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NONSTATIONARY INITIAL BOUNDARY VALUE CONTACT PROBLEMS OF GENERALIZED ELASTOTHERMODIFFUSION

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Abstract. The initial boundary value problems with mixed boundary conditions are considered for a system of partial differential equations of generalized electrothermodiffusion. Approximate solutions are constructed and a mathematical substantiation of the method is given.

It is well known that a wide range of problems of mathematical physics includes boundary value and initial boundary value problems for differential equations. Such problems are rather difficult to solve because of an enormous variety of geometrical forms of the investigated objects and the complexity of boundary conditions. Hence an important and timely task has arisen to develop sufficiently effective methods and tools for solving the above-mentioned problems and obtaining their numerical solutions.

Until recently no methods were known for solving problems of elasticity by means of conjugate fields. However, in the past few years there have appeared and keep on appearing numerous published works dedicated to this topic. The results of our studies in this direction are presented mainly in [1, 2, 3].

In this paper, using the generalized Green–Lindsay theory of elastothermodiffusion as an example, the approaches to the solution of nonstationary initial boundary value contact problems with mixed boundary conditions are described for a system of partial differential equations of this theory in a nonhomogeneous medium. These approaches are based on the method of discrete singularities known as the Riesz–Fischer–Kupradze method. The particular case is treated in [4].

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Let us consider a three-dimensional isotropic elastic medium in which the thermodiffusive process is taking place. The deformed state is described by the displacement vector $v(x,t) = v = (v_1, v_2, v_3)^T = \|v_k\|_{3\times 1}$, temperature change $v_4(x,t)$, and "chemical potential of the medium" $v_5(x,t)$. Here $x = (x_1, x_2, x_3)$ is a point of the Euclidean space $\mathbb{R}^3$, t the time, $C(x,t) = \gamma_2 \operatorname{div} v(x,t) + a_{12} v_4(x,t) + a_2 v_5(x,t)$, $C(x,t)$ the diffusing substance concentration, and T denotes the operation of transposition.

The object of our investigation is a system of partial differential equations of the generalized theory of elastothermodiffusion having the form [4]

$$L\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\Big) V(x,t) = H(x,t), \tag{1}$$

where

$$V = (v_1, v_2, \dots, v_5)^T = (v, v_4, v_5)^T = \|v_k\|_{5\times 1}$$

is an unknown vector V

$$LV = \big\|(LV)_k\big\|_{5\times 1},$$

$$(LV)_k = \begin{cases} \mu \Delta v_k + (\lambda+\mu)\dfrac{\partial}{\partial x_k} \operatorname{div} v - \rho \dfrac{\partial^2 v}{\partial t^2} + \\ \qquad - \displaystyle\sum_{l=1}^{2} \gamma_l \Big(1 + \tau^1 \frac{\partial}{\partial t}\Big) \frac{\partial v_{3+l}}{\partial x_k}, \quad k = 1,2,3, \\ \delta_{k-3} \Delta v_k - a_{k-3}\Big(1 + \tau^0 \dfrac{\partial}{\partial t}\Big)\dfrac{\partial v_k}{\partial t} - \gamma_{k-3} \dfrac{\partial}{\partial t} \operatorname{div} v - \\ \qquad - a_{12} \Big(1 + \tau^0 \dfrac{\partial}{\partial t}\Big) \dfrac{\partial v_{9-k}}{\partial t}, \quad k = 4,5; \end{cases}$$

$H(x,t) = \big(X(x,t), X_4(x,t), X_5(x,t)\big)^T = (X_1, \dots, X_5)^T = \|X_k\|_{5\times 1}$ is a given vector-function; the elastic, thermal, diffusive, and relaxational constants τ^0, τ^1 satisfy the natural restrictions [3]:

$$\mu > 0, \quad 3\lambda + 2\mu > 0, \quad \rho > 0, \quad a_k > 0, \quad \delta_k > 0, \quad \gamma_k > 0, \quad k = 1,2,$$
$$a_1 a_2 - a_{12}^2 > 0, \quad \tau^1 \geq \tau^0 > 0.$$

In particular, the classical case is realized for $\tau^1 = \tau^0 = 0$.

In this theory the differential operator of stresses which is a 5×5 matrix is determined by the formula

$$R_\tau\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}, n(x)\Big) V(x,t) =$$
$$= \bigg(T\Big(\frac{\partial}{\partial x}, n(x)\Big) v(x,t) - n(x) \sum_{l=1}^{2} \gamma_l \Big(1 + \tau^1 \frac{\partial}{\partial t}\Big) v_{3+l}, \delta_1 \frac{\partial v_4}{\partial n}, \delta_2 \frac{\partial v_5}{\partial n}\bigg)^T,$$

where $T(\frac{\partial}{\partial x}, n(x)) \equiv \|\mu\delta_{jk}\frac{\partial}{\partial n} + \lambda n_j \frac{\partial}{\partial x_k} + \mu n_k \frac{\partial}{\partial x_j}\|_{3\times 3}$ is the matrix differential operator of elastic stresses in the classical theory [1], $n(x) = (n_1(x), n_2(x), n_3(x))$ the unit vector δ_{jk} the Kronecker symbol.

Clearly,

$$R_\tau\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}, n(x)\Big)V \equiv R_\tau\Big(\frac{\partial}{\partial x}, 0, n(x)\Big)V - \sum_{k=1}^{2} r^k(\tau)\frac{\partial v_{3+k}}{\partial t},$$

where $r^k(\tau) = \gamma_k \tau^1(n; 0, 0)^T$.

Assume that D is a finite domain with the boundary S which is a compact closed surface of the Liapunov class $\Lambda_2(\alpha)$, $\alpha > 0$ [1]; $D_{11} \subset D$ and $D_{12} \subset D$ are the nonintersecting domains with the Liapunov boundaries S_{11} and S_{12}, respectively; D_{21} and D_{22} are the nonintersecting finite domains outside S with the boundary Liapunov surfaces S_{21} and S_{22}, respectively; D_1 is a finite domain bounded by the surfaces $S \cup S_{11} \cup S_{12}$; D_2 is an infinite domain bounded by the surfaces $S \cup S_{21} \cup S_{22}$. Consider the infinite domain $D_1 \cup S \cup D_2 \equiv \mathbb{R}^3 \setminus \bigcup_{l,k=1}^{2} \overline{D}_{lk}$; S is the contacting surface.

Let D_1 and D_2 be two different homogeneous isotropic physical media. The elastothermodiffusive constants for D_j will be denoted by the lower left indices ${}_j\lambda, {}_j\mu, {}_j\rho, {}_j\gamma_l, \dots, {}_j\tau^1, {}_j\tau^0, \dots$ while the differential operators by ${}_jL(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}), {}_jR_\tau(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}, n(x)), \dots$, where $j = 1, 2$.

Problem A^t. In the infinite cylinder

$$Z_\infty^{1,2} = \Big\{(x,t) : \ x \in D_1 \cup D_2, \ t \in]0, \infty[\Big\}$$

determine a regular vector $V = (v, v_4, v_5)^T : Z_\infty^{1,2} \to \mathbb{R}^5$ of the class $C^1(\overline{Z}_\infty^{1,2}) \cap C^2(Z_\infty^{1,2})$ using the boundary conditions

$$\begin{aligned}
&\forall (x,t) \in Z_\infty^{1,2} : {}_jL\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\Big)V(x,t) = {}_jH(x,t), \quad x \in D_j, \ \ j = 1, 2,\\
&\forall x \in D_j : \ \lim_{t\to+0} V(x,t) = {}_j\varphi^{(0)}(x),\\
&\qquad \lim_{t\to+0} \frac{\partial V(x,t)}{\partial t} = {}_j\varphi^{(1)}(x), \ \ j = 1, 2,\\
&\forall y \in S, \quad t \geq 0 : [V]_S^{\pm} \equiv V^+(y,t) - V^-(y,t) = f(y,t)\\
&\qquad [R_\tau V]_S^{\pm} \equiv \Big[{}_1R_\tau\Big(\frac{\partial}{\partial y}, \frac{\partial}{\partial t}, n\Big)V\Big]^+ - \\
&\qquad -\Big[{}_2R_\tau\Big(\frac{\partial}{\partial y}, \frac{\partial}{\partial t}, n\Big)V(y,t)\Big]^- = F(y,t),\\
&\forall y \in S_{j1}, \quad t \geq 0 : V^+(y,t) = {}_jf(y,t), \quad j = 1, 2,
\end{aligned} \tag{2}$$

$$\forall y \in S_{j2}, \quad t \geq 0 : \Big[{}_jR_\tau\Big(\frac{\partial}{\partial y}, \frac{\partial}{\partial t}, n\Big)V\Big]^+ = {}_jF(y,t), \quad j = 1,2;$$

for large t and $x \in D_2$:

$$\big|D^\alpha_{x,t}V(x,t)\big| \leq \frac{\operatorname{const} e^{\sigma_0 t}}{1+|x|^{1+|\alpha|}}\,, \quad |\alpha| = \overline{0,2}, \quad \sigma_0 \geq 0,$$

$$D^\alpha_{x,t} \equiv \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}\partial t^{\alpha_4}}\,, \quad |\alpha| = \sum_{k=1}^4 \alpha_4,$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is the multi-index.

Here

$$V^+(y,t) = \lim_{D_1 \ni x \to y \in S} V(x,t),$$
$$V^-(y,t) = \lim_{D_2 \ni x \to y \in S} V(x,t),$$
$$\big[{}_1R_\tau V\big]^+ = \lim_{D_1 \ni x \to y \in S} {}_1R_\tau\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}, n(y)\Big)V(x,t),$$
$$\big[{}_2R_\tau V\big]^- = \lim_{D_2 \ni x \to y \in S} {}_2R_\tau\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}, n(y)\Big)V(x,t).$$

In a similar manner we determine the limit values on the surfaces S_{jk}, $j,k = 1,2$; here ${}_j\varphi^{(0)}, {}_j\varphi^{(1)} : D_j \to \mathbb{R}^5$, $f, F : S \times [0,\infty[\to \mathbb{R}^5$, ${}_jH : Z^{1,2}_\infty \to \mathbb{R}^5$, ${}_jf : S_{j1} \times [0,\infty[\to \mathbb{R}^5$, ${}_jF : S_{j2} \times [0,\infty[\to \mathbb{R}^5$ are given real-valued vector-functions of the definite classes [6].

For a classical (regular) solution to exist it is necessary that the condition of "natural agreement" of the initial data be fulfilled. Here these conditions have the form

$$\begin{aligned}
\forall y \in S : {}&{}_1\varphi^{(0)}(y) - {}_2\varphi^{(0)}(y) = f(y,0),\\
&{}_1\varphi^{(1)}(y) - {}_2\varphi^{(1)}(y) = \frac{\partial f(y,t)}{\partial t}\Big|_{t=0},\\
&\Big[{}_1R_\tau\Big(\frac{\partial}{\partial y}, 0, n\Big){}_1\varphi^{(0)}(y) - \sum_{k=1}^2 {}_1r^k(\tau){}_1\varphi^{(1)}_{3+k}(y)\Big] -\\
&-\Big[{}_2R_\tau\Big(\frac{\partial}{\partial y}, 0, n\Big){}_2\varphi^{(0)}(y) - \sum_{k=1}^2 {}_2r^k(\tau){}_2\varphi^{(1)}_{3+k}(y)\Big] = F(y,0),
\end{aligned} \tag{3}$$

$$\forall y \in S_{j1} : {}_j\varphi^{(0)}(y) = {}_jf(y,0), \quad {}_j\varphi^{(1)}(y) = \frac{\partial_j f(y,t)}{\partial t}\Big|_{t=0}, \quad j = 1,2,$$

$$\forall y \in S_{j2} : {}_jR_\tau\Big(\frac{\partial}{\partial y}, 0, n\Big){}_j\varphi^{(0)}(y) -$$

$$-\sum_{k=1}^{2} {}_j r^k(\tau)\, {}_j\varphi^{(1)}_{3+k}(y) = {}_jF(y,0), \quad j=1,2.$$

To investigate the dynamic problem A^t we shall use the Laplace transform with respect to t. However, in our case these "natural conditions" are not sufficient for substantiating the method. So we must additionally use the "higher order" agreement conditions [2, 3] (with $m=\overline{1,7}$):

$$\begin{aligned}
\forall y\in S:\ & \left.\frac{\partial^m f(y,t)}{\partial t^m}\right|_{t=0} = {}_1\varphi^{(m)}(y) - {}_2\varphi^{(m)}(y),\\
& \left.\frac{\partial^m F(y,t)}{\partial t^m}\right|_{t=0} = \Big[{}_1R_\tau\Big(\frac{\partial}{\partial y},0,n\Big){}_1\varphi^{(m)}(y) -\\
&\qquad - \sum_{k=1}^{2} {}_1r^k(\tau)\,{}_1\varphi^{(m+1)}_{3+k}(y)\Big] -\\
&\qquad - \Big[{}_2R_\tau\Big(\frac{\partial}{\partial y},0,n\Big){}_2\varphi^{(m)}(y) -\\
&\qquad - \sum_{k=1}^{2} {}_2r^k(\tau)\,{}_2\varphi^{(m+1)}_{3+k}(y)\Big],\\
\forall y\in S_{j1}:\ & \left.\frac{\partial^m {}_jf(y,t)}{\partial t^m}\right|_{t=0} = {}_j\varphi^{(m)}(y), \quad j=1,2,\\
\forall y\in S_{j2}:\ & \left.\frac{\partial^m {}_jF(y,t)}{\partial t^m}\right|_{t=0} = {}_jR_\tau\Big(\frac{\partial}{\partial y},0,n\Big){}_j\varphi^{(m)}(y) -\\
&\qquad - \sum_{k=1}^{2} {}_jr^k(\tau)\,{}_j\varphi^{(m+1)}_{3+k}(y), \quad j=1,2,
\end{aligned} \tag{4}$$

where $(m\geq 2)$

$$\begin{aligned}
&\Big({}_j\varphi_1^{(m)}(x), {}_j\varphi_2^{(m)}(x), {}_j\varphi_3^{(m)}(x)\Big)^T =\\
&= {}_j\rho^{-1}\Big[{}_j\mu\Delta\Big({}_j\varphi_1^{(m-2)}, {}_j\varphi_2^{(m-2)}, {}_j\varphi_3^{(m-2)}\Big)^T +\\
&+({}_j\lambda+{}_j\mu)\,\mathrm{grad}\,\mathrm{div}\Big({}_j\varphi_1^{(m-2)}, {}_j\varphi_2^{(m-2)}, {}_j\varphi_3^{(m-2)}\Big)^T -\\
&-\sum_{k=1}^{2} {}_j\gamma_k\,\mathrm{grad}\,{}_j\varphi^{(m-2)}_{3+k} - \sum_{k=1}^{2} {}_j\gamma_k\,{}_j\tau^1\,\mathrm{grad}\,{}_j\varphi_{3+k} - \left.\frac{\partial^{m-2}{}_jX}{\partial t^{m-2}}\right|_{t=0}\Big],\\
&{}_ja_1\,{}_j\tau^0\,{}_j\varphi_4^{(m)}(x) + {}_ja_{12}\,{}_j\tau^0\,{}_j\varphi_5^{(m)}(x) =\\
&= {}_j\delta_1\Delta\,{}_j\varphi_4^{(m-2)}(x) - {}_ja_1\,{}_j\varphi_4^{(m-1)}(x) - {}_ja_{12}\,{}_j\varphi_5^{(m-1)}(x) -
\end{aligned} \tag{5}$$

$$-{}_j\gamma_1 \operatorname{div}\left({}_j\varphi_1^{(m-1)}, {}_j\varphi_2^{(m-1)}, {}_j\varphi_3^{(m-1)}\right)^T - \frac{\partial^{m-2}{}_jX_4(x,t)}{\partial t^{m-2}}\bigg|_{t=0},$$

$${}_ja_{12}{}_j\tau^0{}_j\varphi_4^{(m)}(x) + {}_ja_{2}{}_j\tau^0{}_j\varphi_5^{(m)}(x) =$$

$$= {}_j\delta_2\Delta_j\varphi_5^{(m-2)}(x) - {}_ja_{2}{}_j\varphi_5^{(m-1)}(x) - {}_ja_{12}{}_j\varphi_4^{(m-1)}(x) -$$

$$-{}_j\gamma_2 \operatorname{div}\left({}_j\varphi_1^{(m-1)}, {}_j\varphi_2^{(m-1)}, {}_j\varphi_3^{(m-1)}\right)^T - \frac{\partial^{m-2}{}_jX_5(x,t)}{\partial t^{m-2}}\bigg|_{t=0}.$$

Using the method of potential and the theory of multidimensional singular integral equations developed in [1, 2, 3], it can be proved with the aid of the integral Laplace transform, that the structural conditions of agreement (3), (4) are sufficient for the existence of a classical solution of the dynamic problem A^t. The proof is similar to that for the case of a homogeneous medium [3, 5]. Here our aim is to construct a solution agorithm for the above-indicated composite nonhomogeneous medium bounded by several closed surfaces and, using this example, we shall realize the method of an approximate construction of Riesz–Fischer–Kupradze solutions (the method of discrete singularities). Incidentally, in the limiting case this method can serve as a tool for proving existence theorems.

Theorem 1. *If the initial data of Problem A^t satisfy the agreement conditions* (3), (4) *and certain smoothness conditions* (*see* [6]), *then there exists a unique classical solution of the dynamic problem A^t, and in the complex half-plane* $\operatorname{Re}\zeta > \sigma_0^*$ *this solution is represented by the Laplace–Mellin integral*

$$V(x,t) = \frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty} e^{\zeta t}\widehat{V}(x,\zeta)d\zeta,$$

where $\widehat{V}(x,\zeta)$ is a solution of the corresponding elliptic problem, which is represented as a series

$$\widehat{V}(x,\zeta) = \sum_{k=1}^{\infty} a_k(\zeta)_kU(x,\zeta) + U(x,\zeta).$$

This series converges uniformly (with respect to the metric of the space C) in the domain $x \in \overline{D'} \Subset D_1 \cup D_2$; $a_k(\zeta)$, ${}_kU(x,\zeta)$, $U(x,\zeta)$ are the known functions and vector-functions (constructed explicitly), $\zeta = \sigma + iq$, $\sigma > \sigma_0^* > \sigma_0$ and σ_0^* is the known constant.

By the formal application of the Laplace transform

$$\widehat{V}(x,\zeta) = \int\limits_0^\infty e^{-\zeta t}V(x,t)dt \tag{6}$$

the dynamic problem A^t is reduced to the corresponding elliptic problem $\widehat{A}(\zeta)$ with a complex parameter ζ (the spectral problem) for the image $\widehat{V}(x,\zeta)$:

Problem $\widehat{A}(\zeta)$. Determine a regular vector

$$\widehat{V}(x,\zeta)=(\widehat{v},\widehat{v}_4,\widehat{v}_5)^T,\quad \forall\zeta\in\Pi_{\sigma_0^*}\equiv\big\{\zeta:\ \operatorname{Re}\zeta>\sigma_0^*>\sigma_0\big\},$$
$$\widehat{V}\in C^1(\overline{D}_1\cup\overline{D}_2)\cap C^2(D_1\cup D_2),$$

in $D_1\cup D_2$ using the conditions

$$\forall x\in D_j:\ {}_jL\Big(\frac{\partial}{\partial x},\zeta\Big)\widehat{V}(x,\zeta)={}_j\widetilde{H}(x,\zeta),\quad j=1,2, \tag{7}$$

$$\forall y\in S:\ [\widehat{V}]_S^{\pm}\equiv\widehat{V}^{+}(y,\zeta)-\widehat{V}^{-}(y,\zeta)=\widehat{f}(y,\zeta),$$
$$[R_\tau\widehat{V}]_S^{\pm}\equiv\Big[{}_1R_\tau\Big(\frac{\partial}{\partial y},\zeta,n\Big)\widehat{V}(y,\zeta)\Big]^{+}- -\Big[{}_2R_\tau\Big(\frac{\partial}{\partial y},\zeta,n\Big)\widehat{V}(y,\zeta)\Big]^{-}=\widetilde{F}(y,\zeta), \tag{8}$$

$$\forall y\in S_{j1}:\ \widehat{V}^{+}(y,\zeta)={}_j\widehat{f}(y,\zeta),\quad j=1,2, \tag{9}$$

$$\forall y\in S_{j2}:\ \Big[{}_jR_\tau\Big(\frac{\partial}{\partial y},\zeta,n\Big)\widehat{V}^{+}(y,\zeta)\Big]^{+}={}_j\widetilde{F}(y,\zeta),\quad j=1,2, \tag{10}$$

where

$$\widetilde{H}=\Big({}_j\widetilde{X},{}_j\widetilde{X}_4,{}_j\widetilde{X}_5\Big)^T,$$
$${}_j\widetilde{X}={}_j\widehat{X}-{}_j\rho\Big({}_j\varphi_1^{(1)},{}_j\varphi_2^{(1)},{}_j\varphi_3^{(1)}\Big)^T-{}_j\rho\zeta\Big({}_j\varphi_1^{(0)},{}_j\varphi_2^{(0)},{}_j\varphi_3^{(0)}\Big)^T- -{}_j\tau^1\sum_{k=1}^{2}\gamma_k\operatorname{grad}{}_j\varphi_{3+k}^{(0)},$$
$${}_j\widetilde{X}_4={}_j\widehat{X}_4-{}_ja_1{}_j\varphi_4^{(0)}(x)-{}_ja_1{}_j\tau^0\big({}_j\varphi_4^{(1)}+\zeta_j\varphi_4^{(0)}\big)- -{}_ja_{12}{}_j\varphi_5^{(0)}-{}_ja_{12}{}_j\tau^0\big({}_j\varphi_5^{(1)}+\zeta_j\varphi_5^{(0)}\big)- -{}_j\gamma_1\operatorname{div}\Big({}_j\varphi_1^{(0)},{}_j\varphi_2^{(0)},{}_j\varphi_3^{(0)}\Big)^T,$$
$${}_j\widetilde{X}_5={}_j\widehat{X}_5-{}_ja_2{}_j\varphi_5^{(0)}(x)-{}_ja_2{}_j\tau^0\big({}_j\varphi_5^{(1)}+\zeta_j\varphi_5^{(0)}\big)- -{}_ja_{12}{}_j\varphi_4^{(0)}-{}_ja_{12}{}_j\tau^0\big({}_j\varphi_4^{(1)}+\zeta_j\varphi_4^{(0)}\big)- -{}_j\gamma_2\operatorname{div}\Big({}_j\varphi_1^{(0)},{}_j\varphi_2^{(0)},{}_j\varphi_3^{(0)}\Big)^T,$$
$$\widetilde{F}=\widehat{F}-{}_1\gamma_{11}\tau^1n_0(y){}_1\varphi_4^{(0)}+{}_2\gamma_{12}\tau^1n_0(y){}_2\varphi_4^{(0)}-$$

$$-{}_1\gamma_{21}\tau^1 n_0(y)_1\varphi_5^{(0)} + {}_2\gamma_{22}\tau^1 n_0(y)_2\varphi_5^{(0)},$$

$$_j\widetilde{F} = {}_j\widehat{F} - {}_j\gamma_{1j}\tau^1 n_0(y)_j\varphi_4^{(0)} - {}_j\gamma_{2j}\tau^1 n_0(y)_j\varphi_5^{(0)}, \quad j=1,2,$$

$$n_0(y) = (n(y),0,0)^T$$

$$\left|D_x^\beta \widehat{V}(x,\zeta)\right| \leq \frac{\text{const}}{1+|x|^{1+|\beta|}}\ , \quad |\beta| = \overline{0,2},$$

$\beta = (\beta_1,\beta_2,\beta_3)$ is the multi-index.

Now we introduce the following Green matrix-functions.

Let ${}_jG_\tau(x,y;\zeta,{}_j\mathbb{R}^3)$ be the Green tensor (5×5 matrix) for the differential operator ${}_jL(\frac{\partial}{\partial x},\zeta)$, the infinite homogeneous domain ${}_j\mathbb{R}^3 = \mathbb{R}^3\backslash(\overline{D}_{j1}\cup\overline{D}_{j2})$ and mixed boundary conditions (9), (10) on $S_{j1}\cup S_{j2}$, respectively.

We have

$${}_jG_\tau(x,y;\zeta,{}_j\mathbb{R}^3) = {}_j\Phi_\tau(x-y,\zeta) - {}_jg_\tau(x,y;\zeta,{}_j\mathbb{R}^3),$$

where ${}_j\Phi_\tau(x-y,\zeta)$ is a matrix of fundamental solutions of the operator ${}_jL(\frac{\partial}{\partial x},\zeta)$ which is constructed explicitly in terms of elementary functions [4, 7], and ${}_jg_\tau(x,y;\zeta,{}_j\mathbb{R}^3)$ are regular matrices.

Let $\widehat{V}(x,\zeta)$ be a regular solution of Problem $\widehat{A}(\zeta)$. Applying the Green matrices ${}_jG_\tau$, the formula of general representation of a regular vector takes, in view of the contact and boundary conditions, the form

$$\begin{aligned}
2\chi_{D_1}(x)\widehat{V}(x,\zeta) = &\int_S {}_1G_\tau(x,z;\zeta,{}_1\mathbb{R}^3)\Big[{}_1R_\tau\Big(\frac{\partial}{\partial z},\zeta,n\Big)\widehat{V}(z,\zeta)\Big]^+ d_zS - \\
&-\int_S \Big[{}_1\widetilde{R}_{\tau 1}G_\tau^T(x,z;\zeta,{}_1\mathbb{R}^3)\Big]^T \widehat{V}^+(z,\zeta)d_zS - \\
&-\int_{D_1} {}_1G_\tau(x,z;\zeta,{}_1\mathbb{R}^3)_1\widetilde{H}(z,\zeta)dz - \\
&-\int_{S_{11}} \Big[{}_1\widetilde{R}_{\tau 1}G_\tau^T(x,z;\zeta,{}_1\mathbb{R}^3)\Big]^T \widehat{f}(x,\zeta)d_zS + \\
&+\int_{S_{12}} {}_1G_\tau(x,z;\zeta,{}_1\mathbb{R}^3)_1\widetilde{F}(z,\zeta)d_zS, \quad x\in {}_1\mathbb{R}^3\backslash S;
\end{aligned} \tag{11}$$

$$\begin{aligned}
2\chi_{D_2}(x)\widehat{V}(x,\zeta) = &-\int_S {}_2G_\tau(x,z;\zeta,{}_2\mathbb{R}^3)\Big[{}_1R_\tau\Big(\frac{\partial}{\partial z},\zeta,n\Big)V(z,\zeta)\Big]^+ d_zS + \\
&+\int_S \Big[{}_2\widetilde{R}_{\tau 2}G_\tau^T(x,z;\zeta,{}_2\mathbb{R}^3)\Big]^T \widehat{V}^+(z,\zeta)d_zS -
\end{aligned}$$

$$
\begin{aligned}
&-\int\limits_{D_2} {}_2G_\tau(x,z;\zeta,{}_2\mathbb{R}^3)\,{}_2\widetilde{H}(z,\zeta)dz+\\
&+\int\limits_{S} {}_2G_\tau(x,z;\zeta,{}_2\mathbb{R}^3)\widetilde{F}(z,\zeta)d_zS-\\
&-\int\limits_{S}\Big[{}_2\widetilde{R}_\tau\Big(\frac{\partial}{\partial z},\zeta,n\Big){}_2G_\tau^T(x,z;\zeta,{}_2\mathbb{R}^3)\Big]^T\widehat{f}(x,z)d_zS-\\
&-\int\limits_{S_{21}}\Big[{}_2\widetilde{R}_\tau\,{}_2G_\tau^T(x,z;\zeta,{}_2\mathbb{R}^3)\Big]^T\widehat{f}(x,\zeta)d_zS+\\
&+\int\limits_{S_{22}} {}_2G_\tau(x,z;\zeta,{}_2\mathbb{R}^3)\,{}_2\widetilde{F}(z,\zeta)d_zS,\quad x\in{}_2\mathbb{R}^3\backslash S,
\end{aligned}
\tag{12}
$$

where $\chi_{D_j}(x)$ is the characteristic function of the domain D_j, equal to 1 for $x\in D_j$ and to 0 for $x\overline{\in}D_j$, $j=1,2$.

It is easy to check that if the vectors $\widehat{V}^+$ and $({}_1R_\tau\widehat{V})^+$ are found by (11) for $x\in D_2$ and by (11) for $x\in D_1$, then the substitution of these values in (11) for $x\in D_1$ and in (12) for $x\in D_2$ will give a solution of Problem $\widehat{A}(\zeta)$.

To this end we introduce the following matrices:

$$
{}_j\Psi(x,z,\zeta)=
$$
$$
=\left\|\boxed{({}_j\widetilde{R}_{\tau j}G_\tau^T(x,z;\zeta,{}_j\mathbb{R}^3))^T}_{5\times 5},\quad \boxed{-{}_jG_\tau(x,z;\zeta,{}_j\mathbb{R}^3)}_{5\times 5}\right\|_{5\times 10}, \tag{13}
$$
$$
\psi(x,\zeta)=\|\psi_k\|_{10\times 1}=\big(\widehat{V}^+,({}_1R_\tau\widehat{V})^+\big)^T.
$$

It is easy to notice that by virtue of (11) and (12) we shall have

$$
\forall x\in D_2:\ \int\limits_S {}_1\Psi(x,z,\zeta)\psi(z,\zeta)d_zS={}_1\Theta(x), \tag{14}
$$

$$
\forall x\in D_1:\ \int\limits_S {}_2\Psi(x,z,\zeta)\psi(z,\zeta)d_zS={}_2\Theta(x), \tag{15}
$$

where

$$
\begin{aligned}
{}_1\Theta(x)=&-\int\limits_{D_1} {}_1G_\tau(x,z;\zeta,{}_1\mathbb{R}^3)\,{}_1\widetilde{H}(z,\zeta)dz-\\
&-\int\limits_{S_{11}}\Big[{}_1\widetilde{R}_{\tau 1}G_\tau^T(x,z;\zeta_1,\mathbb{R}^3)\Big]^T{}_1\widehat{f}(z,\zeta)d_zS+
\end{aligned}
$$

$$+\int\limits_{S_{12}} {}_1G_\tau(x,z;\zeta,{}_1\mathbb{R}^3)_1\widetilde{F}(z,\zeta)d_zS,$$

$$\begin{aligned}{}_2\Theta(x)=&\int\limits_{D_2} {}_2G_\tau(x,z;\zeta,{}_2\mathbb{R}^3)_2\widetilde{H}(z,\zeta)dz-\\&-\int\limits_{S} {}_2G_\tau\widetilde{F}d_zS+\int\limits_{S}\Big[{}_2\widetilde{R}_{\tau 2}G_\tau^T\Big]^T\widehat{f}d_zS+\\&+\int\limits_{S_{21}}\Big[{}_2\widetilde{R}_{\tau 2}G_\tau^T\Big]^T{}_2\widehat{f}d_zS-\int\limits_{S_{22}} {}_2G_\tau(x,z;\zeta,{}_2\mathbb{R}^3)_2\widetilde{F}d_zS\end{aligned}$$

are given vector-functions.

Now we shall construct auxiliary surfaces in the manner as follows: $\widetilde{S}_1$ is the closed surface which lies strictly inside the domain D_1 covering the surface $S_{11}\cup S_{12}$; $\widetilde{S}_2$ is the closed surface lying strictly inside the domain D_2 covering the surface S. Clearly,

$$\widetilde{S}_1\cap S=\varnothing,\quad \widetilde{S}_2\cap S=\varnothing.$$

Let $\{{}_jx^k\}_{k=1}^\infty$, $j=1,2$, be an everywhere dense countable set of points on the auxiliary surface $\widetilde{S}_j$, $j=1,2$.

By (14) and (15) we have

$$\int\limits_S {}_1\Psi({}_2x^k,z,\zeta)\psi(z,\zeta)d_zS={}_1\Theta({}_2x^k),\quad k=\overline{1,\infty},\tag{16}$$

$$\int\limits_S {}_2\Psi({}_1x^k,z,\zeta)\psi(z,\zeta)d_zS={}_2\Theta({}_1x^k),\quad k=\overline{1,\infty},\tag{17}$$

Denote the rows of the matrix ${}_j\Psi$ considered as ten-component vectors (columns) by ${}_j\Psi^1,{}_j\Psi^2,\dots,{}_j\Psi^5$ (1×10 matrices) and investigate an infinite countable set of vectors

$$\Big\{{}_1\Psi^l({}_2x^k,z,\zeta)\Big\}_{k=1,\,l=1}^{\infty,5}\cup\Big\{{}_2\Psi^l({}_1x^k,z,\zeta)\Big\}_{k=1,\,l=1}^{\infty,5}.\tag{18}$$

Theorem 2. *Set* (18) *is linearly independent and complete in the vector Hilbert space $L_2(S)$, i.e., it forms a basis in this space.*

For the proof of this theorem see [8].

Rewrite (18) as follows:

$$\big\{\psi^k(z)\big\}_{k=1}^\infty,\tag{19}$$

where

$$\psi^k(z)\equiv{}_{a_k}\Psi^{l_k}\big({}_{b_k}x^{q_k},z,\zeta\big),$$

$$a_k = k - 2\Big[\frac{k-1}{2}\Big], \quad b_k = 2\Big[\frac{k+1}{2}\Big] - k + 1,$$
$$l_k = \Big[\frac{k+1}{2}\Big] - 5\left[\frac{\frac{k+1}{2} - 1}{5}\right], \quad q_k = \left[\frac{\frac{k+1}{2} + 4}{5}\right],$$

$[a]$ is the integer part of the number a.

Clearly, by virtue of (16) and (17) the scalar product

$$\big(\psi^k, \overline{\psi}\big) = \int\limits_S [\psi^k]^T \psi\, ds = \big(\psi, \overline{\psi}^k\big)$$

is known for any k; namely,

$$\int\limits_S [\psi^k]^T \psi\, ds = {}_{a_k}\Theta_{l_k}\big({}_{b_k}x^{q_k}\big), \quad k = \overline{1, \infty}.$$

It is obvious that the complex-conjugate system

$$\Big\{\overline{\psi}^k(z)\Big\}_{k=1}^{\infty} \tag{20}$$

is complete, too.

Now let us determine the coefficients α_k, $k = \overline{1, N}$, assuming that the norm reduces to minimum (in $L_2(S)$ with respect to system (20):

$$\min_{\alpha_k} \Big\| \psi(z) - \sum_{k=1}^{N} \alpha_k \overline{\psi}^k(z) \Big\|_{L_2(S)}.$$

As is well known, for this it is necessary and sufficient that

$$\Big(\psi(z) - \sum_{k=1}^{N} \alpha_k \overline{\psi}^k(z), \overline{\psi}^j(z)\Big) = 0, \quad j = \overline{1, N}.$$

Hence we come to an algebraic system of equations

$$\sum_{k=1}^{N} \alpha_k \big(\overline{\psi}^k, \overline{\psi}^j\big) = \big(\psi, \overline{\psi}^j\big), \quad j = \overline{1, N},$$

with the right-hand side known and Gram determinant differing from zero, which determines the coefficients α_k. Therefore by virtue of the property of the space $L_2(S)$ we have

$$\lim_{N\to\infty} \Big\| \psi(z) - \sum_{k=1}^{N} \alpha_k \overline{\psi}^k(z) \Big\|_{L_2(S)} = 0. \tag{21}$$

Let us introduce the notation

$$\overset{N}{\psi}(z)=\sum_{k=1}^{N}\alpha_k\overline{\psi}^{\,k}(z),$$

$$_N\widehat{V}^+=\Big(\overset{N}{\psi}_1,\overset{N}{\psi}_2,\dots,\overset{N}{\psi}_5,\Big)^T\equiv\sum_{k=1}^{N}\alpha_k\Big(\overline{\psi}_1^{\,k},\overline{\psi}_2^{\,k},\dots,\overline{\psi}_5^{\,k},\Big)^T,$$

$$_N\big({}_1R_\tau\widehat{V}\big)^+=\Big(\overset{N}{\psi}_6,\overset{N}{\psi}_7,\dots,\overset{N}{\psi}_{10},\Big)^T\equiv\sum_{k=1}^{N}\alpha_k\Big(\overline{\psi}_6^{\,k},\overline{\psi}_7^{\,k},\dots,\overline{\psi}_{10}^{\,k},\Big)^T.$$

We have in the sense of the metric of the space $L_2(S)$

$$\psi(z)=\lim_{N\to\infty}\overset{N}{\psi}(z),\quad \widehat{V}^+=\lim_{N\to\infty}{}_N\widehat{V}^+,$$
$$\big({}_1R_\tau\widehat{V}\big)^+=\lim_{N\to\infty}{}_N\big({}_1R_\tau\widehat{V}\big)^+.$$

By substituting the obtained approximate values in (11) for $x\in D_1$ and in (12) for $x\in D_2$ and denoting the substitution result by ${}_N\widehat{V}(x,\zeta)$ we have

$$\begin{aligned}\forall x\in D_1:\ 2{}_N\widehat{V}(x,\zeta)=&\int_S {}_1G_\tau(x,z;\zeta,{}_1\mathbb{R}^3)\Big[\sum_{k=1}^{N}\alpha_k\Big(\overline{\psi}_6^{\,k},\dots,\overline{\psi}_{10}^{\,k}\Big)^T\Big]\,d_zS-\\ &-\int_S\Big[{}_1\widetilde{R}_{\tau}\,{}_1G_\tau^T(x,z;\zeta,{}_1\mathbb{R}^3)\Big]^T\times\\ &\times\Big[\sum_{k=1}^{N}\alpha_k\Big(\overline{\psi}_1^{\,k},\dots,\overline{\psi}_5^{\,k}\Big)^T\Big]\,d_zS+{}_1\Theta(x),\end{aligned}\tag{22}$$

$$\begin{aligned}\forall x\in D_2:\ 2{}_N\widehat{V}(x,\zeta)=&-\int_S {}_2G_\tau(x,z;\zeta,{}_2\mathbb{R}^3)\Big[\sum_{k=1}^{N}\alpha_k\Big(\overline{\psi}_6^{\,k},\dots,\overline{\psi}_{10}^{\,k}\Big)^T\Big]\,d_zS+\\ &+\int_S\Big[{}_2\widetilde{R}_{\tau}\,{}_2G_\tau^T(x,z;\zeta,{}_2\mathbb{R}^3)\Big]^T\times\\ &\times\Big[\sum_{k=1}^{N}\alpha_k\Big(\overline{\psi}_1^{\,k},\dots,\overline{\psi}_5^{\,k}\Big)^T\Big]\,d_zS-{}_2\Theta(x).\end{aligned}\tag{23}$$

Thus determining the difference between (11)–(22) and (12)–(23) and applying the Cauchy–Buniakovsky inequality we find by virtue of (21) that for any $\varepsilon>0$ there is a positive number $N(\varepsilon)$ such that for $N>N(\varepsilon)$ we shall have

$$\Big|\widehat{V}(x,\zeta)-{}_N\widehat{V}(x,\zeta)\Big|<\varepsilon,$$

where $x \in \overline{D}\,' \Subset D_1 \cup D_2$ (a strictly internal subdomain), $\widehat{V}(x,\zeta)$ is an exact solution of the problem, i.e.,

$$\widehat{V}(x,\zeta) = \lim_{N\to\infty} {}_N\widehat{V}(x,\zeta), \quad x \in \overline{D}\,'$$

tends to the limit uniformly (with respect to the metric C) in $\overline{D}\,'$.

As for the convergence with respect to the metric L_2, one can easily verify that the relation

$$\lim_{N\to\infty} \left\| \widehat{V}(x,\zeta) - {}_N\widehat{V}(x,\zeta) \right\|_{L_2(D_1\cup D_2)} = 0$$

is valid.

This method can be generalized to other more complicated problems.

The foregoing discussion clearly implies that the solution of Problem A^t is constructed by means of the Green tensor ${}_jG_\tau(x,y;\zeta,{}_j\mathbb{R}^3)$ whose approximate values of can be constructed explicitly.

We shall give an algorithm of constructing Green tensors.

Clearly, to construct the tensor ${}_jG_\tau$ it is sufficient to construct the matrix ${}_jg_\tau$. We have

$${}_jg_\tau(x,x^0;\zeta,{}_j\mathbb{R}^3) = \left\| {}_j\overset{(1)}{g}_\tau, {}_j\overset{(2)}{g}_\tau, \dots, {}_j\overset{(5)}{g}_\tau \right\|_{5\times5},$$

where the column-vectors ${}_j\overset{(s)}{g}_\tau(x,x^0;\zeta,{}_j\mathbb{R}^3)$, $s=\overline{1,5}$, are a solution of the following problem: In the infinite domain ${}_j\mathbb{R}^3 \equiv \mathbb{R}^3 \setminus (\overline{D}_{j1} \cup \overline{D}_{j2})$ determine the vector ${}_j\overset{(s)}{g}_\tau$ by the conditions $(j=1,2)$

$$\begin{aligned}
&\forall x \in {}_j\mathbb{R}^3: {}_jL\Big(\frac{\partial}{\partial x},\zeta\Big){}_j\overset{(s)}{g}_\tau(x,x^0;\zeta,{}_j\mathbb{R}^3) = 0 \quad (x^0 \in {}_j\mathbb{R}^3),\\
&\forall y \in S_{j1}: {}_j\overset{(s)}{g}_\tau(y,x^0;\zeta,{}_j\mathbb{R}^3) = {}_j\overset{(s)}{\Phi}_\tau(y-x^0,\zeta),\\
&\forall y \in S_{j2}: \Big[{}_jR_\tau\Big(\frac{\partial}{\partial y},\zeta,n\Big){}_j\overset{(s)}{g}_\tau(y,x^0;\zeta,{}_j\mathbb{R}^3)\Big]^+ =\\
&\qquad = {}_jR_\tau\Big(\frac{\partial}{\partial y},\zeta,n\Big){}_j\overset{(s)}{\Phi}_\tau(y-x^0,\zeta),
\end{aligned}$$

where $s \in \{1,\dots,5\}$ is a fixed number.

Let us construct auxiliary domains and surfaces in the manner as follows: $\widetilde{D}_{jk}$, $k=1,2$, is the domain wholly lying in D_{jk}, $\widetilde{S}_{jk}$ are the boundaries of $\widetilde{D}_{jk}$, and ${}_j\widetilde{S} = \bigcup\limits_{k=1}^{2} \widetilde{S}_{jk}$.

We introduce a 5×5 matrix

$${}_jM_\tau(y-x,\zeta) = \left\| {}_j\overset{(1)}{M}_\tau, {}_j\overset{(2)}{M}_\tau, \dots, {}_j\overset{(5)}{M}_\tau \right\|_{5\times5},$$

where

$$
{}_jM_\tau(y-x,\zeta)=\begin{cases} {}_j\Phi_\tau(y-x,\zeta), & y\in S_{j1},\\ {}_jR_\tau\Big(\dfrac{\partial}{\partial y},\zeta,n\Big){}_j\Phi_\tau(y-x,\zeta), & y\in S_{j2}, \end{cases}
$$
$$
x\in R^3.
$$

Let $\{{}_j\widetilde{x}^k\}_{k=1}^{\infty}$ be an everywhere dense countable set of points on an auxiliary surface ${}_j\widetilde{S}$.

Theorem 3. *A countable set of vectors*

$$
\Big\{{}_j\overset{(l)}{M}_\tau\big(y-{}_j\widetilde{x}^k,\zeta\big)\Big\}_{k=1,\,l=1}^{\infty,\,5},\quad y\in\bigcup_{m=1}^{2}S_{jm} \tag{24}
$$

is linearly independent and complete in the Hilbert space $L_2(\bigcup\limits_{m=1}^{2}S_{jm})$.

For the proof see [3, pp. 57, 123].

Enumerate the elements of (24) as follows:

$$
\begin{gathered}
{}_j\overset{k}{\psi}(y)={}_j\overset{(p_k)}{M}{}_\tau\Big(y-{}_j\widetilde{x}^{[\frac{k+4}{5}]},\zeta\Big),\quad k=\overline{1,\infty},\\
p_k=k-5\Big[\frac{k-1}{5}\Big],
\end{gathered} \tag{25}
$$

and assume $\{{}_j\overset{k}{\varphi}(y)\}_{k=1}^{\infty}$ to be the system obtained from (25) by orthonormalization on $\bigcup\limits_{m=1}^{2}S_{jm}$ by the Schmidt mehod, i.e.,

$$
{}_j\overset{k}{\varphi}(y)=\sum_{s=1}^{k}{}_ja_{ks}{}_j\overset{s}{\psi}(y),\quad y\in\bigcup_{m=1}^{2}S_{jm},\quad k=\overline{1,\infty}, \tag{26}
$$

where ${}_ja_{ks}$ are the orthonormalization coefficients.

It is easy to verify that

$$
{}_j\overset{k}{\varphi}(y)=\begin{cases} \displaystyle\sum_{s=1}^{k}{}_ja_{ks}{}_j\overset{(p_s)}{\Phi}\Big(y-{}_j\widetilde{x}^{[\frac{s+4}{5}]},\zeta\Big), & y\in S_{j1},\\ \displaystyle\sum_{s=1}^{k}{}_ja_{ks}{}_jR_\tau\Big(\frac{\partial}{\partial y},\zeta,n\Big){}_j\overset{(p_s)}{\Phi}\Big(y-{}_j\widetilde{x}^{[\frac{s+4}{5}]},\zeta\Big), & y\in S_{j2}. \end{cases} \tag{27}
$$

We introduce the notation

$$
{}_j\overset{(s)}{\Omega}{}^{(1)}(z,x^0)={}_j\overset{(s)}{\Phi}{}_\tau(z-x^0,\zeta),\quad z\in S_{j1},
$$

$$ {}_j\overset{(s)}{\Omega}{}^{(2)}(z,x^0) = {}_jR_\tau\Big(\frac{\partial}{\partial y},\zeta,n\Big)_j\overset{(s)}{\Phi}{}_\tau(z-x^0,\zeta), \quad z\in S_{j2}. $$

Let

$$ {}_j\overset{(s)}{\Omega}(z,x^0) = {}_j\overset{(s)}{\Omega}{}^{(k)}(z,x^0), \quad z\in S_{jk}, \quad k=1,2. $$

Clearly, by the property of the matrixt of fundamental solutions we have

$$ {}_j\overset{(s)}{\Omega}(\cdot,x^0)\in C^\infty(S_{j1}\cup S_{j2}), \quad x^0\in {}_j\mathbb{R}^3. $$

By decomposing ${}_j\overset{(s)}{\Omega}(z,x^0)$ in a Fourier series with respect to the complete orthonormal system of vectors (26) we obtain

$$ {}_j\overset{(s)}{\Omega}(z,x^0) \approx \sum_{k=1}^{\infty} {}_j\overset{(s)}{\Omega}{}_k(x^0)\, {}_j\overset{k}{\varphi}(z), \quad z\in \bigcup_{l=1}^{2} S_{jl}, $$

where

$$ {}_j\overset{(s)}{\Omega}{}_k(x^0) = \int\limits_{S_{j1}\cup S_{j2}} \Big[{}_j\overset{(s)}{\Omega}{}_k(y,x^0)\Big]^T\, {}_j\overline{\overset{k}{\varphi}}(y)\,d_yS. $$

Therefore by the property of the space $L_2(\bigcup_{l=1}^{2} S_{jl})$ we have

$$ \lim_{N\to\infty}\Big\| {}_j\overset{(s)}{\Omega}(z,x^0) - \sum_{k=1}^{N} {}_j\overset{(s)}{\Omega}{}_k(x^0)\, {}_j\overset{k}{\varphi}(z)\Big\|_{L_2(\bigcup_{l=1}^{2} S_{jl})} = 0. $$

We introduce the vectors

$$ {}_j\overset{(s)}{g}{}_\tau^{(N)}(x,x^0) = \sum_{k=1}^{N}\sum_{m=1}^{k} {}_j\overset{(s)}{\Omega}{}_k(x^0)\, {}_ja_{km}\, {}_j\overset{(p_m)}{\Phi}{}_\tau\Big(x - {}_j\widetilde{x}^{[\frac{m+4}{5}]},\zeta\Big), $$

$$ x,x^0\in {}_j\mathbb{R}^3, \quad s=\overline{1,5}. $$

Clearly, all the coefficients are uniquely defined. Now it is easy to show that the desired approximate value of the Green tensor is written as

$$ \begin{aligned} {}_j\overset{(s)}{G}{}_\tau^{(N)}(x,x^0;\zeta,{}_j\mathbb{R}^3) &= {}_j\overset{(s)}{\Phi}{}_\tau(x-x^0,\zeta) - {}_j\overset{(s)}{g}{}_\tau^{(N)}(x,x^0) = \\ &= {}_j\overset{(s)}{\Phi}{}_\tau(x-x^0,\zeta) - \sum_{k=1}^{N}\sum_{m=1}^{k} {}_j\overset{(s)}{\Omega}{}_k(x^0)\, {}_ja_{km}\times \\ &\quad\times {}_j\overset{(p_m)}{\Phi}{}_\tau\Big(x - {}_j\widetilde{x}^{[\frac{m+4}{5}]},\zeta\Big), \end{aligned} \tag{28} $$

where $j=1,2$, $s=\overline{1,5}$, $x,x^0\in\mathbb{R}^3$, $N>0$ is a natural number. For the mathematical substantiation of this fact see [3, pp. 122–125].

We would like to emphasize that for fixed N the Green tensor ${}_jG_\tau^{(N)}$ is given explicitly by a finite number of quadratures.

Finally, on substituting (28) in (22) and (23), we respectively have

$$\begin{aligned}\forall x\in D_1:\ 2_{N,\widetilde N}\widehat V(x,\zeta) = & \int_S {}_1G_\tau^{(\widetilde N)}(x,z;\zeta,{}_1\mathbb{R}^3)\Big[\sum_{k=1}^N \alpha_k\big(\overline\psi{}_6^k,\dots,\overline\psi{}_{10}^k\big)^T\Big]d_zS-\\ & -\int_S\Big[{}_1\widetilde R_{\tau 1}\overset{T}{G}{}_\tau^{(\widetilde N)}(x,z;\zeta,{}_1\mathbb{R}^3)\Big]^T\times\\ & \times\Big[\sum_{k=1}^N\alpha_k\big(\overline\psi{}_1^k,\dots,\overline\psi{}_5^k\big)^T\Big]d_zS+{}_{1,\widetilde N}\Theta(x),\end{aligned}\tag{29}$$

$$\begin{aligned}\forall x\in D_2:\ 2_{N,\widetilde N}\widehat V(x,\zeta) = & -\int_S {}_1G_\tau^{(\widetilde N)}(x,z;\zeta,{}_2\mathbb{R}^3)\times\\ & \times\Big[\sum_{k=1}^N \alpha_k\big(\overline\psi{}_6^k,\dots,\overline\psi{}_{10}^k\big)^T\Big]d_zS+\\ & +\int_S\Big[{}_2\widetilde R_{\tau 2}\overset{T}{G}{}_\tau^{(\widetilde N)}(x,z;\zeta,{}_2\mathbb{R}^3)\Big]^T\times\\ & \times\Big[\sum_{k=1}^N\alpha_k\big(\overline\psi{}_1^k,\dots,\overline\psi{}_5^k\big)^T\Big]d_zS-{}_{2,\widetilde N}\Theta(x).\end{aligned}\tag{30}$$

Thus we obtain the following relation with respect to the metric of the space C in $\overline{D}\,'\Subset D_1\cup D_2$:

$$\widehat V(x,\zeta)=\lim_{\substack{N\to\infty\\ \widetilde N\to\infty}}{}_{N,\widetilde N}\widehat V(x,\zeta).\tag{31}$$

As for an estimate in the closed domain $\overline{D}_1\cup\overline{D}_2$, these estimates hold in the space $L_2(\overline{D}_1\cup\overline{D}_2)$.

Remark. Fundamental differential operators of the theory developed here have the form

$$\begin{aligned}P_{\tau(k)}\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial t},n(x)\Big)V(x,t)= & \Big(T\Big(\frac{\partial}{\partial x},n(x)\Big)v(x,t)-\\ & -n(x)\sum_{l=1}^2\gamma_l\Big(1+\tau^1\frac{\partial}{\partial t}\Big)v_{3+l},\\ & -(\delta_{1k}+\delta_{2k})v_4+(\delta_{3k}+\delta_{0k})\delta_1\frac{\partial v_4}{\partial n},\end{aligned}$$

$$-(\delta_{1k}+\delta_{3k})v_5+(\delta_{2k}+\delta_{0k})\delta_2\frac{\partial v_5}{\partial n}\Big)^T,$$

$$Q_{(k)}\Big(\frac{\partial}{\partial x},n(x)\Big)V(x,t)=\Big(v,(\delta_{1k}+\delta_{2k})\delta_1\frac{\partial v_4}{\partial n}+(\delta_{3k}+\delta_{0k})v_4,$$
$$(\delta_{1k}+\delta_{3k})\delta_2\frac{\partial v_5}{\partial n}+(\delta_{2k}+\delta_{0k})v_5\Big)^T,$$

where $k=\overline{0,3}$, $R_\tau\equiv P_{\tau(0)}$, δ_{jk} is the Kronecker symbol.

Let ${}_1D_k$, $k=\overline{0,7}$, be the nonintersecting domain within S with the Liapunov boundaries ${}_1S_k$ (accordingly, ${}_1D_k\cap{}_1D_m=\varnothing$, $k\neq m$); ${}_2D_k$ $k=\overline{0,7}$, be the nonintersecting finite domains outside S with the Liapunov boundaries ${}_2S_k$ (accordingly, ${}_2D_k\cap{}_2D_m=\varnothing$, $k\neq m$); ${}_1D$ be a finite domain bounded by the surfaces $S\cup\bigcup\limits_{k=0}^{7}{}_1S_k$, ${}_2D$ be an infinite domain bounded by the surfaces $S\cup\bigcup\limits_{k=0}^{7}{}_2S_k$. Consider a nonhomogeneous infinite domain ${}_1D\cup S\cup{}_2D\equiv\mathbb{R}^3\setminus\bigcup\limits_{j=1}^{2}\cup\bigcup\limits_{k=0}^{7}{}_j\overline{D}_k$, where S is the contacting surface.

The method described can be generalized as well to the case where contact conditions on S and boundary conditions on ${}_jS_k$ have the form

$$\forall y\in S,\quad t\geq 0:\lim_{{}_1D\ni x\to y\in S}{}_1Q_{(k_0)}\Big(\frac{\partial}{\partial x},n(y)\Big)V(x,t)-$$
$$-\lim_{{}_2D\ni x\to y\in S}{}_2Q_{(k_0)}\Big(\frac{\partial}{\partial x},n(y)\Big)V(x,t)=f(y,t),$$
$$\lim_{{}_1D\ni x\to y\in S}{}_1P_{\tau(k_0)}\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial t},n(y)\Big)V(x,t)-$$
$$-\lim_{{}_2D\ni x\to y\in S}{}_2P_{\tau(k_0)}\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial t},n(y)\Big)V(x,t)=F(y,t),$$
$$\forall y\in{}_jS_k,\quad t\geq 0:\lim_{{}_jD\ni x\to y\in{}_jS_k}{}_jQ_{(k)}V(x,t)={}_jf_{(k)}(y,t),$$
$$k=\overline{0,3},\quad j=1,2,$$
$$\forall y\in{}_jS_k,\quad t\geq 0:\lim_{{}_jD\ni x\to y\in{}_jS_k}{}_jP_{\tau(k)}V(x,t)={}_jF_{(k)}(y,t),$$
$$k=\overline{4,7},\quad j=1,2,$$

where $k_0\in\{0,1,\dots,7\}$ is a fixed number.

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