the family  $\overline{B}'$  (writing  $\overline{B} \in LR(\overline{B}')$ ), if there exist  $\delta > 0$  and  $c > 0$  such that for any  $R \in \overline{B}$ ,  $\text{diam } R < \delta$ , there is  $R' \in \overline{B}'$  such that  $R \subset R'$  and  $|R'| < c|R|$ .

We shall agree that  $I^n = [0, 1]^n$  and  $f \in L(I^n)$ , if  $f \in L(\mathbb{R}^n)$  and  $\text{supp } f \subset I^n$ .

## 2. $TI$ -BASES FORMED OF INTERVALS

Let  $B_2$  be the basis in  $\mathbb{R}^2$  for which  $B_2(x)$  ( $x \in \mathbb{R}^2$ ) consists of all two-dimensional intervals containing the point  $x$ .

The theorem below characterizes  $B \subset B_2$ ,  $TI$ -bases for which  $F_B^+ = F_{B_2}^+$ .

**Theorem 1.** *Let  $B \subset B_2$  be a  $TI$ -basis. Then the following conditions are equivalent:*

- (a)  $F_B^+ = F_{B_2}^+$ ;
- (b) for every  $f \in L(\mathbb{R}^2)$ ,  $f \geq 0$ , a.e. on  $\mathbb{R}^2$

$$\overline{D}_B \left( \int f, x \right) = \overline{D}_{B_2} \left( \int f, x \right) \quad \text{and} \quad \underline{D}_B \left( \int f, x \right) = \underline{D}_{B_2} \left( \int f, x \right);$$

- (c)  $\overline{B}_2$  is locally regular with respect to  $\overline{B}$ .

The implication (b)  $\Rightarrow$  (a) is evident. Therefore to prove Theorem 1 it suffices to show that (a)  $\Rightarrow$  (c) and (c)  $\Rightarrow$  (b).

The implication (a)  $\Rightarrow$  (c) follows from the following assertion.

**Theorem 2.** *Let  $B \subset B_2$  be a  $TI$ -basis. If  $\overline{B}_2$  is not locally regular with respect to  $\overline{B}$ , then there exists a function  $f \in L(I^2)$ ,  $f \geq 0$ , such that*

$$\begin{aligned} \overline{D}_{B_2} \left( \int f, x \right) &= \infty \quad \text{a.e. on } I^2, \\ D_B \left( \int f, x \right) &= f(x) \quad \text{a.e. on } I^2. \end{aligned}$$

Before proving Theorem 2 we shall give several lemmas.

For the interval  $I$  we denote by  $\alpha I$  ( $\alpha > 0$ ) the interval  $H(I)$ , where  $H$  is the homothety with the coefficient  $\alpha$  whose center is the center of  $I$ .

**Lemma 1.** *Let  $I \in \overline{B}_2$  and  $h > 1$ . Then  $\{M_{B_2}(h\chi_I) > 1\} \subset (2h + 1)I$  and  $|\{M_{B_2}(h\chi_I) > 1\}| \geq h(\ln h)|I|$ .*

The validity of this lemma can be shown by a direct checking.

Projections of  $I$  onto the  $ox^1$ - and  $ox^2$ -axes will be denoted by  $\text{pr}_1 I$  and  $\text{pr}_2 I$ , respectively. For  $I \in \overline{B}_2$  and  $h > 0$  we have

$$\begin{aligned} R_1(I, h) &= (2h + 1) \text{pr}_1 I \times 3 \text{pr}_2 I, \\ R_2(I, h) &= 3 \text{pr}_1 I \times (2h + 1) \text{pr}_2 I, \\ R(I, h) &= R_1(I, h) \cup R_2(I, h). \end{aligned}$$

It is clear that  $|R(I, h)| \leq 18h|I|$ .

**Lemma 2.** *Let  $B \subset B_2$  be a TI-basis;  $h > 1$ . Suppose that for  $I \in \overline{B}_2$  there is no  $J \in \overline{B}$ ,  $J \supset I$ , such that  $|J| \leq h|I|$ . Then  $\{M_B(h\chi_I) > 1\} \subset R(I, h)$ .*

*Proof.* Let  $J \in \overline{B}$  and  $(1/|J|) \int_J h\chi_I > 1$ . From the condition of the lemma we can easily find that either

$$|\text{pr}_1 J|_1 < |\text{pr}_1 I|_1 \quad \text{or} \quad |\text{pr}_2 J|_1 < |\text{pr}_2 I|_1, \quad (1)$$

where  $|\cdot|_1$  is the Lebesgue measure on  $\mathbb{R}$ .

By Lemma 1,  $J \subset (2h + 1)I$ , which by virtue of (1) implies that either  $J \subset R_1(I, h)$  or  $J \subset R_2(I, h)$ , so that we have  $J \subset R(I, h)$ . It follows from the latter inclusion that  $\{M_B(h\chi_I) > 1\} \subset R(I, h)$ .  $\square$

It can be easily seen that the following two lemmas are valid.

**Lemma 3.** *Let  $I, J \in \overline{B}_2$ ;  $h > 1$ . If either  $|\text{pr}_1 J|_1 \geq h|\text{pr}_1 I|_1$  or  $|\text{pr}_2 J|_1 \geq h|\text{pr}_2 I|_1$ , then  $h|J \cap I| \leq |I \cap R(I, h)|$ .*

**Lemma 4.** *Let  $I, J \in \overline{B}_2$ . If  $J \cap I \neq \emptyset$  and  $J \setminus R(I, h) \neq \emptyset$ , then either  $|\text{pr}_1 J|_1 \geq |\text{pr}_1 I|_1$ , or  $|\text{pr}_2 J|_1 \geq |\text{pr}_2 I|_1$ .*

The following lemma is also valid.

**Lemma 5.** *Let  $B \subset B_2$ ;  $h > 1$ , and let  $I_1, \dots, I_k \in \overline{B}_2$  be the equal intervals. If*

$$\begin{aligned} \{M_B(h\chi_{I_m}) > 1\} &\subset R(I_m, h) \quad (m = \overline{1, k}), \\ R(I_m, h) \cap R(I_{m'}, h) &= \emptyset \quad (m \neq m'), \end{aligned}$$

then

$$\left\{ M_B \left( \sum_{m=1}^k h\chi_{I_m} \right) > 1 \right\} \subset \bigcup_{m=1}^k R(I_m, h). \quad (2)$$

*Proof.* Inequality (2) is equivalent to the following inequality: if  $x \notin \bigcup_{m=1}^k R(I_m, h)$  and  $I \in B(x)$ , then

$$\frac{1}{|I|} \int_I \sum_{m=1}^k h\chi_{I_m} \leq 1. \quad (3)$$

Let  $x \notin \bigcup_{m=1}^k R(I_m, h)$  and  $I \in B(x)$ . Then we may have three cases. Let us consider each of them separately.

(i)  $I$  intersects none of the intervals  $I_m$  ( $m = \overline{1, k}$ ); in this case the validity of (3) is evident.

(ii)  $I$  intersects only one interval  $I_m$  ( $m = \overline{1, k}$ ).

Let  $I_n$  be the interval for which  $I \cap I_n \neq \emptyset$ .  $x \notin R(I_m, h)$ , and therefore by the condition of the lemma,

$$\frac{1}{|I|} \int_I h\chi_{I_n} \leq M_B(h\chi_{I_n})(x) \leq 1,$$

from which, taking into account the equality  $I \cap I_m = \emptyset$  ( $m \neq n$ ), we obtain (3).

(iii)  $I$  intersects more than one interval  $I_m$  ( $m = \overline{1, k}$ ).

It follows from the equality  $R(I_m, h) \cap R(I_{m'}, h) = \emptyset$  ( $m \neq m'$ ) that

$$(2h+1)I_m \cap I_{m'} = \emptyset \quad (4)$$

Denote  $P = \{m \in [1, k] : I \cap I_m \neq \emptyset\}$ . By (4) we have

$$I \cap I_m \neq \emptyset, \quad I \setminus (2h+1)I_m \neq \emptyset \quad (m \in P),$$

which gives either

$$|\text{pr}_1 I|_1 \geq h|\text{pr}_1 I_m|_1 \quad \text{or} \quad |\text{pr}_2 I|_1 \geq h|\text{pr}_2 I_m|_1 \quad (m \in P).$$

Now according to Lemma 3 we can conclude that

$$h|I \cap I_m| \leq |I \cap R(I_m, h)| \quad (m \in P).$$

From the obtained inequality, taking into consideration the pairwise non-intersection of  $R(I_m, h)$  ( $m = \overline{1, k}$ ), we write

$$\int_I \sum_{m=1}^k h\chi_{I_m} = \sum_{m \in P} \int_I h\chi_{I_m} = \sum_{m \in P} h|I \cap I_m| \leq \sum_{m \in P} |I \cap R(I_m, h)| \leq |I|.$$

Thus inequality (3) and Lemma 5 are proved.  $\square$

**Lemma 6.** *Let  $B \subset B_2$ ,  $h > 1$ , and for every  $m \in [1, k]$  let  $\{I_{m,q}\}_{q=1}^{q_m} \subset \overline{B}_2$  be a family of equal intervals. If*

$$\begin{aligned} \{M_B(h\chi_{I_{m,q}}) > 1\} &\subset R_{m,q} \\ (R_{m,q} = R(I_{m,q}, h); m = \overline{1, k}) \quad q = \overline{1, q_m}), \end{aligned} \quad (5)$$

$$R_{m,q} \cap R_{m',q'} = \emptyset \quad (m, q) \neq (m', q'), \quad (6)$$

$$\begin{aligned} |\text{pr}_1 I_{m,1}|_1 &\geq h |\text{pr}_1 I_{m+1,1}|_1, \\ |\text{pr}_2 I_{m,1}|_1 &\geq h |\text{pr}_2 I_{m+1,1}|_1, \end{aligned} \quad (m = \overline{1, k-1}) \quad (7)$$

then

$$\left\{ M_B \left( \sum_{m=1}^k \sum_{q=1}^{q_m} h\chi_{I_{m,q}} \right) > 2 \right\} \subset \bigcup_{m=1}^k \bigcup_{q=1}^{q_m} R_{m,q}.$$

*Proof.* The inclusion we have to prove is equivalent to the following inequality: if  $x \notin \bigcup_{m=1}^k \bigcup_{q=1}^{q_m} R_{m,q}$  and  $I \in B(x)$ , then

$$\frac{1}{|I|} \int_I \sum_{m=1}^k \sum_{q=1}^{q_m} h\chi_{I_{m,q}} \leq 2. \quad (8)$$

Assume  $x \notin \bigcup_{m=1}^k \bigcup_{q=1}^{q_m} R_{m,q}$  and  $I \in B(x)$ . It is clear that (8) is fulfilled for  $I \cap I_{m,q} = \emptyset$  ( $m = \overline{1, k}$ ,  $q = \overline{1, q_m}$ ). Denote otherwise  $n = \min\{m \in [1, k] : \exists I_{m,q} (q = \overline{1, q_m}), I \cap I_{m,q} \neq \emptyset\}$  and consider first the case with  $1 < n < k$ .

By virtue of Lemma 5 (see (5) and (6)), we write

$$M_B \left( \sum_{q=1}^{q_n} h\chi_{I_{n,q}} \right) (x) \leq 1,$$

which implies

$$\int_I \sum_{q=1}^{q_n} h\chi_{I_{n,q}} = \sum_{q=1}^{q_n} h |I \cap I_{n,q}| \leq |I|. \quad (9)$$

For some  $q \in [1, q_n]$   $I \cap I_{n,q} \neq \emptyset$ , it is clear that  $I \setminus R_{n,q} \neq \emptyset$ . Therefore, according to Lemma 4, we have either

$$|\text{pr}_1 I|_1 \geq |\text{pr}_1 I_{n,q}|_1 \quad \text{or} \quad |\text{pr}_2 I|_1 \geq |\text{pr}_2 I_{n,q}|_1.$$

Taking into account that the intervals  $\{I_{m,q}\}_{q=1}^{q_m}$  ( $m = \overline{1, k}$ ) are equal and using (7), we can write that either

$$|\text{pr}_1 I|_1 \geq h |\text{pr}_1 I_{m,q}|_1 \quad (n < m \leq k, q = \overline{1, q_m})$$

or

$$|\text{pr}_2 I|_1 \geq h |\text{pr}_2 I_{m,q}|_1 \quad (n < m \leq k, q = \overline{1, q_m}),$$

whence by Lemma 3 we get

$$h|I \cap I_{m,q}| \leq |I \cap R_{m,q}| \quad (n < m \leq k; q = \overline{1, q_m}). \quad (10)$$

Clearly,  $I \cap I_{m,q} = \emptyset$  ( $n \leq m < n$ ;  $q = \overline{1, q_m}$ ), so that by (6), (9), (10) and  $1 \leq m < n$  we have

$$\begin{aligned} \int_I \sum_{m=1}^k \sum_{q=1}^{q_m} h \chi_{I_{m,q}} &= \sum_{m=1}^k \sum_{q=1}^{q_m} h |I \cap I_{m,q}| = \sum_{m=1}^{n-1} \sum_{q=1}^{q_m} h |I \cap I_{m,q}| + \\ &+ \sum_{q=1}^{q_n} h |I \cap I_{n,q}| + \sum_{m=n+1}^k \sum_{q=1}^{q_m} h |I \cap I_{m,q}| = \\ &= A_1 + A_2 + A_3 \leq 0 + |I| + \sum_{m=n+1}^k \sum_{q=1}^{q_m} |I \cap R_{m,q}| \leq |I| + |I| = 2|I|. \end{aligned}$$

Inequality (8) can be proved in a simpler way for  $n = 1$  or  $n = k$ , since in these cases we do not have the terms  $A_1$  and  $A_3$ .  $\square$

For the basis  $B$  let us define the operator

$$M_B^*(f)(x) = \sup_{R \in B(x)} \frac{1}{|R|} \left| \int_R f \right| \quad (f \in L_{loc}(\mathbb{R}^n), x \in \mathbb{R}^n).$$

**Lemma 7.** *Let the basis  $B$  differentiate the integrals of the functions  $f_k \in L(\mathbb{R}^n)$  ( $k \in \mathbb{N}$ ),  $\sum_{k=1}^{\infty} \|f_k\|_1 < \infty$ . If*

$$\sum_{k=1}^{\infty} |\{M_B^*(f_k) > \lambda_k\}|_e < \infty,$$

where  $\lambda_k > 0$  ( $k \in \mathbb{N}$ ),  $\sum_{k=1}^{\infty} \lambda_k < \infty$  and  $|\cdot|_e$  is an outer measure, then  $B$  differentiates the integral of the function  $\sum_{k=1}^{\infty} f_k$ .

Note that since  $M_B^*(f) \leq M_B(f)$  ( $f \in L(\mathbb{R}^n)$ ), the conclusion of Lemma 7 will be the more so valid when the inequality  $\sum_{k=1}^{\infty} |\{M_B(f_k) > \lambda_k\}|_e < \infty$  is fulfilled.

*Proof.* Let us note that:

(i)  $\sum_{k=1}^{\infty} \|f_k\|_1 < \infty$ ; therefore the set  $A_0 = \{x \in \mathbb{R}^n : \sum_{k=1}^{\infty} |f_k(x)| < \infty\}$  is of measure 0;

(ii)  $B$  differentiates  $\int f_k$  ( $k \in \mathbb{N}$ ), i.e., the sets  $A_k = \{x \in \mathbb{R}^n : D_B(\int f, x) = f(x)\}$  ( $k \in \mathbb{N}$ ) have a complete measure on  $\mathbb{R}^n$ .

(iii)  $\sum_{k=1}^{\infty} |\{M_B^*(f_k) > \lambda_k\}|_e < \infty$ ; therefore the set  $\overline{\lim}_{k \rightarrow \infty} \{M_B^*(f_k) > \lambda_k\}$  is of measure 0.

It follows from (i)–(iii) that the set  $A = \bigcup_{k=1}^{\infty} A_k \setminus (\overline{\lim}_{k \rightarrow \infty} \{M_B^*(f_k) > \lambda_k\} \cup A_0)$  is of complete measure. Let us show that for every  $x \in A$  we have  $D_B\left(\int \sum_{k=1}^{\infty} f_k, x\right) = \sum_{k=1}^{\infty} f_k(x)$ , which will prove the lemma.

Let  $x \in A$  and  $\varepsilon > 0$ . From the inclusion  $x \in A$  we can conclude that:

- 1) there is  $k_0 \in \mathbb{N}$  ( $k_0 = k(x, \varepsilon)$ ) such that  $x \notin \bigcap_{k=k_0}^{\infty} \{M_B^*(f_k) > \lambda_k\}$ ,  
 $\sum_{k=k_0}^{\infty} \lambda_k < \varepsilon/3$  and  $\sum_{k=k_0}^{\infty} |f_k(x)| < \varepsilon/3$ ;
- 2) there is  $\delta > 0$  such that for every  $R \in B(x)$ ,  $\text{diam } R < \delta$

$$\left| \frac{1}{|R|} \int_R \sum_{k=1}^{k_0-1} f_k - \sum_{k=1}^{k_0-1} f_k(x) \right| < \varepsilon/3.$$

According to 1) and 2) we can write that for every  $R \in B(x)$ ,  $\text{diam } R < \delta$ ,

$$\begin{aligned} \left| \frac{1}{|R|} \int_R \sum_{k=1}^{\infty} f_k - \sum_{k=1}^{\infty} f_k(x) \right| &\leq \left| \frac{1}{|R|} \int_R \sum_{k=1}^{k_0-1} f_k - \sum_{k=1}^{k_0-1} f_k(x) \right| + \\ &+ \sum_{k=k_0}^{\infty} \frac{1}{|R|} \left| \int_R f_k \right| + \sum_{k=k_0}^{\infty} |f_k(x)| < \varepsilon/3 + \sum_{k=k_0}^{\infty} M_B^*(f_k)(x) + \\ &+ \varepsilon/3 \leq 2\varepsilon/3 + \sum_{k=k_0}^{\infty} \lambda_k < \varepsilon. \end{aligned}$$

Since  $\varepsilon > 0$  is arbitrary, we conclude that the equality  $D_B\left(\int \sum_{k=1}^{\infty} f_k, x\right) = \sum_{k=1}^{\infty} f_k(x)$  is valid.  $\square$

*Proof of Theorem 2.* Let  $k \geq 3$ . Choose  $h_k$  such that  $\frac{1}{k} \ln \frac{h_k}{k} \geq 18 \cdot 2^{2k}$ .

Let  $\alpha_k = \frac{1}{k} \ln \frac{h_k}{k} / 2^{2(k+2)} h_k$ . Obviously,  $0 < \alpha_k < 1$ .

For  $I \in \overline{B}_2$  and  $h > 0$  denote  $Q^0(I, h) = (2h + 1)I$ . Let  $2^{-m} < |\text{pr}_1 Q^0(I, h)|_1 \leq 2^{-m+1}$  and  $2^{-m'} < |\text{pr}_2 Q^0(I, h)|_1 \leq 2^{-m'+1}$ , where  $m, m' \in \mathbb{N}$ . Denote by  $Q(I, h)$  the interval concentric with  $Q^0(I, h)$ , and  $|\text{pr}_1 Q(I, h)|_1 = 2^{-m+1}$ ,  $|\text{pr}_2 Q(I, h)|_1 = 2^{-m'+1}$ .

For the basis  $B$  denote by  $M_B^{(r)}$  ( $r > 0$ ) the operator

$$M_B^{(r)}(f)(x) = \sup_{R \in B(x), \text{diam } R < r} \frac{1}{|R|} \int_R |f| \quad (f \in L(\mathbb{R}^n), x \in \mathbb{R}^n).$$

$\overline{B}_2 \notin LR(\overline{B})$ , since there is  $I \in \overline{B}_2$  such that:

- 1) there is no  $J \in \overline{B}$ ,  $J \supset I$  for which  $|J| < 2^k h_k |I|$ ;
- 2)  $\text{diam } Q(I, 2^k h_k) < 1/k$ .

Divide  $I^2$  into intervals equal to  $Q(I, 2^k h_k)$  and denote them by  $Q_{1,q}$  ( $1 \leq q \leq q_1$ ). Take  $I_{1,q}$  ( $1 \leq q \leq q_1$ ) equal to  $T_q(I)$ , where  $T_q$  is a shift translating  $Q(I, 2^k h_k)$  to  $Q_{1,q}$ .

Let the families  $\{I_{1,q}\}_{q=1}^{q_1} \cdots \{I_{m,q}\}_{q=1}^{q_m}$  consisting of equal intervals be already constructed. Consider the sets

$$A_m^1 = \bigcup_{j=1}^m \bigcup_{q=1}^{q_j} \{M_{B_2}^{(1/k)}(h_k, \chi_{I_{j,q}}) > k\},$$

$$A_m^2 = \bigcup_{j=1}^m \bigcup_{q=1}^{q_j} R_{j,q} \quad (R_{j,q} = R(I_{j,q}, 2^k h_k)).$$

If  $|A_m^1| > 1 - \frac{1}{k}$ , then we stop the construction. If  $|A_m^1| \leq 1 - \frac{1}{k}$ , then we shall construct the family  $\{I_{m+1,q}\}_{q=1}^{q_{m+1}}$  as follows:

Consider the set  $A_m = I^2 = I^2 \setminus (A_m^1 \cup A_m^2)$  which, obviously, can be represented as  $G_1 \cup G_2$ , where  $G_1$  is open and  $G_2$  consists of a finite number of smooth closed lines. It is clear that there is  $\delta \in (0, 1/k)$  such that if we divide  $I^2$  into equal intervals  $\{J_j\}$  with diameters less than  $\delta$ , then

$$\left| A_m \cap \bigcup_{J_j \subset A_m} J_j \right| \geq \left(1 - \frac{\alpha_k}{4}\right) |A_m|.$$

$\overline{B_2} \notin LR(\overline{B})$ , since there is  $I \in \overline{B_2}$  such that:

- 1) there is no  $J \in \overline{B}$ ,  $J \supset I$  for which  $|J| \leq 2^k h_k |I|$ ;
- 2)  $\text{diam } Q(I, 2^k h_k) < \delta$ ;
- 3)  $|\text{pr}_1 I_{m,1}|_1 \geq 2^k h_k |\text{pr}_1 I|_1$  and  $|\text{pr}_2 I_{m,1}|_1 \geq 2^k h_k |\text{pr}_2 I|_1$ .

Divide  $I^2$  into intervals equal to  $Q(I, 2^k h_k)$ . Denote the intervals included in  $A_m$  by  $Q_{m+1,q}$  ( $1 \leq q \leq q_{m+1}$ ). Take  $I_{m+1,q}$  ( $1 \leq q \leq q_{m+1}$ ) equal to  $T_q(I)$ , where  $T_q$  is a shift translating  $Q(I, 2^k h_k)$  to  $Q_{m+1,q}$ .

By our construction we obtain

$$\left| A_m \cap \bigcup_{q=1}^{q_{m+1}} Q_{m+1,q} \right| \geq \left(1 - \frac{\alpha_k}{4}\right) |A_m|. \quad (11)$$

By Lemma 1,- for  $q = \overline{1, q_{m+1}}$

$$\{M_{B_2}(h_k \chi_{I_{m+1,q}}) > k\} \subset \left(2 \frac{h_k}{k} - 1\right) I_{m+1,q} \subset Q_{m+1,q}, \quad (12)$$

$$|\{M_{B_2}(h_k \chi_{I_{m+1,q}}) > k\}| > \frac{h_k}{k} \ln \frac{h_k}{k} |I_{m+1,q}|. \quad (13)$$

Since  $\text{diam } Q_{m+1,q} < 1/k$  ( $q = \overline{1, q_{m+1}}$ ), because of (12) we have

$$\{M_{B_2}(h_k \chi_{I_{m+1,q}}) > k\} = \{M_{B_2}^{(1/k)}(h_k \chi_{I_{m+1,q}}) > k\} \quad (q = \overline{1, q_{m+1}}). \quad (14)$$



It can be easily seen that (see (13), (14))

$$|\{M_{B_2}^{(1/k)}(h_k \chi_{I_{m+1,q}}) > k\}| \geq \alpha_k |Q_{m+1,q}| \quad (q = \overline{1, q_{m+1}}),$$

which by (11) readily implies

$$\begin{aligned} \left| \bigcup_{q=1}^{q_{m+1}} \{M_{B_2}^{(1/k)}(h_k \chi_{I_{m+1,q}}) > k\} \right| &\geq \alpha_k \left| \bigcup_{q=1}^{q_{m+1}} Q_{m+1,q} \right| \geq \\ &\geq \alpha_k \left(1 - \frac{\alpha_k}{4}\right) |A_m| > \frac{\alpha_k}{2} |A_m|. \end{aligned} \quad (15)$$

According to the construction and Lemma 2 we have for  $q = \overline{1, q_{m+1}}$

$$\{M_B(h_k \chi_{I_{m+1,q}}) > 1/2^k\} \subset R_{m+1,q} \quad (R_{m+1,q} = R(I_{m+1,q}, 2^k h_k)).$$

On account of our choice of  $h_k$ , because of (13) and (14) we write

$$|\{M_{B_2}^{(1/k)}(h_k \chi_{I_{m+1,q}}) > k\}| > 2^k |R_{m+1,q}| \quad (q = \overline{1, q_{m+1}}),$$

whence

$$\left| \bigcup_{q=1}^{q_{m+1}} \{M_{B_2}^{(1/k)}(h_k \chi_{I_{m+1,q}}) > k\} \right| > 2^k \left| \bigcup_{q=1}^{q_{m+1}} R_{m+1,q} \right|. \quad (16)$$

Let us show that for sufficiently large  $mk$  the construction ceases, i.e., we shall have the inequality

$$\left| \bigcup_{m=1}^{m_k} \bigcup_{q=1}^{q_m} \{M_{B_2}^{(1/k)}(h_k \chi_{I_{m+1,q}}) > k\} \right| > 1 - \frac{1}{k}. \quad (17)$$

Assume the contrary, i.e.,  $|A_m^1| \leq 1 - \frac{1}{k}$  ( $m \in \mathbb{N}$ ). Introduce the notation

$$\begin{aligned} A_m^3 &= \bigcup_{q=1}^{q_m} \{M_{B_2}^{(1/k)}(h_k \chi_{I_{m,q}}) > k\}, \\ &\hspace{15em} (m \in \mathbb{N}) \\ A_m^4 &= \bigcup_{q=1}^{q_m} R_{m,q}. \end{aligned}$$

Clearly, (15) and (16) will be valid for all  $m \in \mathbb{N}$ , and therefore we write

$$\begin{aligned} |A_{m+1}^3| &> \frac{\alpha_k}{2} |A_m| = \frac{\alpha_k}{2} \left(1 - \sum_{j=1}^m |A_j^3 \cup A_j^4|\right) \geq \\ &\geq \frac{\alpha_k}{2} \left(1 - \sum_{j=1}^m \left(1 + \frac{1}{2^k}\right) |A_j^3|\right) \geq \frac{\alpha_k}{2} \left(1 - \frac{1}{2^k} - \sum_{j=1}^m |A_j^3|\right), \end{aligned}$$

which by the equality  $|A_m^1| = \sum_{j=1}^m |A_m^3|$  implies that  $|A_m^1| > 1 - \frac{1}{2^{k-1}} > 1 - \frac{1}{k}$  for sufficiently large  $m$ . From this contradiction we conclude that (17) is valid.

Consider the function  $f_k = \sum_{m=1}^{m_k} \sum_{q=1}^{q_m} h_k \chi_{I_{m,q}}$ . Clearly,  $f_k \in L(I^2)$  and  $f_k \geq 0$ . From (17) we get

$$|\{M_{B_2}^{(1/k)}(f_k) > k\}| > 1 - \frac{1}{k}. \quad (18)$$

From the construction we can easily see that  $2^k h_k$  and the families  $\{I_{1,q}\}_{q=1}^{q_1}, \dots, \{I_{m_k,q}\}_{q=1}^{q_{m_k}}$  satisfy all the conditions of Lemma 6. Therefore by Lemma 6 we write

$$\begin{aligned} & \{M_B(f_k) > 1/2^{k-1}\} = \{M_B(2^k f_k) > 2\} = \\ & = \left\{ M_B \left( \sum_{m=1}^{m_k} \sum_{q=1}^{q_m} 2^k h_k \chi_{I_{m,q}} \right) > 2 \right\} \subset \bigcup_{m=1}^{m_k} \bigcup_{q=1}^{q_m} R_{m,q}. \end{aligned} \quad (19)$$

Obviously, (16) is fulfilled for all  $m \in [1, m_k - 1]$ , so that taking the construction into account, we have

$$2^k \left| \bigcup_{m=1}^{m_k} \bigcup_{q=1}^{q_m} R_{m,q} \right| < \left| \bigcup_{m=1}^{m_k} \bigcup_{q=1}^{q_m} \{M_{B_2}^{(1/k)}(h_k \chi_{I_{m,q}}) > k\} \right| \leq 1. \quad (20)$$

Due to (19) and (20) we write

$$\left| \{M_B(f_k) > 1/2^{k-1}\} \right| < \frac{1}{2^k}. \quad (21)$$

It can be easily seen that

$$\frac{h_k}{k} \ln \frac{h_k}{k} \left| \bigcup_{m=1}^{m_k} \bigcup_{q=1}^{q_m} I_{m,q} \right| < \left| \bigcup_{m=1}^{m_k} \bigcup_{q=1}^{q_m} \{M_{B_2}^{(1/k)}(h_k \chi_{I_{m,q}}) > k\} \right| \leq 1,$$

Hence, due to our choice of  $h_k$ , we obtain

$$\|f_k\|_1 = h_k \left| \bigcup_{m=1}^{m_k} \bigcup_{q=1}^{q_m} I_{m,q} \right| < 1 / \frac{1}{k} \ln \frac{h_k}{k} < \frac{1}{2^k}. \quad (22)$$

Consider the function  $f = \sum_{k=3}^{\infty} f_k$ . Clearly,  $f \in L(I^2)$  (see (22)) and  $f \geq 0$ .

It is obvious that if  $x \in \overline{\lim_{k \rightarrow \infty} \{M_{B_2}^{(1/k)}(f_k) > k\}}$ , then  $\overline{D}_{B_2} \left( f \sum_{k=3}^{\infty} f_k, x \right) = \infty$ . Because of (18)  $\overline{\lim_{k \rightarrow \infty} \{M_{B_2}^{(1/k)}(f_k) > k\}}$  has a complete measure on  $I^2$ . Therefore the latter equality holds almost everywhere on  $I^2$ .

It is clear that  $B$  differentiates  $\int f_k$  ( $k \in \mathbb{N}$ ). Moreover, owing to (21) we get  $\sum_{k=3}^{\infty} |\{M_B(f_k) > 1/2^{k-1}\}| < \sum_{k=3}^{\infty} \frac{1}{2^k} < \infty$ , which by Lemma 7 implies that  $B$  differentiates  $\int f$ .  $\square$

After making some technical changes in the proof of Theorem 2 we can obtain the following generalization.

**Theorem 3.** *Let  $B \subset B_2$  be a TI-basis. If  $\overline{B}_2$  is not locally regular with respect to  $\overline{B}$  then for any function  $f \in L \setminus L \ln^+ L(I^2)$ ,  $f \geq 0$ , there is a Lebesgue measure-preserving and invertible mapping  $\omega : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ,  $\{x : \omega(x) \neq x\} \subset I^2$  such that*

$$\begin{aligned} \overline{D}_{B_2} \left( \int f \circ \omega, x \right) &= \infty \text{ a.e. on } I^2, \\ D_B \left( \int f \circ \omega, x \right) &= (f \circ \omega)(x) \text{ a.e. on } I^2. \end{aligned}$$

To prove the implication (c)  $\Rightarrow$  (b) let us show the validity of

**Lemma 8.** *Let  $B \subset B_2$  be a TI-basis. If  $\overline{B}_2$  is locally regular with respect to  $\overline{B}$ , then for every  $f \in L(\mathbb{R}^2)$*

$$|\{M_{B_2}^{(r)}(f) > \lambda\}| \leq 25 \left| \left\{ M_B^{(cr)}(f) > \frac{\lambda}{4c} \right\} \right| \quad (\lambda > 0, 0 < r < \delta),$$

where  $\delta$  and  $c$  are the constants from the definition of local regularity of  $\overline{B}_2$  with respect to  $\overline{B}$ .

*Proof.* Note first that if  $I$  and  $I'$  ( $I \subset I'$ ) are the one-dimensional intervals and  $I^*$  is either the left or the right half of the interval  $I$ , then for the point  $y \in I'$  the set  $E = \{z \in \mathbb{R} : (I' + z - y) \supset I^*\}$  possesses the following properties:

- 1)  $E$  is an interval;
- 2)  $E \cap I' \neq \emptyset$ ;
- 3)  $|E|_1 = |I'|_1 - \frac{1}{2}|I|_1 \geq \frac{1}{2}|I'|_1$ .

It follows from these properties that

- 4)  $5E \supset I'$

is likewise valid.

Let  $f \in L(\mathbb{R}^2)$ . Denote by  $P$  the set of all two-dimensional intervals  $E \in \overline{B}_2$  such that  $E \subset \left\{ M_B^{(cr)}(f) > \frac{\lambda}{4c} \right\}$  and show the inclusion

$$\{M_{B_2}^{(r)}(f) > \lambda\} \subset \bigcup_{E \in P} 5E \quad (\lambda > 0, 0 < r < \delta). \quad (23)$$

Let  $x \in \{M_{B_2}^{(r)}(f) > \lambda\}$  ( $\lambda > 0$ ,  $0 < r < \delta$ ). Then there is  $I \in B_2(x)$  such that  $\text{diam } I < r$  and

$$\frac{1}{|I|} \int_I |f| > \lambda. \quad (24)$$

Since  $\overline{B_2} \in LR(\overline{B})$ , there is an interval  $I'$  from  $\overline{B}$  such that  $I \subset I'$  and  $|I'| \leq c|I|$ . Then it is obvious that  $\text{diam } I' \leq c \text{diam } I < cr$ . Let  $y \in I'$  be the point for which  $I' \in B(y)$ . Divide the interval  $I$  into four equal intervals. Because of (24) the integral mean will be greater than  $\lambda$  at least on one of these intervals. Denote this interval by  $I^*$ . Thus

$$\frac{1}{|I^*|} \int_{I^*} |f| > \lambda. \quad (25)$$

Consider the set  $E = \{z \in \mathbb{R}^2 : (I' + z - y) \supset I^*\}$ . By virtue of the above discussion we can easily note that

- 1)  $E \in \overline{B_2}$ ;
- 2)  $E \cap I' \neq \emptyset$ ;
- 3)  $|E| \geq \frac{1}{4}|I'|$ ;
- 4)  $5E \supset I'$ .

Since  $x \in I \subset I'$ , then to prove (23) it suffices to show that  $E \in P$ . Let  $z \in E$ . Then  $(I' + z - y) \supset I^*$ . Since  $B$  is the  $TI$ -basis and  $I' \in B(y)$ , we have  $(I' + z - y) \in B(z)$ . Moreover, owing to (25), we have

$$\frac{1}{|I' + z - y|} \int_{I' + z - y} |f| \geq \frac{1}{|I'|} \int_{I^*} |f| \geq \frac{1}{c|I|} \int_{I^*} |f| = \frac{1}{c4|I^*|} \int_{I^*} |f| > \frac{\lambda}{4c},$$

from which by the inequality  $\text{diam}(I' + z - y) < cr$ , we conclude that  $z \in \left\{M_B^{(cr)}(f) > \frac{\lambda}{4c}\right\}$ . Hence  $E \in P$ . Thus inclusion (23) is proved.

Now, using (23) and Lemma 1 from [2], we will have

$$\left| \{M_{B_2}^{(r)}(f) > \lambda\} \right| \leq \left| \bigcup_{E \in P} 5E \right| \leq 25 \left| \bigcup_{E \in P} E \right| \leq 25 \left| \left\{ M_B^{(cr)}(f) > \frac{\lambda}{4c} \right\} \right|$$

for  $\lambda > 0$ ,  $0 < r < \delta$ .  $\square$

The basis  $B$  is said to be regular if there is a constant  $c < \infty$  such that for every  $R \in \overline{B}$  there exists a cubic interval  $Q$  with the following properties:  $Q \supset R$ ,  $|Q| \leq c|R|$ .

According to the well-known Lebesgue theorem on differentiability of integrals, the regular basis  $B$  differentiates  $L$ , i.e.,  $D_B(\int f, x) = f(x)$  a.e. on  $\mathbb{R}^n$  for every  $f \in L(\mathbb{R}^n)$ .

We shall say that the basis  $B$  possesses property  $(E)$  if for every  $f \in L(\mathbb{R}^n)$ ,  $\underline{D}_B(\int f, x) \leq f(x) \leq \overline{D}_B(\int f, x)$  a.e. on  $\mathbb{R}^n$ .

Obviously, the basis  $B$ , containing a regular subbasis, possesses the property (E).

It directly follows from the relation  $\overline{B}_2 \in LR(\overline{B})$  that  $B$  contains a regular subbasis, and hence possesses property (E). Now, to prove the implication (b)  $\Rightarrow$  (c), it suffices to consider the following assertion.

**Lemma 9.** *Let  $B$  be a subbasis of  $B_2$  with property (E). Suppose that there exist positive constants  $c_1, c_2, c_3$  and  $\delta$  such that for every  $f \in L(\mathbb{R}^2)$ ,*

$$|\{M_{B_2}^{(r)}(f) > \lambda\}| \leq c_1 \left| \left\{ M_B^{(c_2 r)}(f) > \frac{\lambda}{c_3} \right\} \right| \quad (\lambda > 0, 0 < r < \delta).$$

Then for every  $f \in L(\mathbb{R}^2)$ ,  $f \geq 0$ ,

$$\underline{D}_B \left( \int f, x \right) = \underline{D}_{B_2} \left( \int f, x \right) \quad \text{and} \quad \overline{D}_B \left( \int f, x \right) = \overline{D}_{B_2} \left( \int f, x \right),$$

a.e. on  $\mathbb{R}^2$ .

The proof of Lemma 9 is based on the well-known Besicovitch theorem on possible values of upper and lower derivative numbers (see [1], Ch.IV, §3) and is carried out analogously to the proof of Theorem 1 from [2].

### 3. ON A POSSIBILITY OF EXTENDING THEOREM 1 TO MORE GENERAL BASES

It should be noted that beyond the scope of  $TI$ -bases Theorem 1 becomes invalid. Moreover, even for bases which are similar enough to  $TI$ -bases the local regularity of  $\overline{B}_2$  with respect to  $\overline{B}$  is insufficient for the equality  $F_B^+ = F_{B_2}^+$  to be fulfilled.

A basis  $B$  will be called a  $TI^*$ -basis if for every  $x \in \mathbb{R}^n$ ,  $R \in B(x)$  and  $y \in \mathbb{R}^n$  there is a translation  $T$  such that  $T(R) \in B(y)$ .

The following theorem is valid.

**Theorem 4.** *There is a  $TI^*$ -basis  $B \subset B_2$  with the properties:*

- 1)  $\overline{B}_2$  is locally regular with respect to  $\overline{B}$ , and  $B(O) = B_2(O)$ ;
- 2) there is a function  $f \in L(I^2)$ ,  $f \geq 0$  such that

$$\overline{D}_{B_2} \left( \int f, x \right) = \infty \quad \text{a.e. on } I^2,$$

$$D_B \left( \int f, x \right) = f(x) \quad \text{a.e. on } I^2.$$

*Proof.* Consider the sequences  $\alpha_k \uparrow \infty$ ,  $\alpha_k > 0$  and  $h_k \uparrow \infty$ ,  $h_k > 10$  with the properties

$$\sum_{k=1}^{\infty} \frac{\alpha_k}{h_k} < \infty, \quad \sum_{k=1}^{\infty} \frac{\ln h_k}{h_k} = \infty. \quad (26)$$

Let  $q_k = 2^{m_k}$ , where  $m_k \in \mathbb{N}$  ( $k \in \mathbb{N}$ ) and  $m_k \uparrow \infty$ . For every  $k$ , we divide  $I^2$  into  $q_k^2$  equal square interval and denote them by  $I_{k,q}^2$ . The length of square intervals sides will be denoted by  $\Delta_k$ . For every  $I_{k,q}^2$  let us consider the square interval  $I_{k,q}$  concentric with  $I_{k,q}^2$  having the side length  $\delta_k = \Delta_k/(2h_k + 1)$ . We shall assume  $q_k \uparrow \infty$  so that  $\delta_k > \Delta_{k+1}$  ( $k \in \mathbb{N}$ ).

Let  $f_k = \sup\{\alpha_k h_k \chi_{I_{k,q}} : q = \overline{1, q_k^2}\}$  ( $k \in \mathbb{N}$ ) and  $f = \sum_{k=1}^{\infty} f_k$ . Clearly,  $f \geq 0$ . It can be easily checked that

$$\|f_k\|_1 = \alpha_k h_k \left| \bigcup_{q=1}^{q_k^2} I_{k,q} \right| = \alpha_k h_k \sum_{q=1}^{q_k^2} \frac{1}{(2h_k + 1)^2} |I_{k,q}^2| < \frac{\alpha_k}{h_k},$$

which according to (26) implies

$$\|f\|_1 \leq \sum_{k=1}^{\infty} \|f_k\|_1 < \sum_{k=1}^{\infty} \frac{\alpha_k}{h_k} < \infty,$$

i.e.,  $f \in L(I^2)$ .

By Lemma 1 for  $k \in \mathbb{N}$ ,  $q = \overline{1, q_k^2}$  we have

$$\{M_{B_2}(\alpha_k h_k \chi_{I_{k,q}}) > \alpha_k\} \subset (2h_k + 1)I_{k,q} = I_{k,q}^2, \quad (27)$$

and

$$|\{M_{B_2}(\alpha_k h_k \chi_{I_{k,q}}) > \alpha_k\}| \geq h_k \ln h_k |I_{k,q}|.$$

Thus we write

$$\begin{aligned} \left| \bigcup_{q=1}^{q_k^2} \{M_{B_2}(\alpha_k h_k \chi_{I_{k,q}}) > \alpha_k\} \right| &= \sum_{q=1}^{q_k^2} |\{M_{B_2}(\alpha_k h_k \chi_{I_{k,q}}) > \alpha_k\}| \geq \\ &\geq \sum_{q=1}^{q_k^2} h_k \ln h_k |I_{k,q}| = \sum_{q=1}^{q_k^2} h_k \ln h_k \frac{|I_{k,q}^2|}{(2h_k + 1)^2} > \frac{\ln h_k}{9h_k} \end{aligned} \quad (28)$$

for  $k \in \mathbb{N}$ .

For every  $k \in \mathbb{N}$  let us consider those intervals  $I_{k+1,q}^2$  ( $q = \overline{1, q_{k+1}^2}$ ) which are contained in  $\bigcup_{q=1}^{q_k^2} \{M_{B_2}(\alpha_k h_k \chi_{I_{k,q}}) > \alpha_k\}$ . Denote their union by  $O_k$ . It can be easily seen that if  $\{q_k\}$  tends rapidly enough to  $\infty$ , then

$$|O_k| \geq \frac{1}{2} \left| \bigcup_{q=1}^{q_k^2} \{M_{B_2}(\alpha_k h_k \chi_{I_{k,q}}) > \alpha_k\} \right| \quad (k \in \mathbb{N}). \quad (29)$$

Using the property of dyadic intervals, we can see that  $\{O_k\}$  is the sequence of independent sets, i.e., for every  $n \geq 2$  and for arbitrary pairwise different natural numbers  $k_1, k_2, \dots, k_n$ ,

$$\left| \bigcap_{m=1}^n O_{k_m} \right| = \prod_{m=1}^n |O_{k_m}|.$$

Owing to (26), (28), and (29),  $\sum_{k=1}^{\infty} |O_k| > \sum_{k=1}^{\infty} \frac{1}{2} \frac{\ln h_k}{9h_k} = \infty$ . Using now Borel–Kantelly’s lemma, we get

$$\overline{\lim}_{k \rightarrow \infty} O_k, \text{ has a complete measure in } I^2. \quad (30)$$

From (27) we have for  $k \in \mathbb{N}$ ,  $q = \overline{1, q_k^2}$

$$\{M_{B_2}(\alpha_k h_k \chi_{I_{k,q}}) > \alpha_k\} = \{M_{B_2}^{(\Delta_k)}(\alpha_k h_k \chi_{I_{k,q}}) > \alpha_k\},$$

which for  $k \in \mathbb{N}$  implies

$$\begin{aligned} O_k &\subset \bigcup_{q=1}^{q_k^2} \{M_{B_2}(\alpha_k h_k \chi_{I_{k,q}}) > \alpha_k\} = \\ &= \bigcup_{q=1}^{q_k^2} \{M_{B_2}^{(\Delta_k)}(\alpha_k h_k \chi_{I_{k,q}}) > \alpha_k\} \subset \{M_{B_2}^{(\Delta_k)}(f_k) > \alpha_k\}. \end{aligned}$$

Because of (30), from the latter inclusion we obtain

$$\overline{\lim}_{k \rightarrow \infty} \{M_{B_2}^{(\Delta_k)}(f_k) > \alpha_k\} \text{ has a complete measure in } I^2. \quad (31)$$

Since  $\alpha_k \uparrow \infty$ ,  $\Delta_k \downarrow 0$  ( $k \rightarrow \infty$ ), we can easily see that if  $x \in \overline{\lim}_{k \rightarrow \infty} \{M_{B_2}^{(\Delta_k)}(f_k) > \alpha_k\}$ , then  $\overline{D}_{B_2}(f, x) = \infty$ . By virtue of (31) we conclude that the latter equality holds a.e. on  $I^2$ .

Assume  $k$  to be fixed. For every  $I_{k,q}$  ( $q = \overline{1, q_k^2}$ ) let us consider the intervals  $J_{k,q}^1$  ( $\text{pr}_1 J_{k,q}^1 = 3 \text{pr}_1 I_{k,q}$ ,  $\text{pr}_2 J_{k,q}^1 = (0, 1)$ ) and  $J_{k,q}^2$  ( $\text{pr}_1 J_{k,q}^2 = (0, 1)$ ,  $\text{pr}_2 J_{k,q}^2 = 3 \text{pr}_2 I_{k,q}$ ). Denote  $G_k^i = \bigcup_{q=1}^{q_k^2} J_{k,q}^i$  ( $i = \overline{1, 2}$ ),  $G_k = G_k^1 \cup G_k^2$ .

It can be easily verified (see (26)) that  $\sum_{k=1}^{\infty} |G_k| < \infty$  which implies that  $\overline{\lim}_{k \rightarrow \infty} G_k$  has zero measure.

Determine now an unknown basis  $B$ . If  $x \in \left( \overline{\lim}_{k \rightarrow \infty} G_k \right) \cup \mathbb{Q}^2 \cup (\mathbb{R}^2 \setminus I^2)$  ( $\mathbb{Q}$  is a set of rational numbers), then we put  $B(x) = B_2(x)$ . For  $x \in I^2 \setminus \left( \overline{\lim}_{k \rightarrow \infty} G_k \cup \mathbb{Q}^2 \right)$  choose  $B(x)$  such that if  $d_I$  is the length of the lesser

side of the interval  $I \ni x$ , and  $d_I \notin \bigcup_{k \geq k(x)} [\delta_k, \Delta_k/2]$  (for  $x \notin \overline{\lim}_{k \rightarrow \infty} G_k$ ,  $k(x)$  is the number with the property  $x \notin G_k$  for  $k \geq k(x)$ ), then  $I \in B(x)$ , while if  $d_I \in [\delta_k, \Delta_k/2]$  for some  $k \geq k(x)$ , then  $I \in B(x)$  iff either  $I \cap G_k^1 = \emptyset$  or  $I \cap G_k^2 = \emptyset$ .

It can be easily verified that for every  $I \in \overline{B}_2$  and  $x \in \mathbb{R}^2$  there is a translation  $T$ ,  $T(I) \in B(x)$ ; this implies that  $B$  is the  $TI^*$ -basis. The local regularity of  $\overline{B}_2$  with respect to  $\overline{B}$  is also easily checked. By proving the inclusion  $f \in F_B^+$  the proof of the theorem will be completed.

Denote  $A_k = \{x \in \mathbb{R}^n : D_B(\int f_k, x) = f_k(x)\}$  ( $k \in \mathbb{N}$ ). Clearly,  $A_k$  ( $k \in \mathbb{N}$ ) has a complete measure on  $\mathbb{R}^2$ . Denote  $A = \left(I^2 \cap \bigcap_{k=1}^{\infty} A_k\right) \setminus \left(\overline{\lim}_{k \rightarrow \infty} G_k \cup \mathbb{Q}^2\right)$ . Since  $f(x) = 0$  for  $x \notin I^2$ ,  $B$  differentiates  $\int f$  on  $\mathbb{R}^2 \setminus I^2$ ;  $A$  is the set of a complete measure on  $I^2$ . Therefore to prove the inclusion  $f \in F_B^+$ , it suffices to show the differentiability of  $\int f$  with respect to  $B$  on the set  $A$ .

Let  $x \in A$  and  $\varepsilon > 0$ . Find  $\delta > 0$  for which  $\left|(1/|I|) \int_I f - f(x)\right| < \varepsilon$  when  $I \in B(x)$ ,  $\text{diam } I < \delta$ . Consider  $k'(x) \geq k(x)$  for which

$$\sum_{k \geq k'(x)} 16 \|f_k\|_1 < \varepsilon/2. \quad (32)$$

It is clear that  $\sum_{k=1}^{\infty} f_k(x) = \sum_{k=1}^{k'(x)-1} f_k(x)$ , and  $B$  differentiates  $\int \sum_{k=1}^{k'(x)-1} f_k$  at the point  $x$ . This implies that there is  $\delta \in (0, \Delta_{k'(x)}/2)$  such that

$$\left| \frac{1}{|I|} \int_I \sum_{k=1}^{k'(x)-1} f_k - \sum_{k=1}^{k'(x)-1} f_k(x) \right| < \varepsilon/2, \text{ for } I \in B(x), \text{ diam } I < \delta. \quad (33)$$

Consider an arbitrary interval  $I \in B(x)$ ,  $\text{diam } I < \delta$ . For  $k_I \in \mathbb{N}$  let  $\Delta_{k_I+1}/2 \leq d_I \leq \Delta_{k_I}/2$ . Clearly,  $k_I \geq k'(x)$ . By virtue of the inequality  $\Delta_{k+1} < \delta_k$  ( $k \in \mathbb{N}$ ) and the condition  $x \notin G_k$  ( $k \geq k(x)$ ), we have  $I \cap \text{supp } f_k = \emptyset$  for  $k'(x) \leq k \leq k_I - 1$ , which gives

$$\int_I f_k = 0 \text{ for } k'(x) \leq k \leq k_I - 1. \quad (34)$$

Analogously,  $\int_I f_{k_I} = 0$  if  $d_I < \delta_{k_I}$ , while if  $\delta_{k_I} \leq d_I \leq \Delta_{k_I}/2$ , then  $\int_I f_{k_I} = 0$  because  $I \cap \text{supp } f_k = \emptyset$  by construction of the basis  $B$ . Thus

$$\int_I f_{k_I} = 0. \quad (35)$$

Let is show that

$$|I \cap \text{supp } f_k| \leq 16|I| |\text{supp } f_k| \text{ for } k \geq k_I + 1. \quad (36)$$



To this end it is sufficient to consider the following easily verifiable facts:

- 1)  $d_I \geq \Delta_k/2$  for  $k \geq k_I + 1$ ;
- 2) let  $P_k$  be a set of vertices of the squares  $I_{k,q}^2$  ( $q = \overline{1, q_k^2}$ ). If the vertices of the interval  $J \subset I^2$  belong to the set  $P_k$ , then  $|J \cap \text{supp } f_k| = |I| |\text{supp } f_k|$ ;
- 3) for every interval  $J \subset I^2$ ,  $d_J \geq \Delta_k/2$  there is an interval  $J' \supset J$ ,  $|J'| \leq 16|J|$  with vertices at the points of the set  $P_k$ .

By (36) we can write that for  $k \geq k_I + 1$ ,

$$\int_I f_k = \alpha_k h_k |I \cap \text{supp } f_k| \leq 16\alpha_k h_k |I| |\text{supp } f_k| = 16|I| \|f_k\|_1,$$

which, owing to (32) and the inequality  $k_I \geq k'(x)$ , implies

$$\frac{1}{|I|} \int_I \sum_{k \geq k_I+1} f_k = \frac{1}{|I|} \sum_{k \geq k_I+1} \int_I f_k \leq \frac{1}{|I|} \sum_{k \geq k_I+1} 16|I| \|f_k\|_1 < \varepsilon/2. \quad (37)$$

By virtue of relations (33)–(35) and (37) (it should also be noted that  $f_k(x) = 0$  for  $k \geq k'(x)$ ), we have

$$\begin{aligned} & \left| \frac{1}{|I|} \int_I \sum_{k=1}^{\infty} f_k - \sum_{k=1}^{\infty} f_k(x) \right| \leq \left| \frac{1}{|I|} \int_I \sum_{k=1}^{k'(x)-1} f_k - \sum_{k=1}^{k'(x)-1} f_k(x) \right| + \\ & + \frac{1}{|I|} \int_I \sum_{k=k'(x)}^{k_I} f_k + \frac{1}{|I|} \int_I \sum_{k \geq k_I+1} f_k + \sum_{k \geq k'(x)} f_k(x) < \frac{\varepsilon}{2} + 0 + \frac{\varepsilon}{2} + 0 = \varepsilon. \end{aligned}$$

Due to the arbitrariness of  $x \in A$  and  $\varepsilon > 0$ , we conclude that  $D_B \left( \int f, x \right) = f(x)$  for every  $x \in A$ .  $\square$

#### 4. REMARKS

(1) Let us consider one application of the obtained results. Let  $B_Z$  be the basis in  $\mathbb{R}^2$  for which  $B_Z(x)$  ( $x \in \mathbb{R}^2$ ) consists of all intervals  $I \ni x$ ,  $D_I^2 \leq d_I \leq D_I \leq 1$ , where  $D_I$  and  $d_I$  are the lengths, respectively, of the greater and of the lesser side of  $I$ . This basis was introduced by Zygmund and it was he who initiated (see [1], Ch. VI, §4) the study of the differential properties of  $B_Z$ .

Morion showed (see [1], Appendix IV) that for the integral classes  $B_Z$  behaves like  $B_2$ , i.e.,  $B_Z$  does not differentiate a wider integral class than  $L \ln^+ L$ .

A question arises: does there exist in general a function whose integral does not differentiate  $B_2$  and differentiates  $B_Z$ ? From the above proven theorems the answer is positive.

$B_Z$  is the  $TI$ -basis. The fact that  $\overline{B}_2 \notin LR(\overline{B}_Z)$  is easily verified. By Theorem 1 this implies that the strict inclusion  $F_{B_2}^+ \subset F_{B_Z}^+$  holds. Moreover,

Theorem 3 implies that by perturbing the values  $\omega$  of any function  $f \in L \setminus L \ln^+ L(I^2)$ ,  $f \geq 0$ , one can get the function  $f \circ \omega$  with the following properties:  $\overline{D}_{B_2}(\int f \circ \omega, x) = \infty$  a.e. on  $I^2$ , and  $B_Z$  differentiates  $\int f \circ \omega$ .

Thus, for the integral classes the bases  $B_2$  and  $B_Z$  behave similarly, while for individual nonnegative functions the basis  $B_Z$  behaves better than the basis  $B_2$ .

(2) The bases  $B$  and  $B'$  are said to be positive equivalent ( $B \stackrel{\pm}{\Leftrightarrow} B'$ ) if for every  $f \in L(\mathbb{R}^n)$ ,  $f \geq 0$ ,  $\overline{D}_B(\int f, x) = \overline{D}_{B'}(\int f, x)$  and  $\underline{D}_B(\int f, x) = \underline{D}_{B'}(\int f, x)$  a.e. on  $\mathbb{R}^n$  (i.e., condition (b) in Theorem 1 means that the bases  $B$  and  $B'$  are positive equivalent).

$B$  is said to be a Busemann–Feller type basis ( $BF$ -basis) if for any  $R \in \overline{B}$  and  $x \in R$  we have  $R \in B(x)$ .

We shall say that  $B$  exactly differentiates  $\varphi(L)$  (writing  $B \in D(\varphi(L))$ ) if  $B$  differentiates  $\varphi(L)$  and does not differentiate a wider integral class than  $\varphi(L)$ .

For integral classes the behavior of  $B \subset B_2$ ,  $BF$ ,  $TI$ -bases was studied by Stokoloc in [3], where he introduced the property ( $S$ ) and proved that if  $B$  possesses the property ( $S$ ), then  $B \in D(L \ln^+ L)$ , and if  $B$  does not possess this property, then  $B \in D(L)$ .

One can easily see that ignoring the  $BF$  property, this result remains valid. Thus, for the integral classes  $B \subset B_2$ , the  $TI$ -basis behaves like  $B_2$  or  $B_1$  ( $B_1$  is the basis formed of square intervals). The  $B_Z$ -basis illustrates that an analogous fact does not hold for nonnegative individual functions and, generally speaking, if we combine Stokoloc's assertion and Theorem 1, then we shall have:

- 1) if  $\overline{B}_2 \in LR(\overline{B})$ , then  $B \stackrel{\pm}{\Leftrightarrow} B_2$ ;
- 2) if  $\overline{B}_2 \notin LR(\overline{B})$  and  $B$  possesses the property ( $S$ ), then  $B \in D(L \ln^+ L)$  and  $F_{B_2}^+ \subset F_B^+$  (strictly);
- 3) if  $B$  does not possess the property ( $S$ ), then  $B \in D(L)$ .

(3) Let  $B \subset B_2$  be a  $TI$ -basis. Consider the intervals  $I \in \overline{B}$  of the type  $I = (0, x^1) \times (0, x^2)$ . Denote the set of points  $(x^1, x^2)$  by  $A_B$ . The set  $A_B$  indicates how rich the family  $\overline{B}$  is.

One can easily prove the following criterion of local regularity of  $\overline{B}_2$  with respect to  $\overline{B}$ :  $(\overline{B}_2 \in LR(\overline{B})) \Leftrightarrow (\exists m, k_0 \in \mathbb{N} : A_B \cap ([1/m^k, 1/m^{k-1}) \times [1/m^{k'}, 1/m^{k'-1})) \neq \emptyset$  for  $k, k' > k_0$ ).

(4) The basis  $B$  is said to be invariant ( $HI$ -basis) with respect to homotheties if for every  $x \in \mathbb{R}^n$  and every homothety  $H$  centered at  $x$  we have  $B(x) = \{H(R) : R \in B(x)\}$ .

If  $\overline{B}$  is locally regular with respect to  $\overline{B}'$  and, moreover, if  $\delta$  is equal to  $\infty$ , then  $\overline{B}$  is called regular with respect to  $\overline{B}'$ .

For the basis  $B \subset B_2$  define the sets:  $R_{1,B} = \{r > 1 : \exists I \in \overline{B}, |\text{pr}_1 I|_1 = r |\text{pr}_2 I|_1\}$ ,  $R_{2,B} = \{r > 1 : \exists I \in \overline{B}, |\text{pr}_2 I|_1 = r |\text{pr}_1 I|_1\}$ .

For the basis  $B \subset B_2$  which is simultaneously the  $TI$ - and  $HI$  basis one can easily verify that  $(\overline{B}_2 \in LR(\overline{B})) \Leftrightarrow (\overline{B}_2 \text{ is regular with respect to } B) \Leftrightarrow (\exists m \in \mathbb{N} : R_{i,B} \cap [m^k, m^{m+1}) \neq \emptyset, k \in \mathbb{N}, i = \overline{1, 2})$ .

(5) Let  $B_2$  ( $B_2 = B_2(\mathbb{R}^n)$ ) be the basis in  $\mathbb{R}^n$  for which  $B_2(x)$  ( $x \in \mathbb{R}^n$ ) consists of all  $n$ -dimensional intervals containing the point  $x$ .

Theorems 1–4 are also valid for  $B \subset B_2(\mathbb{R}^n)$  ( $n \geq 3$ ),  $TI$ -bases. Proofs for the  $n$ -dimensional case are similar to those for the two-dimensional case.

(6) Let  $\delta_k^1 \downarrow 0, \dots, \delta_k^n \downarrow 0$  ( $k \rightarrow \infty$ ). Denote  $\Delta_{k,m}^i = [(m-1)\delta_k^i, m\delta_k^i)$  ( $i = \overline{1, n}; k, m \in \mathbb{N}$ ). Let  $B$  be the basis in  $\mathbb{R}^n$  for which  $B(x) = \{ \prod_{i=1}^n \Delta_{k^i, m^i}^i : \prod_{i=1}^n \Delta_{k^i, m^i}^i \ni x; k^i, m^i \in \mathbb{N} \}$ . Such bases are sometimes called nets. We call them  $N$ -bases.

Let  $B$  be the  $N$ -basis in  $\mathbb{R}^n$  ( $n \geq 2$ ). Denote by  $B_T$  the least  $TI$ -basis containing  $B$ . For  $B$  the following analogue of Theorem 1 is valid: the following conditions are equivalent: (i)  $F_B^+ = F_{B_2}^+$ ; (ii)  $B \overset{\pm}{\Leftrightarrow} B_2$ ; (iii)  $\overline{B}_2$  is locally regular with respect to  $\overline{B}_T$ .

Obviously, (ii) $\Rightarrow$ (i), the implication (i) $\Rightarrow$ (iii) follows directly from Theorem 1. Thus it remains only to show that (iii) $\Rightarrow$ (i). To this end, note that using Lemma 1 of [2] one can easily obtain the inequality

$$|\{M_{B_T}^{(r)}(f) > \lambda\}| \leq 3^n \left| \left\{ M_B^{(r)}(f) > \frac{\lambda}{2^n} \right\} \right| \quad (f \in L(\mathbb{R}^n); \lambda, r > 0).$$

From this and Lemma 5 we get the upper bound of the distribution function of  $M_{B_2}^{(r)}(f)$  by means of the distribution function of  $M_B^{(cr)}(f)$ . It easily follows from the relation  $\overline{B}_2 \in LR(\overline{B})$  that  $B$  contains a regular subbasis. Hence  $B$  possesses the property (E). Next, we obtain the implication (iii) $\Rightarrow$ (ii) from Lemma 6.

Note that for  $\delta_k^i = 1/2^k$  ( $i = \overline{1, n}; k \in \mathbb{N}$ ) the relation  $B \overset{\pm}{\Leftrightarrow} B_2$  was proved earlier in [2].

(7) The basis  $B$  constructed in Theorem 4 is not a  $BF$ -basis, which is not a casual fact. In particular, the following assertion is valid: let  $B \subset B_2$  ( $B_2 = B_2(\mathbb{R}^n)$ ) be a  $BF$ -basis and let  $\overline{B}_2 \in LR(\overline{B})$ . Then  $B \overset{\pm}{\Leftrightarrow} B_2$ .

To prove the above assertion we have to consider the following facts which easily follow from the Busemann-Feller property of the basis  $B$  and from the relation  $\overline{B}_2 \in LR(\overline{B})$ : (i)  $B$  contains a regular subbasis; (ii) for every  $f \in L(\mathbb{R}^n)$ ,

$$|\{M_{B_2}^{(r)}(f) > \lambda\}| \leq \left| \left\{ M_B^{(cr)}(f) > \frac{\lambda}{c} \right\} \right| \quad (\lambda > 0, 0 < r < \delta),$$

where  $c$  and  $\delta$  are the constants from the definition of local regularity of  $\overline{B}_2$  with respect to  $\overline{B}$ .

From (i) we have that  $B$  possesses the property  $(E)$ . Now, taking into account (ii) and using Lemma 6, we can conclude that the relation  $B \stackrel{\pm}{\Leftrightarrow} B_2$  is valid.

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