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HARMONIC MAPS OVER RINGS

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ABSTRACT. For the torsion-free modules over noncommutative principal ideal domains von Staudt's theorem is proved. Moreover, more general (nonbijective) harmonic maps with the classical definition of harmonic quadruple is calculated.

INTRODUCTION

K. von Staudt stated a theorem which clearly shows that it is important to consider the manner in which the blocks are embedded in order to get information on the surrounding geometrical structure. It could be considered as the spring of the geometric algebra.

The modern flavor of the subject was established by E. Artin [1], R. Baer [2], and J. Dieudonné [3]. These classic studies described the theory over division rings. R. Baer and J. von Neumann pointed out a possible extension of the structural identity between (projective) geometry and linear algebra to the case of a ring, generating intense research activity in the area of geometric algebra over rings. The main problem in this field is to translate the specific maps from the geometrical point of view (perspectivities. collineations, harmonic maps) in algebraic language (by the semilinear isomohphisms) – the fundamental theorems of geometric algebra. A continuing investigation by many scholars over the last 30 years has charted the evolution of the classical setting into a stable form for the general rings. NATO ASI held conferences twice on the subject and published two books [4], [5].

The boundaries of the subject "What is geometric algebra?" were established by Artin, Baer, and Dieudonné. These classic studies described the structure theory, actions, transitivity, normal subgroups, commutators and automorphisms of the classical linear groups (general linear, symplectic, orthogonal, unitary) from the geometrical point of view.

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On the other hand this field of problems is central in the isomorphism theory of the classical groups, which gave way to an extensive isomorphism theory of certain full classical groups. The isomorphism $GL_n(K) \to GL_n(K_1)$ in turn gives rise to isomorphisms between the corresponding elementary groups and the (projective) geometric versions of these groups.

What is the fundamental theorem of geometric algebra? For different geometries it can be stated in various ways. However, in general, the problem is to represent specific geometrical maps by the linear functions, i.e., with the elements of $GL(k, X)$, where X is a k-module. In the classical case when $k = F$ is a field or division ring the following approximate versions of the representations are well known:

(P_1) Perspectivities by linear maps + trivial automorphism of F;

(P_2) Collineations by linear maps + automorphism of F;

(P_3) Harmonic maps by linear maps + automorphism or antiautomorphism of F.

Naturally, for different geometries (affine, projective, symplectic, orthogonal, unitary, etc.) all the above-mentioned versions have a specific flavor. The most developed ring version is the projective case. Recall that the projective geometry $PG(k, X)$ of a torsion-free k-module X can be realized as the lattice of all k-free submodules. In this direction the most significant result is Ojanguren and Sridharans' theorem which generalizes to commutative rings the classical theorem of projective geometry [6].

These and some later results give us a reason to suppose that the theorem of type (P_3) is true for some general noncommutative rings.

Let us formulate the theorem (K. von Staudt's theorem) for the classical case, i.e., for vector spaces over skew fields.

Theorem A (Case $\dim_p A = 1$). *Let X and X_1 be vector spaces over the skew fields F and F_1, respectively,* $\dim_p X = \dim_p X_1 = 1$. *Let $f : P(X) \longrightarrow P(X_1)$ be some map. Then the following alternatives are equivalent:*

(a) *f is bijective and harmonic;*

(b) *f is bijective and f, f^{-1} are harmonic;*

(c) *f is a nontrivial harmonic map;*

(d) *f is bijective and harmonic for the fixed quadruple;*

(e) *there exists either an isomorphism or an anti-isomorphism $\sigma : F \longrightarrow F_1$ and a σ-semilinear isomorphism $\mu : X \longrightarrow X_1$ such that $f(Fx) = F_1\mu(x)$ for all $x \in X$.*

The definition of a semilinear isomorphism with respect to the anti-isomorphism will be given later (Definition 5).

Naturally, for general rings the conditions (a)–(d) are not equivalent and we get the following implications:

$$
\begin{tikzcd}
\text{(d)} \arrow[dr, Rightarrow] & \text{(b)} \arrow[l, Rightarrow] \arrow[r, Rightarrow] \arrow[d, equal] & \text{(a)} \arrow[dl, Rightarrow] \\
& \text{(c)} &
\end{tikzcd}
$$

To prove the fundamental theorem means to show the validity of (e) from one of the conditions (a)–(d).

When trying to extend the concepts of (projective) geometry for a given ring, the following question arises: what is the projective space? It can be defined in two ways:

(i) $\widetilde{P}(X)$ as the set of all k-free direct summands of rank 1.

(ii) $P(X)$ as the set of all k-free submodules of rank 1.

It is known that $\widetilde{P}(X)$ does not always give the desired results. However, taking into consideration $P(X)$, we can get positive results for some general noncommutative rings.

The first generalization of von Staudt's theorem belongs to G. Ancochea [7]. In spite of the foregoing theorem one can extend K. von Staudt's theorem to some special commutative rings, in particular, if k is a commutative local or semilocal ring (N. B. Limaye [8], [9]), or if k is a commutative algebra of finite dimension over a field of sufficiently large order (H. Schaeffer [10]), or if k is a commutative primitive ring (B. R. McDonald [11]). Furthermore, B. V. Limaye and N. B. Limaye [12] generalized the theorem to noncommutative local rings by adopting the definition of a harmonic map. However, for commutative principal ideal domains the Staudt's theorem is invalid [13], [14].

W. Klingenberg in 1956 introduced the idea of "non-injective collineations" between projective spaces of two and three dimensions. In a series of papers F. Veldkamp (partly together with J. C. Ferrar) developed the theory of homomorphisms of ring geometry, which are, roughly speaking, noninjective collineations [15], [16], [17], [18]. The first article of non-injective harmonic maps between projective lines was due to F. Buekenhout [19], after D. G. James [20] got the same result. Buekenhout's work described the situation for division rings. In 1985 C. Bartolone and F. Bartolozzi extended some of Buekenhout's ideas for the ring case [14].

Cross-ratio, harmonic quadruple, and von Staudt's theorem in Moufang planes was studied by V. Havel [21], [22] and J. C. Ferrar [23].

Many interesting and fundamental results according, this and boundary problems were obtained by W. Benz and his scholars [10], [24]–[26]. A. Dress and W. Wenzel constitute an important tool of cross-ratios from a combinatorial point of view [27].

Some other generalizations and related problems can be found in [28]–[41]. For more complete information and an exhaustive bibliography in this area see [14], [28], [35].

Our aim is to calculate more general (nonbijective) harmonic maps satisfying the condition (c) with the classical definition of a harmonic quadruple for some general noncommutative rings and to obtain a complete analog of the classical case. Moreover, using some ideas from [37] we'll consider not only free modules but also the torsion-free ones and calculate σ and μ (i.e., semilinear isomorphism) having given f, X, k, $P(X)$, $PG(k, X)$.

The notions and definitions are standard. k is an arbitrary integral domain with unity; all modules are over k; $PG(k, X)$ is the projective geometry of the k-module X, i.e., the lattice of all k-free submodules; $\mathfrak{M}(X)$ is the complete lattice of all submodules X; $\langle Y \rangle$ denotes the submodule generated by the set Y. Note also that to fix the basic ring k sometimes we'll write $P_k(X)$ and $\mathfrak{M}_k(X)$.

1. Projective Space, Collineation and Cross-Ratio

Let k be a commutative ring with unit. For each k-free module X we can construct a new object (see [6], [14]–[18], [35]), the projective space $\widetilde{P}(X)$ corresponding to X. The elements of $\widetilde{P}(X)$ are k-free direct summands of rank 1. It is clear that each element of $\widetilde{P}(X)$ has the form ke, i.e., is a one-dimensional submodule generated by the unimodular element $e \in X$. Remember that an element e is unimodular if there exists a linear form $\mu : X \longrightarrow k$ such that $\mu(e) = 1$, i.e., the coordinates of e in one of the bases X generate the unit ideal of k. If $e_1, e_2, \dots, e_n, \dots$ is a basis of the k-module X, then $e = \sum a_i e_i$ is unimodular if and only if $\sum k a_i = k$. This definition of the projective space we widen in the following way:

Definition 1. Let k be an integral domain (not necessarily commutative). X is a torsion-free module over k. The projective space $P(X)$ corresponding to X is the set of all k-free submodules of rank 1.

Note that Definition 1 is meaningful for every torsion-free module X and it can happen that for some k-module X, $\widetilde{P}(X) = \varnothing$ while $P(X) \neq \varnothing$. It is also obvious that if $U \subset X$ is a submodule, then e is unimodular in U while e is not unimodular in X. For every k-free submodule $U \hookrightarrow X$ the projective dimension $\dim_p$ will be defined as $\dim_p U = \dim U - 1$. We shall use the terms: "point", "line", "plane" for free submodules of the projective dimensions 0,1,2. We shall condider the zero submodule as an "empty element" of the projective space $P(X)$ with projective dimension -1.

Definition 2. The set of points $\{P_\alpha, \alpha \in \Lambda\}$ of the projective space $P(X)$ will be called *collinear*, if there exists a line $U \hookrightarrow X$ such that $P_\alpha \in U$ for every $\alpha \in \Lambda$ and *strictly collinear* if there exists a line U for which $U = P_\alpha + P_\beta$, for every $\alpha, \beta \in \Lambda$.

If the set of points is strictly collinear, then the line U will be called the *principal* line passing through these points.

In the sequel k^* is the group of units of the ring k. If $s \in k$ is an arbitrary element, then by $[s]$ we denote the set of conjugate elements of the form $t^{-1}st$, where $t \in k^*$.

The points $P, Q \in P(X)$ are independent if $P \cap Q = 0$ and dependent if $P \cap Q \neq 0$.

Let $P_1 = ke_1$, $P_2 = ke_2$ be independent. If $U = P_1 + P_2$ and $P_3 = k(\alpha e_1 + \beta e_2) \hookrightarrow U$ is an arbitrary point, then it is obvious that the points P_1, P_2, P_3 are strictly collinear if and only if $\alpha, \beta \in k^*$. It is also obvious that if P_1, P_2, P_3, P_4 are strictly collinear points and U is a principal line passing through these points, then there exist unimodular elements e_1, e_2 of this line U and an invertible element $s \in k^*$ such that

$$P_1 = ke_1, \quad P_2 = ke_2, \quad P_3 = k(e_1 + e_2), \quad P_4 = k(e_1 + se_2).$$

The element $s \in k^*$ is called the *cross-ratio* of these points. If k is commutative, then s is unique. For the non-commutative situation the cross-ratio is $[s]$. For $s' = tst^{-1}$ we have

$$P_1 = k(te_1), \;\; P_2 = k(te_2), \;\; P_3 = k(te_1 + te_2), \;\; P_4 = k(te_1 + s'(te_2)).$$

On the other hand, if

$$P_1 = k\overline{e}_1, \quad P_2 = k\overline{e}_2, \quad P_3 = k(\overline{e}_1 + \overline{e}_2), \quad P_4 = k(\overline{e}_1 + s_1\overline{e}_2),$$

then we have

$$\begin{gathered}
P_1 = k\overline{e}_1, \quad P_2 = k\overline{e}_2, \quad P_3 = k(\overline{e}_1 + \overline{e}_2), \quad P_= k(\overline{e}_1 + s'te_2)), \\
\overline{e}_1 = \mu_1 e_1, \quad \overline{e}_2 = \mu_2 e_2, \quad \overline{e}_1 + \overline{e}_2 = \mu_3(e_1 + e_2), \\
\overline{e}_1 + s'\overline{e}_2 = \mu_4(e_1 + se_2) \Longrightarrow \mu_1 e_1 + \mu_2 e_2 = \mu_3(e_1 + e_2), \\
\mu_1 = \mu_2 = \mu_3 \Longrightarrow \mu_1 e_1 + s'\mu_2 e_2 = \mu_4 e_1 + \mu_4 s e_2, \\
\mu_1 = \mu_4 \Longrightarrow s'\mu_4 = \mu_4 s.
\end{gathered}$$

For the quadruple of strictly collinear points and their cross-ratio we use the notation

$$[P_1, P_2, P_3, P_4] = [s].$$

We remark that the order of the points P_i is essential.

Let now e_1 and e_2 be generators of a k-free submodule U of rank 2.

Consider the points $k(\alpha_i e_1 + \beta_i e_2)$, $\alpha_i, \beta_i \in k^*$, $1 \leq i \leq 4$. For $i \neq j$ we shall use the notation

$$D_{ij} = \begin{vmatrix} \alpha_i, & \beta_i \\ \alpha_j, & \beta_j \end{vmatrix} = \alpha_i\beta_j - \alpha_j\beta_i; \quad \widetilde{D}_{ij} = \widetilde{\begin{vmatrix} \alpha_i, & \beta_i \\ \alpha_j, & \beta_j \end{vmatrix}} = \alpha_i\alpha_j^{-1} - \beta_i\beta_j^{-1}.$$

Proposition 1. *The points* $k(\alpha_i e_i + \beta_i e_2)$ *are strictly collinear if and only if*

$$D_{ij} \in k^*, \qquad \widetilde{D}_{ij} \in k^*.$$

Proof. We have

$$\begin{aligned}
\beta_i^{-1}(\alpha_i e_1 + \beta_i e_2) &= \beta_i^{-1}\alpha_i e_1 + e_2 = \overline{e}_1,\\
\beta_j^{-1}(\alpha_j e_1 + \beta_j e_2) &= \beta_j^{-1}\alpha_j e_1 + e_2 = \overline{e}_2;\\
\overline{e}_1 - \overline{e}_2 &= (\beta_i^{-1}\alpha_i - \beta_j^{-1}\alpha_j)e_1,\\
\beta_i(\beta_i^{-1}\alpha_i - \beta_j^{-1}\alpha_j)\alpha_j^{-1} &= \alpha_i\alpha_j^{-1} - \beta_i\beta_j^{-1} \in k^*\\
&\Longrightarrow e_1, e_2 \in k(\alpha_i e_1 + \beta_i e_2) + k(e_j e_1 + \beta_j e_2).
\end{aligned}$$

The inclusion $D_{ij} \in k^*$ is proved straightforward. $\square$

Proposition 2. *If* P_1, P_2, P_3, P_4 *are strictly collinear points and* $\alpha_1 = 0$, $\beta_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$, *then*

$$D_{32}D_{42}^{-1} \in [P_1, P_2, P_3, P_4].$$

Proof.

$$\begin{aligned}
k(e_1 + \beta_3 e_2) &= k[(\beta_3 - \beta_2)e_2 + (e_1 + \beta_2 e_2)],\\
k(e_1 + \beta_4 e_2) &= k(\beta_4 e_2 - \beta_2 e_2 + e_1 + \beta_2 e_2\\
&= k[(\beta_4 - \beta_2) + e_2 + e_1 + \beta_2 e_2]\\
&= k[e_2 + (\beta_4 - \beta_2)^{-1}(e_1 + \beta_2 e_2)]\\
&= k[(\beta_3 - \beta_2)e_2 + (\beta_3 - \beta_2)(\beta_4 - \beta_2)^{-1}(e_1 + \beta_2 e_2)]. \qquad \square
\end{aligned}$$

Proposition 3. *If in Proposition* 2, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$, *then*

$$D_{41}D_{42}^{-1}D_{32}D_{31}^{-1} \in [P_1, P_2, P_3, P_4].$$

Proof. Suppose that

$$\begin{aligned}
&e_1 + \beta_3 e_2 = \lambda_1(e_1 + \beta_1 e_2) + \lambda_2(e_1 + \beta_2 e_2)\\
&\quad\Longrightarrow \begin{cases} \lambda_1 + \lambda_2 = 1 \\ \lambda_1\beta_1 + \lambda_2\beta_2 = \beta_3 \end{cases} \Longrightarrow \begin{cases} \lambda_1 = 1 - \lambda_2 \\ \lambda_1\beta_1 + \lambda_2\beta_2 = \beta_3 \end{cases}\\
&\quad\Longrightarrow \beta_1 + \lambda_2(\beta_2 - \beta_1) = \beta_3.
\end{aligned}$$

Consequently, $\lambda_2 = D_{13}D_{12}^{-1}$. In the same way $\lambda_1 = D_{23}D_{12}^{-1}$. From Proposition 1 we have $\beta_i - \beta_j \in k^*$ and $1 - \beta_j \in k^*$. In our conditions for $1 \le i, j, k \le 4$ we have

$$\begin{aligned}
(\beta_i - \beta_j)(\beta_k - \beta_j)^{-1} - 1 &= (\beta_i - \beta_j - \beta_k + \beta_j)(\beta_k - \beta_j)^{-1}\\
&= (\beta_i - \beta_k)(\beta_k - \beta_j)^{-1};
\end{aligned}$$

$$\begin{aligned}(\beta_i-\beta_j)(\beta_k-\beta_j)^{-1}\beta_k-\beta_i &= (\beta_i-\beta_j)(\beta_k-\beta_j)^{-1}\beta_k-\beta_k-(\beta_i-\beta_k)\\ &= [(\beta_i-\beta_j)(\beta_k-\beta_j)^{-1}-1]\beta_k-(\beta_i-\beta_k)\\ &= (\beta_i-\beta_k)(\beta_k-\beta_j)^{-1}\beta_k-(\beta_i-\beta_k)\\ &= (\beta_i-\beta_j)(\beta_k-\beta_j)^{-1}[\beta_k-(\beta_k-\beta_j)]\\ &= (\beta_i-\beta_j)(\beta_k-\beta_j)^{-1}\beta_j.\end{aligned}$$

Taking into account these equations, we find

$$\begin{aligned}&k[-(e_1+\beta_1e_2)+(\beta_1-\beta_3)(\beta_2-\beta_3)^{-1}(e_1+\beta_2e_2)]\\ &\quad= k[((\beta_1-\beta_3)(\beta_2-\beta_3)^{-1}-1)e_1+[(\beta_1-\beta_3)(\beta_2-\beta_3)^{-1}\beta_2-\beta_1]e_2]\\ &\quad= k[(\beta_1-\beta_2)(\beta_2-\beta_3)^{-1}e_1+(\beta_1-\beta_2)(\beta_2-\beta_3)^{-1}\beta_3e_2]\\ &\quad= k(\beta_1-\beta_2)(\beta_2-\beta_3)^{-1}(e_1+\beta_3e_2)]\\ &\quad= k[e_1+\beta_3e_2]=P_3;\\ &k[-(e_1+\beta_1e_2)+(\beta_1-\beta_4)(\beta_2-\beta_4)^{-1}(e_1+\beta_2e_2)]\\ &\quad= k[(-1+(\beta_1-\beta_4)(\beta_2-\beta_4)^{-1})e_1+[(\beta_1-\beta_4)(\beta_2-\beta_4)^{-1}\beta_2-\beta_1]e_2]\\ &\quad= k[(\beta_1-\beta_2)(\beta_4-\beta_2)^{-1}e_1+(\beta_1-\beta_2)(\beta_4-\beta_2)^{-1}\beta_4e_2]\\ &\quad= k(e_1+\beta_4e_2)=P_4=k[-(e_1+\beta_1e_2)\\ &\qquad+(\beta_1-\beta_4)(\beta_2-\beta_4)^{-1}(\beta_2-\beta_3)(\beta_1-\beta_3)^{-1}(\beta_2-\beta_3)^{-1}(e_1+\beta_2e_2)]\\ &\quad= k[-(e_1+\beta_1e_2)+D_{41}D_{42}^{-1}D_{32}D_{31}^{-1}(\beta_1-\beta_3)(\beta_2-\beta_3)^{-1}(e_1+\beta_2e_2)].\end{aligned}$$

Consequently, the equations

$$\begin{aligned}P_1&=k[-(e_1+\beta_1e_2)],\\ P_2&=k[(\beta_1-\beta_3)(\beta_2-\beta_3)^{-1}(e_1+\beta_2e_2)]\end{aligned}$$

complete the proof. □

The set $k^*\subset k$ splits in equivalent classes of conjugate elements. Then for each class $[s_\alpha]$ on the line $U=P_1\cup P_2=P_1\cup P_3=P_2\cup P_3$ we can always find the point P_4 such that $[P_1,P_2,P_3,P_4]=[s]$. In fact, we can find basic elements $e_1,e_2\in U$ such that $P_1=ke_1$, $P_2=ke_2$, $P_3=k(e_1+e_2)$ and then choose the point P_4. The point P_4 is not uniquely defined by the points P_1, P_2, P_3 and the cross-ratio. If the element s belongs to the center of the ring k, then P_4 is unique, which is easy to check by straightforward calculations.

Let P_1,P_2,P_3,P_4 be a quadruple of strictly collinear points on the projective line U. Then the cross-ratio depends on the order of the points. The effect of inversion is illustrated by the equations (see [2])

$$[P_1,P_2,P_3,P_4]=[P_2,P_1,P_3,P_4]^{-1}=[P_1,P_2,P_4,P_3]^{-1},$$
$$[P_1,P_2,P_3,P_4]=1-[P_1,P_3,P_2,P_4].$$

Note that if $A\subset k$ is a subset then $A^{-1}\overset{\text{def}}{=}\{x^{-1},\text{ for all } x\in A\}$. The first two equations we can check from Proposition 3. The generator of the point can always be chosen in such a way that the coefficient of e_1 will be 1. Let

$$P_1=ke_1,\ \ P_2=ke_2,\ \ P_3=k(e_1+e_2),\ \ [P_1,P_2,P_3,P_4]=[s].$$

Choose the basis $\{e_1,e_2\}$ on $\{-e_1,e_1+e_2\}$. Then

$$e_2=-e_1+(e_1+e_2),\ \ se_1+e_2=(1-s)(-e_1)+(e_1+e_2)$$
$$\Longrightarrow [P_1,P_2,P_3,P_4]=1-[s].$$

The quadruple of the strictly collinear points $P_1,P_2,P_3,P_4\in P(X)$ is in a *harmonic* relation if $[P_1,P_2,P_3,P_4]=-1$. Note that this definition implies that $\frac{1}{2}\in k$.

Proposition 4. *Let X_1 and X_2 be torsion-free modules over the rings k_1 and k_2; $\alpha:P(X_1)\longrightarrow P(X_2)=2$ be a bijection, and $\operatorname{rank}X_1=\operatorname{rank}X_2$; then the following statements are equivalent:*

(a) *$P_1,P_2,P_3,P_4\in P(X_1)$ are harmonic if and only if $\alpha(P_1)$, $\alpha(P_2)$, $\alpha(P_3)$, $\alpha(P_4)$ are harmonic;*

(b) *if P_1,P_2,P_3,P_4 are harmonic, then $\alpha(P_1)$, $\alpha(P_2)$, $\alpha(P_3)$, $\alpha(P_4)$ are harmonic, and if $Q_1,Q_2,Q_3,Q_4\in P(X_2)$ are harmonic, then $\alpha^{-1}(Q_1)$, $\alpha^{-1}(Q_2)$, $\alpha^{-1}(Q_3)$, $\alpha^{-1}(Q_4)$ are strictly collinear.*

Proof. (a)$\Longrightarrow$ (b) is obvious. (b) $\Longrightarrow$ (a). Let $\{e_1,e_2\}$ and $\{f_1,f_2\}$ be bases of the lines $U=Q_i+Q_j\subseteq X_2$ and $\alpha^{-1}(U)=\alpha^{-1}(Q_i)+\alpha^{-1}(Q_j)$, $1\le i,j\le 4$. Suppose that

$$\alpha(k_1e_1)=Q_1=k_2f_1,\ \ \alpha(k_1e_2)=Q_2=k_2f_2,$$
$$\alpha(k_1(e_1+e_2))=Q_3=k_2(f_1+f_2),$$
$$\alpha(k_1(e_1+\mu e_2))=Q_4=k_2(f_1-f_2).$$

It is clear that $\mu\in k^*$. Since the triple of the strictly collinear points Q_1,Q_2,Q_3 represents the fourth harmonic point Q_4, we have

$$\alpha(k_1(e_1-e_2))=Q_4=\alpha(k_1(e_1+\mu e_2))\Longrightarrow \mu=-1.$$

The map $f:P(X_1)\longrightarrow P(X_2)$ will be called harmonic if the images of harmonic points are harmonic. $f:P(X_1)\longrightarrow\mathfrak{M}(X_2)$ will be called a collineation if $P_1\subset P_2+P_3$ implies $f(P_1)\subset f(P_2)+f(P_3)$. The map f preserves linear independence if $P_1,\dots,P_\alpha\in P(X_1)$ are independent if and only if $f(P_1), \ldots, f(P_\alpha) \in \mathfrak{M}(X_2)$ are independent, i.e., for every $\beta \in \Lambda$ we have

$$P_\beta \cap \left(\bigcup_{\gamma \in \Lambda, \gamma \neq \beta} P_\gamma \right) = 0 \Longleftrightarrow f(P_\beta) \cap \left(\bigcup_{\gamma \in \Lambda, \gamma \neq \beta} f(P_\gamma) \right) = 0.$$

A collineation which preserves linear independence will be called an LIP-collineation.

Let e be the unimodular element in the k-free submodule A. Then $k_1 e \subset \sum_{i=1}^m k_1 e_i$, where $\{e_i, i = 1, \ldots, m\}$ is some finite subset of the basis A. It is obvious that if f is a collineation, then $f(k_1 e) \subset \sum_{i=1}^m f(k_1 e_i)$.

Recall that the 1–1 map $f : X_1 \longrightarrow X_2$ is a semilinear (σ-semilinear) isomorphism with respect to σ if $\sigma : k_1 \longrightarrow k_2$ is a ring isomorphism and

$$f(ax_1 + bx_2) = \sigma(a) f(x_1) + \sigma(b) f(x_2)$$

for each $a, b \in k_1, x_1, x_2 \in X_2$.

Let $U \subseteq X_1$ be a k_1-free submodule; $f : X_1 \longrightarrow X_2$ be a σ-semilinear map. It is clear that the image of the unimodular element $e \in U$ is unimodular. So we get an induced map, i.e., the projection $P(f) : P(X_1) \longrightarrow P(X_2)$, for which $P(f)(k_1 e) = k_2 f(e)$ for all unimodular elements of all lines of X_1. It is also obvious that $P_1 \subset P_2 + P_3$ implies $P(f)P_1 \subset P(f)P_2 + P(f)P_3$. $\square$

2. Some Facts Concerning Harmonic Maps and Collineations

Let k be a commutative principal ideal domain, F be the quotient field of k. The canonical map $\sigma : k \hookrightarrow F$ induces the semilinear isomorphism

$$\sigma^n : k^n = \underbrace{k + k + \cdots + k}_{n} \longrightarrow F^n = \underbrace{F + F + \cdots + F}_{n}, \quad n \geq 2.$$

This one defines the map $P(\sigma^n) : \widetilde{P}(k^n) \longrightarrow P(F^n)$. When $k = K\langle x \rangle$ is the ring of formal power series in x of some field, then $P(\sigma^n)$ is bijective [6], [13], [14].

Example 1. Let $n \geq 3$ and define the map α by Fig.1.

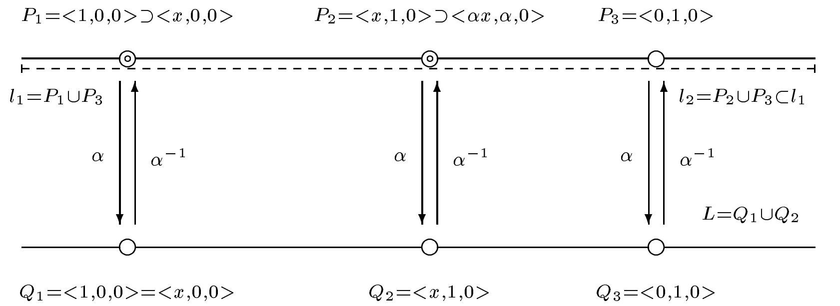

Figure 1

The lines l_1 and l_2 are defined over the ring k and the line L over the field F.

The map $\alpha^{-1} : P(L) \longrightarrow \widetilde{P}(l_1)$ is not a collineation. It is clear that $Q_1 \subset Q_2 + Q_3$. On the other hand,

$$\begin{aligned} &l_1 = P_1 \cup P_2 = P_1 \cup P_3 \supset l_2 = P_2 \cup P_3 \\ &\quad \Longrightarrow \alpha^{-1}(Q_1) = P_1 \not\subseteq \alpha^{-1}(Q_2) \cup \alpha^{-1}(Q_3) = P_2 \cup P_3. \end{aligned}$$

Note that $\langle x, 0, 0\rangle \notin \widetilde{P}(k^n)$.

Example 2. Suppose that $n = 2$ and define the harmonic map $\alpha : \widetilde{P}(l) \to P(L)$ by Fig. 2

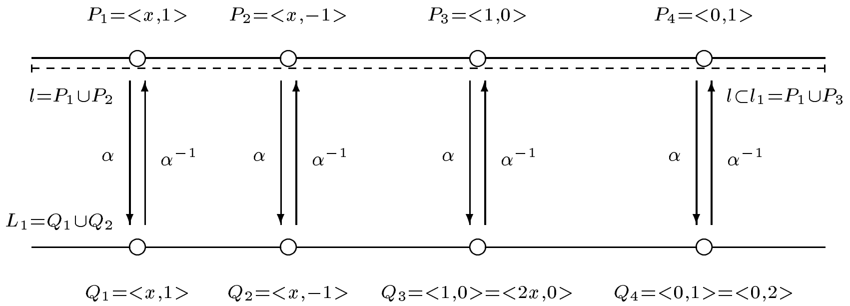

Figure 2

It is easy to see that α is not harmonic, though it is bijective. Note that the lines l and l_1 are defined over the ring k and the line L over the field F. It is obvious that α^{-1} is not harmonic because the points P_1, P_2, P_3, P_4 are not strictly collinear.

For the completion of the picture we shall give an example which shows that for the system of points $\widetilde{P}(X)$ over the principal ideal domain, von Staudt's theorem is not true.

Example 3 (C. Bartolone and F. Di Franco [13]). Let $k = F < x >$, where F is a field, char$F \neq 2$. Define the bijective map $\alpha : \widetilde{P}(k^2) \longrightarrow \widetilde{P}(k^2)$ by the equations

$$\alpha(k(0,1)) = k(0,1), \quad \alpha(k(1,0)) = k(1,0) \quad \text{for} \quad k(f,g) \in \widetilde{P}(k^2),$$

$$\alpha(k(f,g)) = \begin{cases} k(f,g) & \text{if } \deg(f) \equiv \deg(g) \pmod 2, \\ k(-f,g) & \text{if } \deg(f) \not\equiv \deg(g) \pmod 2. \end{cases}$$

This map is harmonic on both sides but is not induced by the semilinear isomorphism [13], [14].

Further, k is a non-commutative left principal ideal domain. Let us investigate the map

$$f : \mathfrak{M}(X) \longrightarrow \mathfrak{M}(X_1),$$

which preserves the lattice-theoretical operation of union ($\cup$-preserving map).

Thus, such map is defined with its restriction on the projective space $P(X)$, so it is natural for the beginning to consider the map $f : P(X) \longrightarrow \mathfrak{M}(X_1)$. Since for our general maps the images of the points are not always points, it is natural to generalize the definition of the harmonic map.

Definition 3. The map $f : P(X) \longrightarrow \mathfrak{M}(X_1)$ will be called *harmonic* if for each quadruple of harmonic points $P_1, P_2, P_3, P_4 \in P(X)$ and their images $f(P_1)$, $f(P_2)$, $f(P_3)$, $f(P_4) \in \mathfrak{M}(X_1)$ there exist $y_1, y_2 \in X_1$ such that

$$\begin{aligned} Q_1 &= k_1 y_1 \hookrightarrow f(P_1), \\ Q_2 &= k_1 y_2 \hookrightarrow f(P_2), \\ Q_3 &= k_1(y_1 + y_2) \hookrightarrow f(P_3), \\ Q_4 &= k_1(y_1 - y_2) \hookrightarrow f(P_4), \end{aligned}$$

i.e., the points Q_1, Q_2, Q_3, Q_4 are in a harmonic relation.

Let F be a quotient field of k. According to U. Brehm [37], consider the tensor product $\overline{X} = F \otimes_k X$ and the canonical map $i : X \longrightarrow F \otimes_k X$. The module X will be considered as a k-submodule of the F-vector space X. It is obvious that $FX = \langle FX \rangle = \overline{X}$.

Suppose as well that F_1 is some skew field and k_1 is a subring of F_1. Let $\overline{X}_1$ be a F_1-vector space and X_1 be a k_1-submodule of $\overline{X}_1$ such that $\langle F_1 X_1 \rangle = \overline{X}_1$.

Proposition 5. *Let* $f : \mathfrak{M}(X) \to \mathfrak{M}(X_1)$ *be* $\cup$*-preserving map and* $\mu : \overline{X} \longrightarrow \overline{X}_1$ *be a semilinear isomorphism with respect to the isomorphism* $\sigma : F \longrightarrow F_1$. *If there exists a subring* $K_1 \hookrightarrow F_1$ *for which*

$$\sigma(k) \subseteq K_1 \subseteq F_1, \quad K_1\mu(X) \subseteq X_1, \ f(kx) = K_1\mu(x),$$

then f *is a LIP-collineation.*

Proof. Let $P_1 = kx_1$, $P_2 = kx_2$, $P_3 = kx_3$, $P_1 \subseteq P_2 + P_3$ then we have

$$\begin{aligned} x_1 = mx_2 + nx_3 &\Longrightarrow \mu(x_1) = \sigma(m)\mu(x_2) + \sigma(n)\mu(x_3) \Longrightarrow K_1\mu(x_1) \\ &\subseteq [K_1\sigma(k)\mu(x_2)] \cup [K_1\sigma(k)\mu(x_3)] \subseteq [K_1\mu(x_2)] \cup [K_1\mu(x_3)] \\ &\Longrightarrow f(kx_1) \hookrightarrow f(kx_2) \cup f(kx_3) \Longrightarrow f \text{ is a collineation,} \end{aligned}$$

so that

$$\begin{aligned} 0 \neq F_1[f(kx)] \cap F_1[f(ky)] &= F_1[K_1\mu(x)] \cap F_1[K_1\mu(y)] \\ &\Longrightarrow \mu(x) \in F_1\mu(y) \Longrightarrow x \in Fy \\ &\Longrightarrow f \text{ preserves linear independence} \quad \square. \end{aligned}$$

Suppose that $f : \mathfrak{M}(X) \to \mathfrak{M}(X_1)$ is an LIP-collineation. Let us observe some general facts concerning collineations and harmonic maps.

(l_1) From the linear independence of f we get $f(0) = 0$. It is also clear that $\dim F_1 f(kx) = 1$ for all $x \in X$. Indeed, let P be a point, i.e., $P = kx$ and $\dim F_1 f(P) = 2$, then all submodules of this point are one-dimensional and have non-zero intersections. Since in $F_1 f(P)$ we can always find two non-incident points, we get a contradiction.

(l_2) Let us show that if $Fx_1 = Fx_2$ for $x_1, x_2 \in F$, then $F_1 f(kx_1) = F_1 f(kX_2)$.

By the condition there exists $\overline{s}, \overline{r} \in F$ such that $\overline{r}x_1 = \overline{s}x_2$. Consequently, we can find $s, r \in k$ for which $rx_1 = sx_2$. So we have

$$\begin{aligned} &k(sx_2) \subseteq k(x_2), \ \ k(sx_2) = k(rx_1) \subseteq k(x_1) \\ &\Longrightarrow f(k(sx_2)) \subseteq f(kx_1) \cap f(kx_2) \\ &\Longrightarrow F_1 f(kx_1) = F_1 f(kx_2). \end{aligned}$$

(l_3) Define the map

$$f_1 : \overline{X} \setminus 0 \longrightarrow \mathfrak{M}_{F_1}(\overline{X}_1)$$

in the following way: for $x \in \overline{X}$, $x \neq 0$,

$$f_1(x) = F_1 f(ky), \quad y \in X \cap (Fx \setminus 0).$$

For each $n \in \mathbb{N}$ and arbitrary $x, y_1, \dots, y_n \in \overline{X} \setminus 0$ from (l_2) we get

$$\begin{aligned} Fx \cap (\cup_{i=1}^n Fy_i) = 0 &\Longrightarrow f_1(x) \cap (\cup_{i=1}^n f_1(y_i)) = 0 \\ &\Longrightarrow \ \text{if} \ \ Fx \neq Fy, \ \ \text{then} \ \ f_1(x) \neq f_1(y). \end{aligned}$$

Since f is a collineation for $x, y_1, y_2 \in \overline{X} \setminus 0$, we have: if $x \in Fy_1 + Fy_2$, then $f_1(x) \subseteq f_1(y_1) + f_1(y_2)$.

(l_4) By induction we can prove that if $x \in Fx_1 + Fx_2 + \cdots + Fx_m$, then $f_1(x) \subseteq f_(x_1) + f_1(x_2) + \cdots + f_1(x_m)$.

For $m = 1, 2$ and $x \in Fx_m$ the statement is obvious. Let $x \notin Fx_m$, then there exist $y, z \in \overline{X}$ such that

$$x \in F(y + z), \;\; y \in Fx_1 + \cdots + Fx_{m-1}, \;\; z \in Fx_m, \;\; y \neq 0.$$

Consequently, by the induction hypothesis we get

$$f_1(y) \subseteq f_1(x_1) + \cdots + f_1(x_{m-1}).$$

On the other hand,

$$\begin{aligned} x \in Fy + Fx_m &\Longrightarrow f_1(x) \subseteq f_1(y) + f_1(x_m) \\ &\Longrightarrow f_1(x) \subseteq f_1(x_1) + f_1(x_2) + \cdots + f_1(x_{m-1}) + f_1(x_m). \end{aligned}$$

(l_5) In the sequel we shall often use the following fact:

Proposition 6. *Let x and y be linear independent elements of $\overline{X}$ and $0 \neq z \in \overline{X}$, $z \in (Fx + Fy) \setminus Fy$. Then there exists $0 \neq d \in F$ such that $F(x + dy) = Fz$.*

It is obvious that $Fz = F(ax + by)$ and $d = a^{-1}b$. It is also obvious that d has only one representation by Fz.

(l_6) Let B be a basis of $\overline{X}$ and x_0 be an arbitrary but fixed element of B. Define

$$\widetilde{\mu} : B \longrightarrow \overline{X}_1, \quad F_1\widetilde{\mu}(x) = f_1(x), \;\; x \in B.$$

So we have

$$\begin{aligned} x_0 + x \in Fx_0 + Fx, \;\; x \in B \setminus x_0 &\Longrightarrow f_1(x_0 + x) \\ &\subseteq f_1(x_0) + f_1(x) = F_1\widetilde{\mu}(x_0) + F_1\widetilde{\mu}(x). \end{aligned}$$

Taking Proposition 6 into consideration, we conclude that there exists $d \in F$, $d \neq 0$ such that

$$f_1(x_0 + x) = F_1(\widetilde{\mu}(x_0) + d\widetilde{\mu}(x)).$$

Definition 4. $\mu(x) \stackrel{\text{def}}{=} d\widetilde{\mu}(x), \;\; x_0 \neq x \in B, \;\; \mu(x_0) \stackrel{\text{def}}{=} \widetilde{\mu}(x_0).$

So for all $x \in B$ we have $f_1(x) = F_1\mu(x)$ and $f_1(x_0 + x) = F_1(\mu(x_0) + \mu(x))$, where $x \in B \setminus x_0$. Consequently, from (l_3) we conclude that $\mu(B) \subseteq \overline{X}_1$ is a linear independent set.

(l_7) Let $a \in F, \;\; x \in B \setminus x_0, \;\; x_0 + ax \notin Fx$. Then from Proposition 6 we can conclude that there exists only one element $\sigma(a, x) \in F_1$ for which

$$f_1(x_0 + ax) = F_1[\mu(x_0) + \sigma(a, x)\mu(x)].$$

Note that the theorem of von Staudt deals with the harmonic maps of the projective line, i.e., it considers the case when $\dim_p X = 1$. In this situation $B = \{x_0, x\}$ and $\sigma(a,x) = \sigma(a)$.

Generally, as we shall show in [9], σ does not depend on x, i.e., $\sigma(a,x) = \sigma(a)$.

So σ is an injective map. From the above we conclude that $\sigma(0) = 0$, $\sigma(1) = 1$.

3. Harmonic Maps Generated by Semilinear Isomorphisms

Let $\frac{1}{2} \in k$, $\dim_p X = 1$ and $f : P(X) \longrightarrow \mathfrak{M}_{k_1}(X_1)$, be a harmonic map (Definition 3). In the previous paragraph we have defined the maps f_1, σ, μ. It is clear that a set-theoretical map f_1 defined on the elements of X can also be considered as the map determined on $P(X)$. Let us show now that σ is either a isomorphism or an anti-isomorphism. Recall that $\sigma : k \longrightarrow k_1$ is an anti-homomorphism if $\sigma(x+y) = \sigma(x) + \sigma(y), \sigma(xy) = \sigma(y)\sigma(x)$ for all $x, y \in k$. We cannot use the classical theorem of K. von Staudt because, on the one hand, f_1 is the map of the projective line over a ring which is not in general a skew field, and on the other hand, f_1 is not bijective.

Consider the lines $l = kx_0 + kx$ and $L = F_1\mu(x_0) + F_1\mu(x)$. On the line l the points

$$k(x_0 + ax), \quad k(x_0 + bx),$$
$$k(x_0 + \frac{a+b}{2}x) = k[2x_0 + (a+b)x], \quad k[(a-b)x] = k\Big(\frac{a-b}{2}x\Big)$$

are in a harmonic relation. According to the definition of f we have

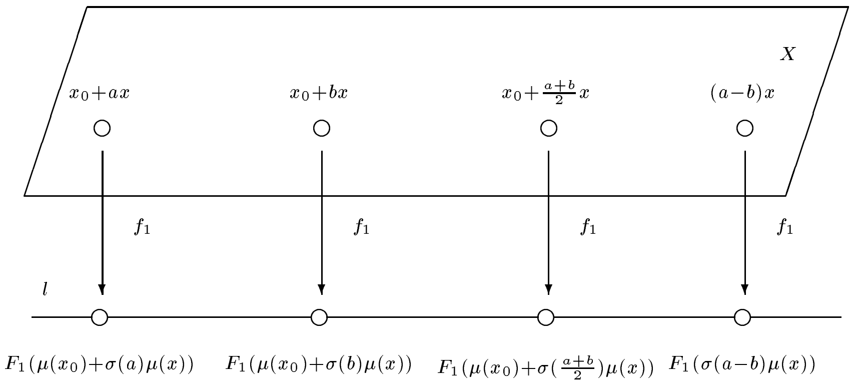

Figure 3

Figure 3 represents the map of the elements of the k-module X on the projective line L over the skew field F_1.

Note that since f is a harmonic map, for the elements $x_1, x_2, x_3, x_4 \in \bar{X}$ we have that if the points kx_1, kx_2, kx_3, kx_4 are in a harmonic relation, then the points $f_1(x_1), f_1(x_2), f_1(x_3), f_1(x_4)$ are also in a harmonic relation. Consequently, the quadruple

$$\begin{gathered} F_1(\mu(x_0)+\sigma(a)\mu(x)), \qquad F_1(\mu(x_0)+\sigma(b)\mu(x)), \\ F_1(\mu(x_0)+\sigma\left(\frac{a+b}{2}\right)\mu(x)), \qquad F_1(\sigma(a-b)\mu(x) = F_1\mu(x) \end{gathered}$$

is harmonic. So, taking Proposition 3 into consideration, we get

$$\begin{gathered} \Big[-\sigma\left(\frac{a+b}{2}\right)+\sigma(b)\Big]\Big[-\sigma\left(\frac{a+b}{2}\right)+\sigma(a)\Big]^{-1}=-1 \Rightarrow \sigma\left(\frac{a+b}{2}\right)-\sigma(b) \\ = -\sigma\left(\frac{a+b}{2}\right)+\sigma(a) \Rightarrow \sigma\left(\frac{a+b}{2}\right) = \frac{\sigma(a)}{2}+\frac{\sigma(b)}{2}. \end{gathered}$$

Suppose that in this equation $b = 0$; then we get $\sigma\left(\frac{a}{2}\right) = \frac{\sigma(a)}{2}$. If now we suppose that $b = a$, then we have

$$\begin{gathered} \sigma\left(\frac{2a}{2}\right) = \sigma(a) = \frac{\sigma(2a)}{2} \Rightarrow \sigma(2a) = 2\sigma(a) \Rightarrow \sigma(a+b) \\ = \sigma\left(\frac{2(a+b)}{2}\right) = 2\sigma\left(\frac{a+b}{2}\right) = 2\left(\frac{\sigma(a)}{2}+\frac{\sigma(b)}{2}\right) = \sigma(a)+\sigma(b). \end{gathered}$$

So σ is an additive isomorphism.

Suppose now that $[\sigma(a)]^{-1} = \sigma(a^{-1})$ for every $a \in F$. Then we have

$$\begin{aligned} a &= a(1-a)(1-a)^{-1} = a(1-a)^{-1} - a^2(1-a)^{-1} \\ &\Rightarrow a + a^2(1-a)^{-1} = 1 + a(1-a)^{-1} - 1 \Rightarrow a^2[a^{-1}+(1-a)^{-1}] \\ &= a[a^{-1}+(1-a)^{-1}] - 1 \Rightarrow a^2 = a - [a^{-1}+(1-a)^{-1}]^{-1} \\ &\Rightarrow \sigma(a^2) = \sigma(a) - [\sigma(a)^{-1} + (1-\sigma(a))^{-1}]^{-1} = [\sigma(a)]^2, \\ ab+ba &= (a+b)^2 - a^2 - b^2 \Rightarrow \sigma(ab)+\sigma(ba) \\ &= [\sigma(a+b)]^2 - [\sigma(a)]^2 - [\sigma(b)]^2 = [\sigma(a)]^2 + [\sigma(b)]^2 \\ &\quad + \sigma(a)\sigma(b) + \sigma(b)\sigma(a) - [\sigma(a)]^2 - [\sigma(b)]^2 \\ &\Rightarrow \sigma(ab)+\sigma(ba) = \sigma(a)\sigma(b)+\sigma(b)\sigma(a). \end{aligned}$$

From (l_6) it is obvious that $\widetilde{\mu}$, and consequently μ can be defined in many different ways, i.e., for every $\alpha \in F$ one can define $\mu_1 = \alpha\mu$, and μ_1 also has the same meaning. Consequently σ is defined for fixed x_0 and for fixed $\mu(x_0) \in \bar{X}_1$. If now we start from x_1 and $\mu(x_1)$, then in the same way we can construct $\tau : F \longrightarrow F_1$.

In fact, $[\tau(a)]^{-1} = \sigma(a^{-1})$. Indeed,

$$\begin{array}{l} f_1(ax_0 + x_1) = F_1[\tau(a)\mu(x_0) + \mu(x_1)] = F_1[\mu(x_0) + [\tau(a)]^{-1}\mu(x_1)] \\ \qquad \Big\| \\ f_1(x_0 + a^{-1}x_1) = F_1[\mu(x_0) + \sigma(a^{-1})\mu(x_1)] \Rightarrow [\tau(a)]^{-1} = \sigma(a^{-1}); \end{array}$$

Similarly,

$$[\sigma(a)]^{-1} = \tau(a^{-1}).$$

So we have to prove that $\sigma(a^{-1}) = [\sigma(a)]^{-1}$. Suppose that $1+a$ and $1-a$ are units of k. Then the points

$$\begin{aligned} &P_1 = k(x_0 + ax_1), \quad P_2 = k(ax_0 + x_1), \\ &P_3 = k[(x_0 + ax_1) + (ax_0 + x_1)] = k[(1+a)(x_0 + x_1)] = k(x_0 + x_1), \\ &P_4 = k[(x_0 + ax_1) - (ax_0 + x_1)] = k[(1-a)x_0 + (a-1)x_1] \\ &\quad = k[(1-a)(x_0 - x_1)] = k(x_0 - x_1) \end{aligned}$$

are in a harmonic relation. On the other hand, consider the points

$$\begin{aligned} Q_1 = k(ax_0 + a^2x_1), &\quad Q_2 = k(a^2x_0 + ax_1), \\ Q_3 = k[(a+a^2)(x_0 + x_1)], &\quad Q_4 = k[(a-a^2)(x_0 - x_1)]. \end{aligned}$$

It is obvious that they are in a harmonic relation while they are strictly collinear, i.e., $Q_i \subseteq Q_j \cup Q_k$, $1 \le i, j, k \le 4$.

We have

$$\begin{gathered} (a+a^2)(x_0 + x_1) + (-a^2x_0 - ax_1) = ax_0 + a^2x_1 \in k(a^2x_0 + ax_1) \\ + k[(a+a^2)(x_0 + x_1)] \Rightarrow Q_1 \in Q_2 \cup Q_3; \\ (a-a^2)(x_0 - x_1) + (a+a^2)(x_0 + x_1) = 2ax_0 + 2a^2x_1 \\ \Rightarrow 2(ax_0 + a^2x_1), ax_0 + a^2x_1 \in k[(a+a^2)(x_0 + x_1)] \\ + k[(a-a^2)(x_0 - x_1)] \Rightarrow Q_1 \in Q_3 \cup Q_4. \end{gathered}$$

All other inclusions can be proved similarly. Further,

$$\begin{aligned} f_1[a(x_0 + ax_1)] &= f_1(x_0 + ax_1) = f_1(ax_0 + a^2x_1) = \\ &= F_1[\mu(x_0) + \sigma(a)\mu(x_1)] = L_1, \\ f_1(ax_0 + x_1) &= f_1(a^2x_0 + ax_1) = F_1[\tau(a)\mu(x_0) + \mu(x_1)] \\ &= F_1[\mu(x_0) + [\tau(a)]^{-1}\mu(x_1)] = L_2, \\ f_1[(a+a^2)(x_0 + x_1)] &= f_1(x_0 + x_1) = F_1[\mu(x_0) + \mu(x_1)] = L_3, \\ f_1[(a-a^2)(x_0 - x_1)] &= f_1(x_0 - x_1) = F_1[\mu(x_0) - \mu(x_1)] = L_4. \end{aligned}$$

Either the quadruple P_1, P_2, P_3, P_4 or the quadruple Q_1, Q_2, Q_3, Q_4 is harmonic, so we get that the quadruple L_1, L_2, L_3, L_4 is also harmonic. For the points L_1, L_2, L_3, L_4 we have

$$D_{41} = \begin{vmatrix} 1, & -1 \\ 1, & \sigma(a) \end{vmatrix}, \qquad D_{42} = \begin{vmatrix} 1, & -1 \\ 1, & [\tau(a)]^{-1} \end{vmatrix},$$
$$D_{32} = \begin{vmatrix} 1, & 1 \\ 1, & [\tau(a)]^{-1} \end{vmatrix}, \qquad D_{31} = \begin{vmatrix} 1, & 1 \\ 1, & \sigma(a) \end{vmatrix}.$$

Let now $\widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3, \widetilde{P}_4$ be arbitrary strictly collinear points over the ring k. Then we can assume that

$$\widetilde{P}_1 = ke_1, \quad \widetilde{P}_2 = ke_2, \quad \widetilde{P}_3 = k(e_1 + e_2),$$
$$\widetilde{P}_4 = k(e_1 + se_2), \quad [\widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3, \widetilde{P}_4] = [s].$$

As $1 - t^{-1}st = t^{-1}(1-s)t$ we can conclude: if s passes through the whole class of conjugate elements, then $1 - s$ is also the whole class of conjugate elements. Consequently, to each class $[s]$ there corresponds the class $[1-s]$. Taking into consideration that

$$\widetilde{P}_4 = k(e_1 + se_2), \quad \widetilde{P}_1 = k(-e_1), \quad \widetilde{P}_2 = [k(e_1 + se_2) + (-e_1)],$$
$$\widetilde{P}_3 = k[(e_1 + se_2) + (1-s)(-e_1)] = k[s(e_1 + e_2)],$$

we conclude that for arbitrary strictly collinear points the equation

$$[\widetilde{P}_4, \widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3] = 1 - [\widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3, \widetilde{P}_4]$$

is true.

Turning back to our consideration, we can check

$$[L_4, L_1, L_2, L_3] = 1 - [L_1, L_2, L_3, L_4] = 2.$$

From Proposition 3 we get $[P_1, P_2, P_3, P_4] = D_{14} D_{24}^{-1} D_{23} D_{13}^{-1}$. Redenote $L_4 = \bar{L}_1, L_1 = \bar{L}_2, L_2 = \bar{L}_3, L_3 = \bar{L}_4$; then we have

$$\begin{aligned}
2 &= [\bar{L}_1, \bar{L}_2, \bar{L}_3, \bar{L}_4] = \bar{D}_{14}[\bar{D}_{24}]^{-1}\bar{D}_{23}[\bar{D}_{13}]^{-1} \\
&= \begin{vmatrix} 1, & -1 \\ 1, & 1 \end{vmatrix} \cdot \begin{vmatrix} 1, & \sigma(a) \\ 1, & 1 \end{vmatrix}^{-1} \cdot \begin{vmatrix} 1, & \sigma(a) \\ 1, & \sigma(a^{-1}) \end{vmatrix} \cdot \begin{vmatrix} 1, & -1 \\ 1, & \sigma(a^{-1}) \end{vmatrix}^{-1} \\
&= 2[1 - \sigma(a)]^{-1}[\sigma(a^{-1}) - \sigma(a)][\sigma(a^{-1} + 1]^{-1} \\
&\Rightarrow [\sigma(a^{-1}) - \sigma(a)][\sigma(a^{-1}) + 1]^{-1} = [1 - \sigma(a)] \\
&\Rightarrow \sigma(a^{-1}) - \sigma(a) = [1 - \sigma(a)][\sigma(a^{-1}) + 1] = 1 + \sigma(a^{-1}) - \sigma(a)\sigma(a^{-1}) - \sigma(a) \\
&\Rightarrow \sigma(a)\sigma(a^{-1}) = 1 \Rightarrow \sigma(a^{-1}) = [\sigma(a)]^{-1}.
\end{aligned}$$

So we have constructed the map σ with the following properties:

$$\begin{aligned}&(1)\quad \sigma(0)=0,\quad \sigma(1)=1;\\&(2)\quad \sigma \text{ is an additive isomorphism};\\&(3)\quad [\sigma(a)]^{-1}=\sigma(a^{-1});\\&(4)\quad \sigma(ab)+\sigma(ba)=\sigma(a)\sigma(b)+\sigma(b)\sigma(a).\end{aligned}$$

Thus, σ satisfies the conditions of the Theorem 1.15 from [1]. Taking into consideration that this theorem formulated for skew fields is also true for general rings we conclude that σ is either an isomorphism or an anti-isomorphism (see, also, [2], [29], [32]–[34]).

Let k_1 and k_2 be arbitrary rings. The map $\sigma : k_1 \longrightarrow k_2$ will be called a semi-isomorphism if it is either an isomorphism or an anti-isomorphism. So for fixed $x_0 \in B$ and $\mu : B \longrightarrow \bar{X}_1$ we can construct a semi-isomorphism $\sigma : F \longrightarrow F_1$ (see (l_6)). If now we replace μ by $\mu_1 = \bar{\alpha}\mu$, this will influence σ.

Definition 5. Let X_1 and X_2 be vector spaces over the skew fields k_1 and $k_2, \dim X_1 = \dim X_2 = 2$ and $\sigma : k_1 \longrightarrow k_2$ is an anti-isomorphism. The map $\mu : X_1 \longrightarrow X_2$ will be called a *semilinear isomorphism* with respect to σ (σ is a semilinear anti-isomorphism), if μ is defined on the basis B (i.e., for e_1, e_2 the images $\mu(e_1)$ and $\mu(e_2)$ are fixed), and then we shall continue as follows:

$$\begin{aligned}&\text{(i)}\quad \mu(ae_i)=\sigma(a)\mu(e_i),\quad i=1,2;\\&\text{(ii)}\quad \mu(a_1e_1+a_2e_2)=[\sigma(a_2)]^{-1}\mu(e_1)+[\sigma(a_1)]^{-1}\mu(e_2).\end{aligned}$$

for each $a, a_1, a_2 \in k$, $a_1, a_2 \neq 0$.

It is clear that $\mu_0 = 0$ and $\mu(e_1 \pm e_2) = \mu(e_1) \pm \mu(e_2)$.

Now let us turn back to our considerations. We have the following alternatives:

(i) σ is an isomorphism. Then

$$\begin{aligned}&f_1(ax_0+a_1x_1)=f(x_0+a_0^{-1}a_1x_1)=F_1[\mu(x_0)+\sigma(a_0^{-1}a_1)\mu(x_1)]\\&=f_1[\mu(x_0)+[\sigma(a_0)]^{-1}\sigma(a_1)\mu(x_0)]=F_1[\sigma(a_0)\mu(x_0)+\sigma(a_1)\mu(x_1)].\end{aligned}$$

(ii) σ is an anti-isomorphism. Then

$$\begin{aligned}&f_1(a_0x_0+a_1x_1)=f_1(x_0+a_0^{-1}a_1x_1)=F_1[\mu(x_0)+\sigma(a_0^{-1}a_1)\mu(x_1)]\\&=F_1[\sigma(a_1^{-1}a_0)\mu(x_0)+\mu(x_1)]=F_1[\mu(x_0)+\sigma(a_1)[\sigma(a_0)]^{-1}\mu(x_1)]\\&=F_1[\sigma(a_0)[\sigma(a_1)]^{-1}\mu(x_0)+\mu(x_1)]=F_1[[\sigma(a_1)]^{-1}\mu(x_0)+[\sigma(a_0)]^{-1}\mu(x_1)].\end{aligned}$$

Thus, for fixed $x_0 \in B$ and $\mu : B \longrightarrow \bar{X}_1$ we have defined the semi-isomorphism σ and the σ-semilinear (anti)-isomorphism μ, though for all $x \in \bar{X}$ it is true that $f_1(x) = F_1\mu(x)$.

Define the subring $K_1 \hookrightarrow F_1$ as follows: $f(kx_0) = K_1\mu(x_0)$. It is clear that K_1 is a k_1-submodule in F_1. Let us show that

(a) $f(kx_1) = K_1\mu(x_1)$,

(b) $f(k(a_0x_0 + a_1x_1)$

$$= \begin{cases} K_1([\sigma(a_1)]^{-1}\mu(x_0)+[\sigma(a_0)]^{-1}\mu(x_1) & \text{if } \sigma \text{ is an anti-isomorphism,} \\ K_1(\sigma(a_0)\mu(x_0)+\sigma(a_1)\mu(x_1) & \text{if } \sigma \text{ is an isomorphism.} \end{cases}$$

(a) The k-points kx_0, kx_1, $k(x_0+x_1)$, $k(x_0-x_1)$ are harmonic and $kx_0 \subset k(x_0 \pm x_1) + kx_1$. From this in general we cannot conclude that $f(kx_0) \subset f(k(x_0 \pm x_1)) + f(kx_1)$, though $F_1\mu(x_0) \subseteq F_1[\mu(x_0) \pm \mu(x_1)] + F_1\mu(x_1)$, and the points $F_1\mu(x_0)$, $F_1\mu(x_1)$, $F_1[\mu(x_0) + \mu(x_1)]$, $F_1[\mu(x_0) - \mu(x_1)]$ are harmonic. By the definition of the harmonic map, in the images $f(kx_0)$, $f(kx_1)$, $f(k(x_0 + x_1)$, $f(k(x_0 - x_1))$, we can find harmonic k_1-points

$$k_1[\alpha_1\mu(x_0)],\;\; k_1[\alpha_2\mu(x_1)],\;\; k_1[\alpha_3(\mu(x_0)+\mu(x_1))],\;\; k_1[\alpha_4(\mu(x_0)-\mu(x_1))].$$

In fact, we can choose the elements $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ in such a way that $\alpha_i = \alpha$, $i = 1, 2, 3, 4$:

$$k_1[\alpha_1\mu(x_0)] \subset k_1[\alpha_2\mu(x_1)] + k_1[\alpha_3(\mu(x_0) - \mu(x_1))] \Rightarrow \alpha_1 = \alpha_3\beta = \alpha_2,$$
$$k_1[\alpha_3(\mu(x_0) - \mu(x_1))] \subset k_1[\alpha_1\mu(x_0)] + k_1[\alpha_2\mu(x_1)] \Rightarrow \alpha_3 = \xi\alpha_1 = \gamma\alpha_2.$$

Consequently, β, ξ, γ are invertible elements so that $\alpha_1 = \alpha_2 = \alpha_3$. The same version is also true for α_4, i.e., $\alpha_i = \alpha_4$. It is obvious that we can choose α such that $k_1[\alpha\mu(x_0)] \subseteq k_1\mu(x_0) \subseteq f(kx_0)$.

Thus we have

$$\begin{array}{ccccc}
k_1[\alpha\mu(x_0)] & \subset & k_1[\alpha(\mu(x_0) \pm \mu(x_1))] & + & k_1[\alpha\mu(x_1)] \\
\cap| & & \cap| & & \cap| \\
k_1\mu(x_0) & & k_1[\mu(x_0) \pm \mu(x_1)] & & k_1\mu(x_1) \\
\cap & & \cap & & \cap \\
f(kx_0) & & f[k(x_0 \pm x_1)] & & f(kx_1) \\
\cap & & \cap & & \cap \\
F_1\mu(x_0) & \subset & F_1[\mu(x_0) \pm \mu(x_1)] & + & F_1\mu(x_1)\ .
\end{array}$$

Let $b \in K_1$ be an arbitrary element. Then we get

$$b[\alpha\mu(x_0)] = \alpha_1[\alpha(\mu(x_0) - \mu(x_1))] + \alpha_2[\alpha\mu(x_1)]$$

$$\Rightarrow b = \alpha_1 = \alpha_2 \Rightarrow b\mu(x_1) \in f(kx_1) \Rightarrow K_1\mu(x_1) \subseteq f(kx_1).$$

Suppose now that $c\mu(x_1) \in f(kx_1)$. Changing the roles to x_0 and x_1, we get $c\mu(x_0) \in f(kx_0) = K_1\mu(x_0) \Rightarrow c \in K_1 \Rightarrow f(kx_1) = K_1\mu(x_1)$. It is clear that $f(k(ax_i)) = K_1\mu(ax_i), i = 0, 1$ are true for arbitrary $a \in K$.

(b) Suppose that σ is an anti-isomorphism. The k-points

$$k(a_0x_0), \quad k(a_1x_1), \quad k(a_0x_0 + a_1x_1), \quad k(a_0x_0 - a_1x_1)$$

are harmonic. So we have

$$f(k(a_ix_i)) \hookrightarrow F_1\mu(a_ix_i), \quad i = 0, 1;$$
$$f[k(a_0x_0 \pm a_1x_1)] \hookrightarrow F_1([\sigma(a_1)]^{-1}\mu(x_0) + [\sigma(a_0)]^{-1}\mu(x_1)].$$

Let $b \in K_1$, then

$$\begin{aligned}
b\mu(a_0x_0) = b\sigma(a_0)\mu(x_0) &\in F_1([\sigma(a_1)]^{-1}\mu(x_0) \\
&+ [\sigma(a_0)]^{-1}\mu(x_1)) + F([\sigma(a_1)]^{-1}\mu(x_0) - [\sigma(a_0)]^{-1}\mu(x_1)) \\
&\Rightarrow b\sigma(a_0)\mu(x_0) = a_1([\sigma(a_1)]^{-1}\mu(x_0) + [\sigma(a_0)]^{-1}\mu(x_1)) \\
&+ a_2([\sigma(a_1)]^{-1}\mu(x_0) - [\sigma(a_0)]^{-1}\mu(x_1)) \\
&\Rightarrow a_1 = a_2 \Rightarrow b\sigma(a_0)\mu(x_0) = 2a_1[\sigma(a_1)]^{-1}\mu(x_0) \Rightarrow 2a_1 = b\sigma(a_0)\sigma(a_1) \\
&\Rightarrow b\sigma(a_0)\sigma(a_1)([\sigma(a_1)]^{-1}\mu(x_0) + [\sigma(a_0)]^{-1}\mu(x_1)) \\
&\in f[k(a_1a_0(a_0x_0 + a_1x_1))] \hookrightarrow F_1\sigma(a_0)\sigma(a_1)\big[[\sigma(a_1)]^{-1})\mu(x_0) \\
&+ [\sigma(a_0)]^{-1}\mu(x_1)\big] \Rightarrow K_1([\sigma(a_1)]^{-1}\mu(x_0) \\
&+ [\sigma(a_0)]^{-1}\mu(x_1)) \subseteq f[k(a_0x_0 + a_1x_1)].
\end{aligned}$$

On the other hand, if

$$c([\sigma(a_1)]^{-1}\mu(x_0) + [\sigma(a_0)]^{-1}\mu(x_1)) \in f[k(a_0x_0 + a_1x_1)],$$

then we get

$$\begin{aligned}
c[\sigma(a_1)]^{-1}\mu(x_0) &= c\mu(a_1^{-1}x_0) \in f[k\mu(a_1^{-1}x_0)] \\
&= K_1\mu(a_1^{-1}x_0) = K_1[\sigma(a_1)]^{-1}\mu(x_0); \\
c[\sigma(a_0)]^{-1}\mu(x_1) &\in K_1[\sigma(a_0)]^{-1}\mu(x_1) \Rightarrow c \in K \\
&\Rightarrow f[k(a_0x_0 + a_1x_1)] = K_1([\sigma(a_1)]^{-1}\mu(x_0) + [\sigma(a_0)]^{-1}\mu(x_1)).
\end{aligned}$$

The case where σ is an isomorphism is easier and can be proved by similar arguments.

If $\alpha \in k, x \in \bar{X} \setminus 0$, then we have

$$\begin{aligned}
K_1\sigma(\alpha)\mu(x) = K_1\mu(\alpha x) &= f[k(\alpha x)] \subseteq f(kx) \\
&= K_1\mu(x) \Rightarrow K_1\sigma(k) \subseteq K_1.
\end{aligned}$$

In general, the constructed subring and the maps μ and σ are not unique. If $0 \neq a \in F$, then $K_2 := K_1 a^{-1}$ is a k_1-submodule and $\mu_1 := a\mu$ is the semilinear (anti)-isomorphism with respect to $\sigma_1 = a\sigma a^{-1}$. In fact

$$K_2\sigma_1(k) = K_1 a^{-1} a\sigma(k) a^{-1} = K_1\sigma(k)a^{-1} = K_1 a^{-1} = K_2$$
$$\Rightarrow K_2\mu_1(x) = K_1 a^{-1} a\mu(x) = K_1\mu(x).$$

Consequently, there exists a ring K_1 such that $1 \in K_1$. In fact, K_1 and μ can be constructed up to a constant factor.

Thus the following inclusions are true:

$$\sigma(k) \hookrightarrow K_1, \quad \sigma(k) \hookrightarrow k_1 \hookrightarrow F_1.$$

By the definition of $f : \mathfrak{M}(x) \longrightarrow (X_1)$ we have $K_1\mu(X) \subseteq X_1$. Thus we prove

Theorem 1 (Representation of Harmonic Maps by the Semilinear Isomorphisms). *Let k be a non-commutative left principal ideal domain, $\frac{1}{2} \in k$ and X be a torsion-free module over k, $\dim_p X = 1$. If $f : P(X) \longrightarrow \mathfrak{M}_{k_1}(X_1)$ is a harmonic map, then there exist a semi-isomorphism $\sigma : F \longrightarrow F_1$, a σ-semilinear (anti)-isomorphism $\mu : \overline{X} \longrightarrow \overline{X}_1$ and a subring $K_1 \hookrightarrow F_1$, $1 \in K_1$, such that $K_1\mu(X) \subseteq X_1$,*

$$\begin{tikzcd}
\sigma(k) \arrow[r, hook] \arrow[d, hook] & K_1 \arrow[d, hook] \\
k_1 \arrow[r, hook] & F_1
\end{tikzcd}$$

and $f(kx) = K_1\mu(x)$ for all $x \in X$.

From the theorem we get: if $f : P(X) \to P(X_1)$ is a bijection, then $K_1 = k_1$ and $k_1\mu(X) = X_1$. So we have

Corollary (K. von Staudt's Theorem). *Let k be a noncommutative left principal ideal domain, $\frac{1}{2} \in k$; X be a torsion-free module over k, $\dim_p X = 1$. The bijection $f : P(X) \longrightarrow P(X_1)$ is harmonic if and only if there exist an isomophism or an anti-isomorphism $\sigma : k \longrightarrow k_1$ and σ-semilinear isomorphism $\mu : \overline{X} \longrightarrow \overline{X}_1$ such that $f(kx) = k_1\mu(x)$ for every $x \in X$.*

Proposition 7. *Let μ and μ_1 be the semilinear (anti)-isomorphisms with respect to σ, $\sigma_1 : F \longrightarrow F_1$ and $\dim \bar{X} \geq 2$. If K and K_1 ($1 \in K, K_1$) are subrings of F such that $K_1\mu_1(x) = K\mu(x)$ for all $x \in X$ and*

$$\begin{tikzcd}
K \arrow[d, hook] & \sigma(k) \arrow[l, hook'] \arrow[r, hook] \arrow[d, hook] & K_1 \arrow[d, hook] \\
F_1 & k_1 \arrow[l, hook'] \arrow[r, hook] & F_1
\end{tikzcd}$$

then there exists an element $a \in K$ such that

$$K_1 = Ka^{-1}, \qquad \mu_1 = a\mu.$$

For this suppose that the points Fx and Fy are distinct. Then there exist $a, b, c \in F$ such that

$$\begin{aligned}
&\mu_1(x) = a\mu(x), \ \ \mu_1(y) = b\mu(y), \ \ \mu_1(x+y) = c\mu(x+y) \\
&\Rightarrow a\mu(x) + b\mu(y) = \mu_1(x) + \mu_1(y) = \mu_1(x+y) \\
&= c\mu(x+y) \Rightarrow a = b = c \Rightarrow \mu_1(z) = a\mu(z).
\end{aligned}$$

Since $FX = \overline{X}$, we get $\mu_1(x) = a\mu(x)$ for all $x \in \bar{X}$. Let $x \in X \setminus 0$; then

$$K\mu(x) = K_1\mu_1(x) = K_1 a\mu(x) \Rightarrow K = K_1 a, K_1 = Ka^{-1}.$$

As $1 \in K_1$, it is clear that $a \in K$.

Some of results presented here were announced in [40], [41].

To conclude we would like to note that the remarkable comprehensive monograph [42] has recently been published, describing the state of the art of this field and prospects for further study.

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