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AN OSCILLATION CRITERION FOR NONLINEAR THIRD-ORDER DIFFERENTIAL EQUATIONS

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Abstract. Sufficient conditions for oscillation of a certain class of nonlinear third-order differential equations are found.

In this paper we considear a nonlinear third-order differential equation of the form

$$y''' + q(t)y' = f(t, y, y', y'') \tag{1}$$

where

(i) $q, q' \in C\big((a,\infty)\big)$ for some a with $0 < a < \infty$,
(ii) $f \in C\big((a,\infty) \times R^3\big)$, $t \in (a,\infty)$, and $y_1, y_2, y_3 \in R$ and $f(t, y_1, y_2, y_3)y_1 < 0$ for all $t \in (a,\infty)$ and all $y_1, y_2, y_3 \in R$ with $y_1 \neq 0$.

By a solution of (1) (proper solution) we mean a function y defined on an interval $[t_0,\infty)$, $t_0 > a$, which has a continuous third derivative with $\sup(|y(s)| : s > t) > 0$ for any $t \in [t_0,\infty)$, and satisfies equation (1). By an oscillatory solution we mean a solution y of (1) that has arbitrarily large zeros. Otherwise the solution is said to be nonoscillatory.

The aim of this paper is to study the oscillatory properties of proper solutions of equation (1) when the operator on the left-hand side of equation (1) is oscillatory. I. T. Kiguradze investigated in [1] the equation

$$u^{(n)} + u^{(n-2)} = f\big(t, u, u', \dots, u^{(n-1)}\big) \tag{2}$$

and our aim is to generalize one of his results concerning equation (1).

Papers [2], [3] investigated similar problems. The equation in [1] is a special case of (1) and in [3] there is an equation with a more general linear operator on the left-hand side, but with a special case of nonlinearity.

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In this paper we use some results of the theory of a third-order linear differential equation [4].

1. First of all we list some results and lemmas and prove one lemma for the linear differential equation

$$y''' + 2A(t)y' + [A'(t) + b(t)]y = 0, \tag{3}$$

where A' and b are continuous functions on (a, ∞) and $b(t) \geq 0$ for $t \in (a, \infty)$ such that $b(t) \not\equiv 0$ on each subinterval.

Let ν_1, ν_2 be linearly independent solutions of the differential equation

$$\nu'' + \frac{1}{2} A\nu = 0. \tag{4}$$

It is known [2] that ν_1^2, ν_2^2,, $\nu_1\nu_2$ form a fundamental set of solutions of the self-adjoint third-order differential equation

$$u''' + 2Au' + A'u = 0. \tag{5}$$

Lemma 1 ([4, Lemma 2.3]). *Let y be a solution of* (3) *defined on* $[t_0, \infty)$, $t_0 > a$, *with* $y(t_0) = y_0$, $y'(t_0) = y_0'$, $y''(t_0) = y_0''$. *Then it can be rewritten in the form*

$$y(t) = u(t) - \int_{t_0}^{t} b(\tau)y(\tau)W(t,\tau)\,d\tau \tag{6}$$

where u is the solution of (5), *and has the same initial values at t_0 as y,*

$$W(t,\tau) = \begin{vmatrix} u_1(t), & u_2(t), & u_3(t) \\ u_1(\tau), & u_2(\tau), & u_2(\tau) \\ u_1'(\tau), & u_2'(\tau), & u_3'(\tau) \end{vmatrix}$$

and u_1, u_2, u_3 form a fundamental set of solutions of equation (5) *with the wronskian equal to* 1 *on* (a, ∞).

Lemma 2 ([4, Corollary 2.3]). *Let the second-order differential equation* (4) *have oscillatory solutions in* $[t_0, \infty)$, $t_0 > a$. *Then the differential equation* (3) *is oscillatory in* $[t_0, \infty)$, *i.e., every solution of* (3) *having a zero is oscillatory in* $[t_0, \infty)$.

Lemma 3 ([4, Theorem 3.6]). *Let* $A(t) \geq 0$, $A'(t) + b(t) \geq d > 0$, $b(t) - A'(t) \geq 0$ *for* $t \in (a, \infty)$. *Then every solution of the differential equation* (3) *is oscillatory in* (a, ∞), *except a solution y (unique up to linear dependence), which satisfies* $y(t) \in L^2[t_0, \infty)$ (*i.e.*, $\int_{t_0}^{\infty} y^2(t)dt < \infty$), $y \to 0$, $y' \to 0$ *as* $t \to 0$, $a < t_0 < \infty$.

Lemma 4 ([4, Theorem 2.17]). *Let the differential equation* (3) *have at least one oscillatory solution in* (a,∞). *A necessary and sufficient condition for a non-trivial solution* y *of* (3) *to be non-oscollatory in* $[t_0,\infty)$, $t_0 > a$ *is that* $F(y(t)) = y(t)y''(t) - \frac{1}{2}y'^2(t) + A(t)y^2(t) > 0$ *for* $t \geq t_0$.

Lemma 5. *Let* $A(t) \geq m > 0$, $A'(t) \leq 0$, $A'(t) + b(t) \geq \frac{d}{t} > 0$ *for* $t \in (a,\infty)$, $a > 0$. *Then every solution of* (3) *is oscillatory in* (a,∞), *except a solution* y (*unique up to linear dependence*), *which satisfies* $y \to 0$, $y' \to 0$, $y'' \to 0$ *as* $t \to \infty$.

Proof. From the supposition $A(t) \geq m > 0$ for $t \in (a,\infty)$ and from Lemma 2 it follows that every solution of (3) with one zero is oscillatory in (a,∞).

Let y be a nonoscillatory solution on (3) and let $y(t) > 0$ for $t \in (a,\infty)$. It fulfills the integral identity

$$F[y(t)] \equiv yy' - \frac{1}{2}\,y'^2 + Ay^2 = k - \int_{t_1}^{t} by^2 d\tau, \tag{7}$$

where $k = y(t_1)y''(t_1) - \frac{1}{2}y'^2(t_1) + A(t_1)y^2(t_1)$, $t_1 > a$.

We obtain identity (7) by multiplying equation (3) by y and integrating term by term from t_1 to t.

Lemma 4 implies that $F[y(t)] > 0$ for $t \in [t_1,\infty)$ and therefore $k > 0$. From (7) it follows that $\int_{t_1}^{\infty} b(\tau)y^2(\tau)d\tau$ converges and from the suppositions $A'(t) + b(t) \geq \frac{d}{t} > 0$, $A'(t) \leq 0$ we have $b(t) - A'(t) \geq \frac{d}{t}$ and at the same time $b(t) \geq \frac{d}{2t}$ for $t > a$. Then we have

$$\infty > \int_{t_1}^{\infty} b(\tau)y^2(\tau)d\tau \geq \frac{d}{2}\int_{t_1}^{\infty} \frac{y^2(\tau)}{\tau} d\tau > 0$$

and therefore

$$\lim_{t\to\infty} \inf y(t) = 0. \tag{8}$$

From (7) it is clear that

$$0 < y''(t) + A(t)y(t) < y''(t) + 2A(t)y(t) \tag{9}$$

for all $t \in [t_1,\infty)$.

Integrating equation (3) term by term from t_1 to t we obtain the identity

$$y'' + 2Ay = k_1 - \int_{t_1}^{t} (b - A')y d\tau, \tag{10}$$

where $k_1 = y''(t_1) + 2A(t_1)y(t_1) > 0$ and therefore $y'' + 2Ay$ is a decreasing function of $t \in [t_1, \infty)$. It follows from (10) and from the fact that $2A(t)y(t) > 0$ for $t > t_1$, that y'' is a function bounded from above on $[t_1, \infty)$. Suppose that y does not converge to zero for $t \to \infty$. From (8) it follows that there exists a sequence $\{t_n\}_{n=1}^{\infty}$, $t_n \to \infty$, such that $y'(t_n) = 0$, $y''(t_n) \geq 0$ for $n = 1, 2, \dots$, and hence we obtain from identity (7) for $t = t_n$ and for $t_n \to \infty$ that $k = \int_{t_1}^{\infty} by^2 dt$ so that identity (7) can be rewritten in the form

$$yy'' - \frac{1}{2}y'^2 + Ay^2 = \int_t^{\infty} by^2 d\tau. \tag{11}$$

Let $\{\bar{t}_n\}_{n=1}^{\infty}$, $\bar{t}_1 > t_1$ be a sequence tending to infinity such that $y'(\bar{t}_n) = 0$, $y''(\bar{t}_n) \leq 0$. Let us prove that $y(\bar{t}_n) \to 0$ for $n \to= \infty$. Two cases are possible:

(a) Let $y''(t) + A(t)y(t) \to 0$ for $t \to \infty$. In this case $y(t_n) \to 0$ for $n \to \infty$, $A'(t) \leq 0$ for $t \in (a, \infty)$ and therefore $y''(t_n) \to 0$ for $n \to \infty$. Identity (10) is in this case of the form

$$y'' + 2Ay = \int_t^{\infty} (b - A')y d\tau \tag{12}$$

because $k_1 = \int_{t_1}^{\infty} (b - A')y d\tau$. Then there is

$$y''(\bar{t}_n) + A(\bar{t}_n)y(\bar{t}_n).$$

If we suppose that $\lim_{n\to\infty} y(\bar{t}_n) > 0$, we obtain for $\bar{t}_n \to \infty$ a contradiction and therefore $y(\bar{t}_n) \to 0$ for $\bar{t}_n \to \infty$.

(b) Let $y'' + Ay$ no limit equal to zero at infinity. The function $y'' + 2Ay$ is decreasing and in this case $\lim_{t\to\infty}[y''(t) + 2A(t)y(t)] = \alpha > 0$ and therefore

$$\int_{t_1}^{\infty} \frac{y''(\tau) + 2A(\tau)y(\tau)}{\tau} d\tau \leq \int_{t_1}^{\infty} \frac{\alpha}{\tau} d\tau = \infty. \tag{13}$$

Identity (7) implies that $F[y(t)] > 0$ for all $t > t_1$, and then applying identity (10) we get

$$\begin{aligned} 0 > -2F[y(t)] &= y'^2(t) + 2A(t)y^2(t) - 2y(t)[y''(t) + 2A(t)y(t)] > \\ &> y'^2(t) + 2my^2(t) - 2k_1 y(t) \end{aligned}$$

and therefore we have $my^2(t) - k_1 y(t) < 0$, $y'^2(t) - 2k_1 y(t) < 0$ for $t > t_1$ from which $y(t) < \frac{k_1}{m}$ and $|y'(t)| < \frac{\sqrt{2}k_1}{\sqrt{m}}$ for all $t > t_1$. From identity (10) it follows that $\int_{t_1}^{\infty} \frac{y(\tau)}{\tau} d\tau < \infty$. It is ease to prove that the integrals $\int_{t_1}^{\infty} \frac{y'(\tau)}{\tau} d\tau$

and $\int_{t_1}^{\infty} \frac{y''(\tau)}{\tau} d\tau$ converge, too. Applying this result to the function $\frac{y''+2Ay}{t}$ we get

$$\int_{t_1}^{\infty} \frac{y''(\tau)+2A(\tau)y(\tau)}{\tau} d\tau = \int_{t_1}^{\infty} \frac{y''(\tau)}{\tau} d\tau + \int_{t_1}^{\infty} \frac{2A(\tau)y(\tau)}{\tau} d\tau$$

and this is in contradiction with (13) and therefore we have $y(t) \to 0$ for $t \to \infty$. Now it is necessary to prove that $y'(t) \to 0$ for $t \to \infty$. It follows from the above identity, i.e., $y'^2(t) - 2k_1 y(t) < 0$. It remains to prove that $y''(t) \to 0$ for $t \to \infty$.

From the inequality $y''(\bar{t}_n) + 2A(\bar{t}_n)y(\bar{t}_n) > 0$ for $n = 1, 2, \ldots,$ and from (10) for $t = \bar{t}_n$ it follows that $y''(\bar{t}_n) \to 0$ for $n \to \infty$ and, since $k_1 = \int_{t_1}^{\infty} (b - A')y d\tau$, identity (10) is in this case of the form (12). From (12), for $t \to \infty$, we obtain $y''(t) \to 0$.

The uniqueness of solutions without zeros can be proved by the same arguments as in Theorem 3.6 [4] and therefore the proof of the uniqueness is omitted. □

Remark 1. If in Lemma 3 we suppose $A(t) \geq m > 0$ and $A'(t) \leq 0$ for every $t > a$, then we can prove by the same arguments as in Lemma 5 that $y''(t) \to 0$ for $t \to \infty$.

3. The aim of this section is to generalize the following results of Kiguradze [1] for $n = 3$.

Theorem A [1, Corollary 1.5]. *Let f have the property* (ii) *and let $f(t, y_1, y_2, y_3) \operatorname{sgn} y_1 \leq -\frac{\varepsilon}{t}|y_1|$, where $\varepsilon > 0$, $t > 0$. Then the differential equation*

$$y''' + y' = f(t, y, y', y'')$$

has the property A, i.e., each of its solutions is either oscillatory or satisfies the conditions $y(t) \to 0$, $y' \to 0$ and $y'' \to 0$ for $t \to \infty$.

The following simple result proves that equation (1) can have oscillatory solutions if the operator on the left-hand side of (1) is oscillatory.

Theorem 1. *Let q and q' have the property* (i) *and let $q(t) \geq 0$, $q'(t) \leq 0$ for $t \in (a, \infty)$. Let further the equation*

$$\nu'' + \frac{1}{4}q(t)\nu = 0 \tag{14}$$

be oscillatory on (a, ∞) and let the function f have the property (ii)*. Then every solution y of equation* (1) *defined on $[t_0, \infty)$, $t_0 > a$, with one zero is oscillatory on $[t_0, \infty)$.*

Proof. From the supposition that equation (14) is oscillatory and from Lemma 2 it follows that the self-adjoint equation

$$u''' + q(t)u' + \frac{1}{2}q'(t)u = 0 \tag{15}$$

is oscillatory on (a, ∞), i.e., each solution of (15) with one zero at $t_0 > a$ is oscillatory on $[t_0, \infty)$. On the other hand, if ν_1, ν_2 form a fundamental set of solutions of (14), then ν_1^2, ν_2^2, $\nu_1\nu_2$ form a fundamental set of solutions of (15). Equation (1) can be rewritten in the form

$$y''' + q(t)y' + \frac{1}{2}q'(t)y = f(t, y, y', y'') + \frac{1}{2}q'(t)y. \tag{16}$$

Applying Lemma 1 to equation (16) we obtain the relation

$$y(t) = u(t) + \int_{t_0}^{t} [f(\tau, y(\tau), y'(\tau), y''(\tau)) + \frac{1}{2}q'(\tau)y(\tau)]W(t,\tau)d\tau, \tag{17}$$

where the solution y of (1) and the solution u of (15) satisfy the same initial condition at $t_0 > a$ and $W(t,\tau)$ is of the form of Lemma 1. Clearly, $W(t,\tau) \geq 0$ for $t \geq \tau \geq t_0$.

Let $y(t_0) = 0$, $y'(t_0) = y_0'$, $y''(t_0) = y_0''$ and let at least one of the numbers y_0', y_0'' be different from zero. Clearly, $u(t_0) = 0$ and u is oscillatory. Let $t_1 > t_0$ be the first zero of u to the right of t_0. Then from (15) we obtain a contradiction. If $y(t_0) = y'(t_0) = y_0''(t_0) = 0$, then $u(t) \equiv 0$ and then either $y(t) \equiv 0$ for $t > t_0$, or it must have at least one zero to the right of t_0. This follows from (17). Identity (7) for equation (1) and for the above solution y has the form

$$yy'' - \frac{1}{2}y'^2 + \frac{1}{2}qy^2 = \int_{t_0}^{t} [f(\tau, y(\tau), y'(\tau), y''(\tau))y(\tau) + \frac{1}{2}q'y^2(\tau)]d\tau.$$

From this identit it follows that y cannot have a double zero to the right of t_0. If t_k is the last zero of y to the right of t_0, then applying Lemma 1 we obtain the assertion of Theorem 1 in this case, too. □

Let $\varphi \in C^2((a,\infty))$ and let $\varphi(t) \neq 0$ for $t \in (a, \infty)$. Construct the linear differential equation

$$\begin{aligned} y''' + q(t)y' + \Big[\frac{1}{2}q'(t) - \\ -\Big(\frac{f(t,\varphi(t),\varphi'(t),\varphi''(t))}{\varphi(t)} + \frac{1}{2}q'(t)\Big) + \frac{1}{2}q'(t))\Big]y = 0. \end{aligned} \tag{18}$$

Applying Lemma 3 we prove

Theorem 2. *Let* (i), (ii) *hold. Let further* $q(t) \geq 0$, $-\frac{f(t,\varphi(t),\varphi'(t),\varphi''(t))}{\varphi(t)} \geq d > 0$, *and* $\frac{f(t,\varphi(t),\varphi'(t),\varphi''(t))}{\varphi(t)} + q'(t) \leq 0$ *for* $t \in (a,\infty)$ *and for every* $\varphi \in C^2((a,\infty))$, $\varphi \neq 0$.

Then every solution y *of* (1) *is either oscillatory or* $y(t) \neq 0$ *on* $[t_0,\infty)$, $t_0 > a$ *and then it has the following properties:* $\int_{t_0}^{\infty} y^2(t)dt < \infty$, $y \to 0$, $y' \to 0$ *for* $t \to \infty$.

Proof. Let y_1 be a solution of (1) and let $y_1(t) \neq 0$ for $t \in [t_0,\infty)$, $t_0 > a$. Put $\varphi(t) = y_1(t)$ for $t \in [t_0,\infty)$ into equation (18). Then equation (18) has only one solution without zeros (up to linear dependence). y_1 is a solution of (17) without zeros on $[t_0,\infty)$ and at the same time it is a solution of (1) and therefore $\int_{t_0}^{\infty} y_1^2(t)dt < \infty$, $y_1 \to 0$, $y_1' \to 0$ for $t \to \infty$. □

Corollary 1. *Let the suppositions of Theorem* 2 *be fulfilled and let, moreover,* $q'(t) \leq 0$ *and* $q(t) \geq m > 0$ *for* $t \in (a,\infty)$. *Then every solution of* (1) *with one zero on* (a,∞) *is oscillatory to the right of this zero and every solution* y *without zeros on* $[t_0,\infty)$ *has the properties* $y \to 0$, $y' \to 0$, $y'' \to 0$ *for* $t \to \infty$ *and* $\int_{t_0}^{\infty} y^2(t)dt < \infty$.

Proof. Theorem 1 implies that every solution of (1) with one zero is oscillatory to the right of this zero and the properties of solutions y without zeros follow from Theorem 2 and Remark 1. □

Remark 2. Corollary 1 defines a set of differential equations of the form (1) that have the property A (Theorem A). It is a certain generalization of the result of Kiguradze [1].

A complete generalization of Theorem A of Kiguradze [1] gives the following theorem proved by Lemma 5.

Theorem 3. *Let* (i), (ii) *hold. Let* $q'(t) \leq 0$, $q(t) \geq m > 0$ *for* $t \in (a,\infty)$, $a > 0$. *Let further* $-\frac{f(t,\varphi(t),\varphi'(t),\varphi''(t))}{\varphi(t)} \geq \frac{d}{t} > 0$ *for* $t \in (a,\infty)$ *and for every* $\varphi \in C^2((a,\infty))$, $\varphi(t) \neq 0$ *for* $t \in (a,\infty)$. *Then every solution of* (1) *with one zero is oscillatory to the right of this zero and every solution* y *of* (1) *defined on* $[t_0,\infty)$, $t_0 > a$, *without zeros on this interval has the property* $y \to 0$, $y' \to 0$, $y'' \to 0$ *for* $t \to \infty$ (*i.e., equation* (1) *has the property A*).

Proof. From Theorem 1 it follows that every solution of (1) with one zero is oscillatory to the right of this zero.

Let y_1 be a solution of (1) defined on $[t_0,\infty)$, $t_0 > a$ and let $y_1(t) \neq 0$ on this interval. If we put $\varphi(t) = y_1(t)$ into equation (18), then we see that the coefficients of this equation fulfill the suppositions of Lemma 5 and therefore equation (18) has only one solution without zeros on $[t_0,\infty)$ (up to a linear dependence). y_1 is a solution of (18) without zeros on $[t_0,\infty)$ and at the same time it is a solution of (1). By Lemma 5 y_1 has the property $y \to 0$, $y' \to 0$, $y'' \to 0$ for $t \to \infty$. $\square$

Remark 3. Theorem A is a special case of Theorem 3.

References

1. I. T. Kiguradze, An oscillation criterion for a class of ordinary differential equations. (Russian) *Differentsyal'nye Uravnenia* **28**(1992), No. 2, 207–220.

2. M. Greguš, On the oscillatory behavior of certain third-order nonlinear differential equation. *Arch. Math.* (*Brno*) **28**(1992), No. 3–4, 221–228.

3. M. Greguš and J. R. Greaf, On a certain nonautonomous nonlinear third-order differential equation. (*To appear*).

4. M. Greguš, Third-order linear differential equations. *Reidel Publishing Company, Dordrecht,* 1982.

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