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WEIGHTED L_Φ INTEGRAL INEQUALITIES FOR MAXIMAL OPERATORS

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Abstract. Sufficient (almost necessary) conditions are given on the weight functions $u(\cdot)$, $v(\cdot)$ for

$$\Phi_2^{-1}\Big[\int\limits_{\mathbb{R}^n}\Phi_2\big(C_2(M_sf)(x)\big)u(x)dx\Big]\leq\Phi_1^{-1}\Big[C_1\int\limits_{\mathbb{R}^n}\Phi_1(|f(x)|)v(x)dx\Big]$$

to hold when Φ_1, Φ_2 are φ-functions with subadditive $\Phi_1\Phi_2^{-1}$, and M_s $(0\leq s<n)$, is the usual fractional maximal operator.

§ 1. Introduction

Let $n\in\mathbb{N}^*$, $0\leq s<n$. The fractional maximal operator M_s of order s is defined as

$$(M_sf)(x)=\sup\Big\{|Q|^{\frac{s}{n}-1}\int\limits_Q|f(y)|dy;\ Q\text{ is a cube with }Q\ni x\Big\},\quad x,y\in\mathbb{R}^n.$$

Throughout this paper, Q will denote a cube with the sides parallel to the coordinate planes.

As in [1], a real function $\Phi(\cdot)$ defined on $[0,\infty[$ is called a φ-function if it is a nondecreasing continuous function which satisfies $\Phi(0)=0$ and $\lim_{s\to\infty}\Phi(s)=\infty$. The φ-function $\Phi(\cdot)$ is subadditive if $\Phi(t_1+t_2)\leq\Phi(t_1)+\Phi(t_2)$ for all $t_1,t_2\in[0,\infty[$. Let $u(\cdot)$, $v(\cdot)$ be weight functions (i.e., nonnegative locally integrable functions). In this paper we study the integral inequality

$$\Phi_2^{-1}\Big[\int\limits_{\mathbb{R}^n}\Phi_2\big(C_2(M_sf)(x)\big)u(x)dx\Big]\leq\Phi_1^{-1}\Big[C_1\int\limits_{\mathbb{R}^n}\Phi_1(|f(x)|)v(x)dx\Big]\quad(1)$$

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for all functions $f(\cdot)$. Here $\Phi_1(\cdot)$, $\Phi_2(\cdot)$ are φ-functions with subadditive $(\Phi_1\Phi_2^{-1})(\cdot)$, and C_1, C_2 are nonnegative constants which do not depend on each function $f(\cdot)$. For convenience we also denote this inequality by $M_s : L_v^{\Phi_1} \to L_u^{\Phi_2}$.

As mentioned in a recent monograph of Kokilashvili and Krbec [2], this is an open problem to characterize a pair of weight functions $u(\cdot)$, $v(\cdot)$ for which $M_s : L_v^{\Phi_1} \to L_u^{\Phi_2}$ holds. Such an integral inequality can be useful in studying the boundary value problems for quasilinear partial differential equations (see [3]).

In the Lebesgue case, i.e., $\Phi_1(t) = t^p$, $\Phi_2(t) = t^q$ with $1 < p \leq q < \infty$, inequality (1) can be rewritten as

$$\left(\int_{\mathbb{R}^n} (M_s f)^q(x)u(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_{\mathbb{R}^n} |f(x)|^p v(x)dx\right)^{\frac{1}{p}}. \tag{2}$$

Sawyer [4] characterized the weight functions $u(\cdot)$, $v(\cdot)$ for which (2) held. One of the crucial keys he used to solve this problem is to note the equivalence of (2) to

$$\left(\int_{\mathbb{R}^n} (M_s \sigma g)^q(x)u(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_{\mathbb{R}^n} |g(x)|^p \sigma(x)dx\right)^{\frac{1}{p}}, \tag{3}$$

where $\sigma(\cdot) = v^{-\frac{1}{p-1}}(\cdot)$. Indeed, $\int_{\mathbb{R}^n} |(f\sigma^{-1})(x)|^p\sigma(x)dx = \int_{\mathbb{R}^n} |f(x)|^p v(x)dx$. Thus in the Orlicz setting, inequality (3) leads naturally to the following first generalization:

$$\Phi_2^{-1}\left[\int_{\mathbb{R}^n} \Phi_2\big(C_2(M_s\sigma g)(x)\big)u(x)dx\right] \leq \Phi_1^{-1}\left[C_1\int_{\mathbb{R}^n} \Phi_1(|g(x)|)\sigma(x)dx\right] \tag{4}$$

where $\sigma(\cdot)$ is some weight function. Such integral inequalities were studied by L. Qinsheng [5], and, independently, by the author [6]. Roughly speaking, inequality (4) is not technically too far from the Lebesgue setting, and so this problem can be handled by Sawyer's ideas [4]. The second generalization of (2) is the two-weight modular inequality (1) or $M_s : L_v^{\Phi_1} \to L_u^{\Phi_2}$ introduced above. This inequality is more difficult to study than (4). Indeed, from the latter, we cannot easily derive (1), since contrary to the Lebesgue case, there is no obvious connection between $\int_{\mathbb{R}^n} \Phi_1|(f(x)\sigma^{-1})(x)|)\sigma(x)dx$ and $\int_{\mathbb{R}^n} \Phi_1(|f(x)|)v(x)dx$. The problem $M : L_w^{\Phi} \to L_w^{\Phi}$, where $M = M_0$ is the classical Hardy–Littlewood maximal operator, was studied by Kerman and Torchinsky [7]. They considered an N-function $\Phi(\cdot)$ with $\Phi(\cdot), \Phi^*(\cdot) \in \Delta_2$. Recall that an N-function is a convex φ-function satisfying $\lim_{t\to 0} \frac{\Phi(t)}{t} = \lim_{t\to\infty} \frac{t}{\Phi(t)} = 0$, and the associated conjugate function $\Phi^*(\cdot)$ is defined by

$\Phi^*(t) = \sup_{s>0}\{st - \Phi(s)\}$. The condition $\Phi(\cdot) \in \Delta_2$ means $\Phi(2t) \le C\Phi(t)$ for all $t > 0$.

As in the Lebesgue setting, the problem $M : L_v^\Phi \to L_u^\Phi$ for $u \neq v$ is completely different. Using the Lebesgue arguments [4], Chen [8] and Sun [9] studied this problem independently. They obtained the following result:

Suppose $\Phi(\cdot)$ is an N-function with $\Phi(\cdot), \Phi^*(\cdot) \in \Delta_2$ and there is a weight function $\sigma(\cdot)$ satisfying the extra-assumption:

$$\sup_{y\in\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \Phi\big(\big[\tau_y^{-1}\widetilde{N}_{\tau_y\sigma}(\tau_y f)\big](x)\big)v(x)dx \le C\int\limits_{\mathbb{R}^n} \Phi(|f(x)|)v(x)dx \qquad (*)$$

for all functions $f(\cdot)$; then $M : L_v^\Phi \to L_u^\Phi$ holds if and only if

$$\int\limits_{\mathbb{R}^n} \Phi\big((M\varepsilon\sigma 1\!\!1_Q)(x)\big)u(x)dx \le c\int\limits_{\mathbb{R}^n} \Phi\big((\varepsilon\sigma 1\!\!1_Q)(x)\big)v(x)dx \qquad (5)$$

for all cubes Q and $\varepsilon > 0$.

Here $(\tau_y f)(x) = f(x-y)$, $\widetilde{N}_\sigma f(x) = \sigma(x)N_\sigma(f\sigma^{-1})(x)$, and N_σ is the maximal function defined by

$$(N_\sigma g)(x) = \sup\bigg\{|Q|_\sigma^{-1}\int\limits_Q |g(y)|\sigma(y)dy;\ Q \text{ is a cube with } Q \ni x\bigg\}.$$

Unfortunately, this result has two main drawbacks. Firstly, for a general N-function $\Phi(\cdot)$, no exact condition is known to be imposed on a weight function $v(\cdot)$ for which there is another weight function $\sigma(\cdot)$ satisfying the extra-condition $(*)$. Secondly, as in the Lebesgue setting, the Sawyer's condition (5) is not easy to check since it is expressed in terms of the maximal function M itself. Thus it was a challenging problem for specialists in weighted inequalities to obtain a sufficient (almost necessary) condition on weight functions $u(\cdot)$, $v(\cdot)$ which ensures $M_s : L_v^{\Phi_1} \to L_u^{\Phi_2}$. An attempt in this direction was made by the author in [10]. For completeness, we recall here the result he obtained.

Let $1 \le r < \infty$, $\Psi_1(t) = \Phi_1(t^{\frac{1}{r}})$ be an N-function and $(\Phi_1\Phi_2^{-1})(\cdot)$ be subadditive. Moreover, let us assume $\Phi_2(\cdot), \Psi_1^*(\cdot) \in \Delta_2$. Then $M_s : L_v^{\Phi_1} \to L_u^{\Phi_2}$ holds if and only if

$$\Phi_2^{-1}\bigg[\Big(A_2\varepsilon|Q|^{\frac{s}{n}}\int\limits_Q u(y)dy\Big)\bigg] \le \Phi_1^{-1}\bigg[\Big(A_1\Phi_1(\varepsilon)\int\limits_Q v(y)dy\Big)\bigg] \qquad (**)$$

with constants A_1 and A_2 for all cubes Q and all $\varepsilon > 0$, whenever the weight function $v(\cdot)$ belongs to the Muckenhoupt class A_p for $1 < p \le r$.

With the definition given below, $w \in A_p$ if and only if $(w, w) \in A_1(0, p, p)$. Although the test condition $(**)$ is more computable than the above Sawyer one (5), the restriction on the weight function $v(\cdot)$ is an inconvenience.

Therefore our main purpose in this paper is to derive $M_s : L_v^{\Phi_1} \to L_u^{\Phi_2}$ by a similar test condition without restrictions on the weight functions $u(\cdot)$, $v(\cdot)$. This condition is denoted by $(u, v) \in A_r(s, \Phi_1, \Phi_2)$ $(r \geq 1)$ and means

$$\frac{1}{|Q|} \int\limits_Q (\Phi_1^{\frac{1}{r}})^* \left[|Q|^{\frac{s}{n}} \frac{A_2}{\Phi_2^{-1}\left[\frac{\Phi_2 \Phi_1^{-1}(A_1 \varepsilon^{-1}|Q|)}{\varepsilon^{-1}|Q|} \frac{1}{\frac{1}{|Q|}\int_Q \varepsilon u(y)dy} \right]} \frac{1}{(\varepsilon v(x))^{\frac{1}{r}}} \right] \times$$
$$\times (\varepsilon v(x))^{\frac{1}{r}} dx \leq 1 \tag{6}$$

for all cubes Q and all $\varepsilon > 0$. The presence of $\varepsilon > 0$ can be explained by the lack of homogeneities of the φ-functions $\Phi_1(\cdot)$ and $\Phi_2(\cdot)$. Assuming that $\Phi_1^{\frac{1}{r}}(\cdot)$ is an N-function for some $r > 1$, we will prove that $(u, v) \in A_r(s, \Phi_1, \Phi_2)$ is the sufficient condition in order that $M_s : L_v^{\Phi_1} \to L_u^{\Phi_2}$ (see Proposition 3). This sufficient condition we introduce is almost necessary in the sense that $(u, v) \in A_1(s, \Phi_1, \Phi_2,)$ is a necessary condition for this embedding (see Proposition 1). Our result includes one result due to Perez [11]. We were really inspired by this author's technique, which we will develop in the Orlicz setting. In order to include the modular inequality (4), we also deal with the general embedding $M_s : L_v^{\Phi_1}(w_1) \to L_u^{\Phi_2}(w_2)$, i.e.,

$$\Phi_2^{-1}\left[\int\limits_{\mathbb{R}^n} \Phi_2\big(C_2 w_2(x)(M_s f)(x)\big) u(x)dx \right] \leq$$
$$\leq 1\Phi_1^{-1}\left[C_1 \int\limits_{\mathbb{R}^n} \Phi_1\big(w_1(x)|f(x)|\big) v(x)dx \right]. \tag{7}$$

Let us consider again the inequality $M_s : L_v^{\Phi_1} \to L_u^{\Phi_2}$. The corresponding weak version $M_s : L_v^{\Phi_1} \to L_u^{\Phi_2,\infty}$ is defined by

$$\Phi_2^{-1}\left[\Phi_2(C_2\lambda) \int\limits_{\{(M_s f)(x) > \lambda\}} u(x)dx \right] \leq \Phi_1^{-1}\left[C_1 \int\limits_{\mathbb{R}^n} \Phi_1(|f(x)|) v(x)dx \right] \tag{8}$$

for all functions $f(\cdot)$ and all $\lambda > 0$. Using the ideas from [12], we will prove that if $(\Phi_1 \Phi_2^{-1})(\cdot)$ is subadditive and $\Phi_1(\cdot)$ is an N-function, then $M_s : L_v^{\Phi_1} \to L_u^{\Phi_2,\infty}$ holds if and only if $(u, v) \in A_1(s, \Phi_1, \Phi_2,)$. This weak inequality was already solved and considered by many authors [2], [13] for the case $s = 0$ and $\Phi_1(\cdot) = \Phi_2(\cdot)$. We emphasize that in this paper we never use the Δ_2 condition for the φ-functions $\Phi_1(\cdot)$ and $\Phi_2(\cdot)$.

We leave open the problem $I_s : L_v^{\Phi_1} \to L_u^{\Phi_2}$, where I_s $(0 < s < n)$ is the fractional integral defined by $I_s f(x) = \int_{\mathbb{R}^n} |x-y|^{s-n} f(y)dy$. A characterization of the weight functions $u(\cdot)$, $v(\cdot)$ for which the corresponding weak inequality was proved was done by the author in [14].

We state our main results in §2, and we will give the basic lemma in the next section. Weak inequalities are proved in §4, and §5 is devoted to proving strong inequalities. The last section deals with our test condition (6).

§ 2. The Results

Weak Inequalities

We first characterize the weight functions $u(\cdot)$, $v(\cdot)$ for which the weak inequality (8) holds.

Proposition 1. *Let $0 \le s < n$, $\Phi_2(\cdot)$ be a φ-function and $\Phi_1(\cdot)$ be an N-function with subadditive $(\Phi_1\Phi_2^{-1})(\cdot)$.*

Assume that the weak inequality $M_s : L_v^{\Phi_1} \to L_u^{\Phi_2,\infty}$ is satisfied for some constants $C_1, C_2 > 0$. Then $(u,v) \in A_1(s,\Phi_1,\Phi_2)$ with the constants $A_1 = C_1$ and $A_2 = C_2$.

Conversely, let $(u,v) \in A_1(s,\Phi_1,\Phi_2)$ for some constants $A_1, A_2 > 0$. Then $M_s : L_v^{\Phi_1} \to L_u^{\Phi_2,\infty}$ is satisfied with the constants $C_1 = NA_1$, $C_2 = CA_2$.

Here $N = N(n)$, $C = C(s,n)$ are nonnegative constants which depend respectively on n and s, n.

In fact, this weak inequality can be considered as the particular case of $M_s : L_v^{\Phi_1}(w_1) \to L_u^{\Phi_2,\infty}(w_2)$, i.e.,

$$\Phi_2^{-1}\Bigg[\int\limits_{\{(M_s f)(x)>\lambda\}} \Phi_2\big(C_2\lambda w_2(x)\big)u(x)dx\Bigg] \le \\ \le \Phi_1^{-1}\Bigg[C_1\int\limits_{\mathbb{R}^n} \Phi_1\Big(w_1(x)|f(x)|\Big)v(x)dx\Bigg] \tag{9}$$

for all functions $f(\cdot)$ and all $\lambda > 0$.

Here $v(\cdot)$, $u(\cdot)$, $w_1(\cdot)$, $w_2(\cdot)$ are weight functions. Recall (see [4]) that for each N-function $\Phi(\cdot)$ and for each weight function $w(\cdot)$, the quantity $\|\cdot\|_{\Phi,w}$ defined by

$$\|f(\cdot)\|_{\Phi,w} = \sup\Bigg\{\int\limits_{\mathbb{R}^n} |f(y)g(y)|w(y)dy;\ \int\limits_{\mathbb{R}^n} \Phi^*(|g(y)|)w(y)dy \le 1\Bigg\}$$

yields a norm in the Orlicz space L_w^{Φ} (which is the set of all measurable functions $f(\cdot)$ satisfying $\int_{\mathbb{R}^n} \Phi(\lambda|f(y)|)w(y)dy < \infty$ for some $\lambda > 0$). And for each cube Q we also define $\|f(\cdot)\|_{\Phi,Q,w}$ as $\|f(\cdot)\|_{\Phi,|Q|^{-1}1\!\!\mathrm{I}_Q w(\cdot)}$.

Our characterization of the weight functions $u(\cdot)$, $v(\cdot)$, $w_1(\cdot)$, $w_2(\cdot)$ for the weak inequality (9) can be stated as

Theorem 2. *Let $0 \leq s < n$, $\Phi_2(\cdot)$ be a φ-function and $\Phi_1(\cdot)$ be an N-function with subadditive $(\Phi_1\Phi_2^{-1})(\cdot)$.*

Suppose that the weak inequality $M_s : L_v^{\Phi_1}(w_1) \to L_u^{\Phi_2,\infty}(w_2)$ is satisfied for some constants $C_1, C_2 > 0$. Then $(u,v) \in A_1(s,\Phi_1,\Phi_2,w_1,w_2)$, i.e.,

$$\Phi_2^{-1}\left[\int_Q \Phi_2\Big(A_2 w_2(x)|Q|^{\frac{s}{n}}\Big\|\frac{1}{w_1(\cdot)\varepsilon v(\cdot)}\Big\|_{\Phi_1^*,Q,\varepsilon v}\Big)u(x)dx\right] \leq \Phi_1^{-1}\Big[A_1\varepsilon^{-1}|Q|\Big]$$

for all cubes Q and all $\varepsilon > 0$. Here $A_1 = C_1$, and $A_2 = C_2$.

Conversely, if $(u,v) \in A_1(s,\Phi_1,\Phi_2,w_1,w_2)$, for some constants $A_1, A_2 > 0$, then $M_s : L_v^{\Phi_1}(w_1) \to L_u^{\Phi_2,\infty}(w_2)$ holds with the constants $C_1 = NA_1$, $C_2 = CA_2$.

For $w_2(\cdot) = 1$, the condition $(u,v) \in A(s,\Phi_1,\Phi_2,w_1,w_2)$ can be reduced to the easy form

$$\frac{1}{|Q|}\int_Q \Phi_1^*\left[|Q|^{\frac{s}{n}}\frac{A_2'}{\Phi_2^{-1}\left[\frac{\Phi_2\Phi_1^{-1}(A_1'\varepsilon^{-1}|Q|)}{\varepsilon^{-1}|Q|}\frac{1}{\frac{1}{|Q|}\int_Q \varepsilon u(y)dy}\right]}\frac{1}{w_1(x)\varepsilon v(x)}\right] \times$$
$$\times \varepsilon v(x)dx \leq 1.$$

Indeed, for each N-function $\Psi(\cdot)$ and each weight function $w(\cdot)$ we have $\|f\|_{[\Psi],w} \leq 1$ equivalent to $\int_{\mathbb{R}^n}\Psi(|f(x)|)w(x)dx \leq 1$. Observe that $(u,v) \in A_1(s,\Phi_1,\Phi_2,1,1)$ is the same as the condition $(u,v) \in A_1(s,\Phi_1,\Phi_2)$ introduced above.

Strong Inequalities

Proposition 3. *Let $0 \leq s < n$, $\Phi_1(\cdot)$, $\Phi_2(\cdot)$ be φ-functions with subadditive $(\Phi_1\Phi_2^{-1})(\cdot)$.*

Suppose $\Phi_1(\cdot)$ is an N-function and the strong inequality $M_s : L_v^{\Phi_1} \to L_u^{\Phi_2}$ is satisfied for some constants C_1, $C_2 > 0$. Then $(u,v) \in A_1(s,\Phi_1,\Phi_2)$ with the constants $A_1 = C_1$ and $A_2 = C_2$.

Conversely, we assume that $\Phi_1^{\frac{1}{r}}(\cdot)$ is an N-function for some $r > 1$ and $(u,v) \in A_r(s,\Phi_1,\Phi_2)$ for some constants $A_1, A_2 > 0$. Then $M_s : L_v^{\Phi_1} \to L_u^{\Phi_2}$ is satisfied with the constants $C_1 = N(n)A_1$, $C_2 = c(s,n)A_2$.

In the Lebesgue case $\Phi_1(t) = t^p$ $(p > 1)$ we take r as $r = \frac{p}{(p's)'} = \frac{ps-p+1}{s}$ $(s > 1)$. Here $r > 1$, and $\Phi_1^{\frac{1}{r}}(t) = t^{(p's)'}$. Thus our proposition includes a Perez' result [5]. In fact, the strong inequality considered here is the particular case of $M_s : L_v^{\Phi_1}(w_1) \to L_u^{\Phi_2}(w_2)$ or inequality (7). Thus the main result of this paper is

Theorem 4. *Let $0 \le s < n$, $\Phi_1(\cdot)$, $\Phi_2(\cdot)$ be φ-functions with subadditive $(\Phi_1\Phi_2^{-1})(\cdot)$.*

Let Φ_1 be an N-function and suppose that the strong inequality $M_s : L_v^{\Phi_1}(w_1) \to L_u^{\Phi_2}(w_2)$ is satisfied for some constants $C_1, C_2 > 0$. Then

$$(u,v) \in A_1(s, \Phi_1, \Phi_2, w_1, w_2),$$

with the constants $A_1 = C_1$ and $A_2 = C_2$.

Conversely, we assume that $\Phi_1^{\frac{1}{r}}(\cdot)$ is an N-function for some $r > 1$ and $(u,v) \in A_r(s, \Phi_1, \Phi_2, w_1, w_2)$ for some constants $A_1, A_2 > 0$, i.e.,

$$\Phi_2^{-1}\left[\int_Q \Phi_2\Big(A_2 w_2(x)|Q|^{\frac{s}{n}}\Big\|\frac{1}{w_1(\cdot)(\varepsilon v(\cdot))^{\frac{1}{r}}}\Big\|_{(\Phi_1^{\frac{1}{r}})^*, Q, (\varepsilon v(\cdot))^{\frac{1}{r}}}\Big)u(x)dx\right] \le$$

$$\le \Phi_1^{-1}\Big[A_1\varepsilon^{-1}|Q|\Big]$$

for all cubes Q and all $\varepsilon > 0$. Then $M_s : L_v^{\Phi_1}(w_1) \to L_u^{\Phi_2}(w_2)$ is satisfied with the constants $C_1 = N(n)A_1$, $C_2 = c(s,n)A_2$.

For $w_2(\cdot) = 1$, the condition $(u,v) \in A_r(s, \Phi_1, \Phi_2, w_1, w_2)$ can be reduced to the easy form

$$\frac{1}{|Q|}\int_Q (\Phi_1^{\frac{1}{r}})^*\left[|Q|^{\frac{s}{n}}\frac{A_2}{\Phi_2^{-1}\left[\frac{\Phi_2\Phi_1^{-1}(A_1\varepsilon^{-1}|Q|)}{\varepsilon^{-1}|Q|}\frac{1}{\frac{1}{|Q|}\int_Q \varepsilon u(y)dy}\right]}\frac{1}{w_1(x)(\varepsilon v(x))^{\frac{1}{r}}}\right]\times$$

$$\times(\varepsilon v(x))^{\frac{1}{r}}dx \le 1.$$

Due to the Hölder inequality, it is clear that for each $r > 1$

$$\Big\|\frac{1}{w_1(\cdot)\varepsilon v(\cdot)}\Big\|_{\Phi_1^*, Q, \varepsilon v(\cdot)} \le \Big\|\frac{1}{w_1(\cdot)[\varepsilon v(\cdot)]^{\frac{1}{r}}}\Big\|_{(\Phi_1^{\frac{1}{r}})^*, Q, [\varepsilon v(\cdot)]^{\frac{1}{r}}};$$

thus $(u,v) \in A_1(s, \Phi_1, \Phi_2, w_1, w_2)$ becomes a necessary and sufficient condition for the strong inequality $M_s : L_v^{\Phi_1}(w_1) \to L_u^{\Phi_2}(w_2)$ if for some $r > 1$

$$\Big\|\frac{1}{w_1(\cdot)\varepsilon v(\cdot)}\Big\|_{\Phi_1^*, Q, \varepsilon v(\cdot)} \ge C\Big\|\frac{1}{w_1(\cdot)[\varepsilon v(\cdot)]^{\frac{1}{r}}}\Big\|_{(\Phi_1^{\frac{1}{r}})^*, Q, [\varepsilon v(\cdot)]^{\frac{1}{r}}}.$$

The latter inequality was studied by Perez [5] in the Lebesgue case.

As we have seen in the first part of Proposition 3, $(u,v) \in A_1(s, \Phi_1, \Phi_2)$ is a necessary condition in order that $M_s : L_v^{\Phi_1} \to L_u^{\Phi_2}$. The argument we will use to get Theorem 4 also involves

Proposition 5. *Let $0 \leq s < n$, $\Phi_1(\cdot)$ be an N-function, $\Phi_2(\cdot)$ a φ-function with subadditive $(\Phi_1\Phi_2^{-1})(\cdot)$. Suppose $(u,v) \in A_1(s,\Phi_1,\Phi_2)$. Then*

$$\Phi_2^{-1}\left[\int_{\mathbb{R}^n} \Phi_2\big(C_2(M_s f)(x)\big)u(x)dx\right] \leq \Phi_1^{-1}\left[C_1 \int_{\mathbb{R}^n} \Phi_1(|f(x)|)v(x)dx\right]$$

for all functions $f(\cdot)$ satisfying the reverse Hölder inequality

$$\left(\frac{1}{|Q|_\omega}\int_Q \Big[\Phi_1(|f(x)|)v(x)\frac{1}{\omega(x)}\Big]\omega(x)dx\right)^{\frac{1}{r}} \leq$$

$$\leq c\frac{1}{|Q|_\omega}\int_Q \Big[\Phi_1(|f(x)|)v(x)\frac{1}{\omega(x)}\Big]^{\frac{1}{r}}\omega(x)dx, \tag{10}$$

where $r > 1$ and $\omega(\cdot)$ is an A_∞ weight function.

Of course, C_1, C_2 will depend on c, n and on the constants of the condition $A_1(s,\Phi_1,\Phi_2)$.

Now we will discuss the means for checking the test condition $(u,v) \in A_r(s,\Phi_1,\Phi_2,\, w_1,1)$. For simplicity we will consider only the case where $\Phi_1(\cdot) = \Phi_2(\cdot) = \Phi(\cdot)$, since the general case can be treated likewise with minor modifications. Let us denote by $\mathcal{A}[Q,\varepsilon,A_1,A_2]$ the quantity

$$\frac{1}{|Q|}\int_Q (\Phi^{\frac{1}{r}})^*\left[|Q|^{\frac{s}{n}}\frac{A_2}{\Phi^{-1}\left[A_1\frac{1}{\frac{1}{|Q|}\int_Q u\varepsilon}\right]}\frac{1}{w_1(x)(\varepsilon v(x))^{\frac{1}{r}}}\right](\varepsilon v(x))^{\frac{1}{r}}dx.$$

With this notation, $(u,v) \in A_r(s,\Phi,\Phi,w_1,1)$ if and only if there are A_1, $A_2 > 0$ for which $\sup_{\varepsilon>0}\sup_Q \mathcal{A}[Q,\varepsilon,A_1,A_2] \leq 1$. It is clear that

$$c_0 \sup_{\varepsilon>0}\sup_{R>0} \mathcal{A}[B(0,R),\varepsilon,c_1A_1,c_2A_2] \leq \sup_{\varepsilon>0}\sup_{Q} \mathcal{A}[Q,\varepsilon,A_1,A_2]$$

where $B(0,R)$ is the ball centered at the origin and with radius $R > 0$, and c_0, c_1, c_2 are nonnegative constants which depend on s and n.

In applications it is easier to compute $\mathcal{A}[B(0,R),\varepsilon,c_1A_1,c_2A_2]$ than the expression $\mathcal{A}[Q,\varepsilon,A_1,A_2]$, especialy when $u(\cdot)$, $v(\cdot)$ are radial weight functions. Thus it is interesting to know when the reverse of the last inequality is satisfied. In order to answer this question, we say that a weight function $w(\cdot)$ satisfies the growth condition $(\mathcal{C})$, when there are constants c, $C > 0$ such that

$$\sup_{R<|x|\leq 2R} w(x) \leq \frac{C}{R^n}\int_{|y|\leq cR} w(y)dy$$

for all $R > 0$. Many of the usual weight functions $w(\cdot)$ satisfy this growth condition, since both nonincreasing and nondecreasing radial functions are admissible. This is also the case when the weight $w(\cdot)$ is essentially constant on annuli, i.e., $w(y) \le cw(x)$ for $\frac{1}{2}|y| \le |x| \le 2|y|$.

Now we can state

Proposition 6. *Let $1 \le r < \infty$, and $(\Phi^{\frac{1}{r}})(\cdot)$ be an N-function. Assume that $u(\cdot)$ is a weight satisfying $(\mathcal{C})$. Also suppose that the weights $(\Phi^{\frac{1}{r}})^*\Big(\lambda \frac{1}{w_1(\cdot)[v(\cdot)]^{\frac{1}{r}}}\Big)[v(\cdot)]^{\frac{1}{r}}$ satisfy the growth condition $(\mathcal{C})$ uniformly in $\lambda > 0$. Then there are nonnegative constants c_0, c_1, c_2 such that*

$$\sup_{\varepsilon>0}\sup_{Q}\mathcal{A}[Q,\varepsilon,A_1,A_2] \le c_0 \sup_{\varepsilon>0}\sup_{R>0}\mathcal{A}[B(0,R),\varepsilon,c_1A_1,c_2A_2].$$

Indeed, c_0, c_1, c_2 depend on the constants in the growth conditions involving $u(\cdot)$, $v(\cdot)$, $w_1(\cdot)$ but not on these individual weights.

We only have to prove Theorems 2, 4, and Propositions 5, 6. We first begin by the basic result underlying the proof of these results.

§ 3. A Basic Lemma

Lemma 7. *Let $r \ge 1$, $\Phi_1^{\frac{1}{r}}(\cdot)$ be an N-function.*

Suppose $(u, v) \in A_r(s, \Phi_1, \Phi_2, w_1, w_2)$ for some constants A_1, $A_2 > 0$. Then

$$\Phi_2^{-1}\Bigg[\int_Q \Phi_2\Big(C_2 w_2(x)|Q|^{\frac{s}{n}}\Big[\frac{1}{|Q|}\int_Q |f(y)|dy\Big]\Big)u(x)dx\Bigg] \le$$
$$\le \Phi_1^{-1}\Bigg[C_1|Q|\Big(\frac{1}{|Q|}\int_Q \Big[\Phi_1(w_1(x)|f(x)|)v(x)\Big]^{\frac{1}{r}}dx\Big)^r\Bigg] \tag{11}$$

for all cubes Q and all locally integrable functions $f(\cdot)$ with supports contained in Q. Here $C_1 = A_1$, $C_2 = A_2$.

Conversely, inequality (11) implies $(u, v) \in A_r(s, \Phi_1, \Phi_2, w_1, w_2)$ with the constants $A_1 = C_1$, $A_2 = C_2$.

Let $\Phi(\cdot)$ be an N-function and $w(\cdot)$ a weight function. The Hölder inequality asserts that

$$\int_{\mathbb{R}^n} |f(y)g(y)|w(y)dy \le \|f\|_{[\Phi],w}\|g\|_{\Phi^*,w},$$

where $\|f\|_{[\Phi],w} = \inf\{s > 0;\ \int_{\mathbb{R}^n}\Phi(\lambda^{-1}|f(y)|)w(y)dy \le 1\}$.

Proof of Lemma 7. Suppose the condition $(u, v) \in A_r(s, \Phi_1, \Phi_2, w_1, w_2)$ is satisfied for some constants $A_1, A_2 > 0$.

Let Q be a cube and $f(\cdot)$ a locally integrable function whose support is contained in Q. Without loss of generality we can assume

$$0 < \int\limits_Q [\Phi_1(w_1(x)|f(x)|)v(x)]^{\frac{1}{r}} dx < \infty.$$

Note that $w_1(\cdot)v(\cdot)$ is not identically zero on Q. Then by the Hölder inequality we have

$$\frac{1}{|Q|}\int\limits_Q |f(x)|dx = \frac{1}{|Q|}\int\limits_Q \big(w_1(x)|f(x)|\big)\Big(\frac{1}{w_1(x)(\varepsilon v(x))^{\frac{1}{r}}}\Big)(\varepsilon v(x))^{\frac{1}{r}} dx \leq$$

$$\leq \big\|w_1(\cdot)f(\cdot)\big\|_{[\Phi_1^{\frac{1}{r}}],Q,(\varepsilon v)^{\frac{1}{r}}} \Big\|\frac{1}{w_1(\cdot)(\varepsilon v(\cdot))^{\frac{1}{r}}}\Big\|_{(\Phi_1^{\frac{1}{r}})^*,Q,(\varepsilon v)^{\frac{1}{r}}}$$

for all $\varepsilon > 0$. Choosing $\varepsilon > 0$ such that

$$\Big(\frac{1}{|Q|}\int\limits_Q \big[\Phi_1(w_1(x)|f(x)|)\varepsilon v(x)\big]^{\frac{1}{r}} dx\Big)^r = 1$$

or $\varepsilon^{-1} = (\frac{1}{|Q|}\int_Q [\Phi_1(w_1(x)|f(x)|)v(x)]^{\frac{1}{r}} dx)^r$, we have

$$\big\|w_1(\cdot)f(\cdot)\big\|_{[\Phi_1^{\frac{1}{r}}],Q,(\varepsilon v)^{\frac{1}{r}}} \leq 1.$$

Next, using the condition $(u, v) \in A_r(s, \Phi_1, \Phi_2, w_1, w_2)$ and the above two inequalities, we get

$$\mathcal{M} = \Phi_2^{-1}\Big[\int\limits_Q \Phi_2\Big(A_2 w_2(x)|Q|^{\frac{s}{n}}\Big[\frac{1}{|Q|}\int\limits_Q |f(y)|dy\Big]\Big)u(x)dx\Big] \leq$$

$$\leq \Phi_2^{-1}\Big[\int\limits_Q \Phi_2\Big(A_2 w_2(x)|Q|^{\frac{s}{n}}\Big\|\frac{1}{w_1(\cdot)(\varepsilon v(\cdot))^{\frac{1}{r}}}\Big\|_{(\Phi_1^{\frac{1}{r}})^*,Q,(\varepsilon v)^{\frac{1}{r}}}\Big)u(x)dx\Big] \leq$$

$$\leq \Phi_1^{-1}\big[A_1|Q|\varepsilon^{-1}\big] = \Phi_1^{-1}\Big[A_1|Q|\Big(\frac{1}{|Q|}\int\limits_Q \big[\Phi_1(w_1(x)|f(x)|)v(x)\big]^{\frac{1}{r}} dx\Big)^r\Big].$$

Therefore inequality (11) is satisfied with the constants $C_1 = A_1$, $C_2 = A_2$.

Conversely, suppose that inequality (11) is satisfied for some constants C_1, $C_2 > 0$. Let Q be a cube and $\varepsilon > 0$. By the definition of the Orlicz's norm, there is a nonnegative function $g(\cdot)$ such that

$$\frac{1}{|Q|}\int\limits_Q [\Phi_1(w_1(x)g(x))\varepsilon v(x)]^{\frac{1}{r}} dx \leq 1$$

and therefore

$$\Big\|\frac{1}{w_1(\cdot)(\varepsilon v(\cdot))^{\frac{1}{r}}}\Big\|_{(\Phi_1^{\frac{1}{r}})^*,Q,(\varepsilon v)^{\frac{1}{r}}} = \frac{1}{|Q|}\int\limits_Q g(y)dy.$$

Now using inequality (11) with such a function $g(\cdot)$ we obtain

$$\mathcal{M} = \Phi_2^{-1}\bigg[\int\limits_Q \Phi_2\Big(C_2 w_2(x)|Q|^{\frac{s}{n}}\Big\|\frac{1}{w_1(\cdot)(\varepsilon v(\cdot))^{\frac{1}{r}}}\Big\|_{(\Phi_1^{\frac{1}{r}})^*,Q,(\varepsilon v)^{\frac{1}{r}}}\Big)u(x)dx\bigg] \leq$$

$$\leq \Phi_2^{-1}\bigg[\int\limits_Q \Phi_2\Big(C_2|Q|^{\frac{s}{n}}\Big[\frac{1}{|Q|}\int\limits_Q g(y)dy\Big]w_2\Big)u(x)dx\bigg] \leq$$

$$\leq \Phi_1^{-1}\bigg[C_1|Q|\Big(\frac{1}{|Q|}\int\limits_Q \big[\Phi_1(w_1(x)g(x))v(x)\big]^{\frac{1}{r}}dx\Big)^r\bigg] \leq \Phi_1^{-1}\big[C_1|Q|\varepsilon^{-1}\big].$$

Thus $(u,v) \in A_r(s,\Phi_1,\Phi_2,w_1,w_2)$ with the constants $A_1 = C_1$, $A_2 = C_2$. $\square$

§ 4. Proof of Theorem 2

Assume that $\Phi_1(\cdot)$ is an N-function, and the weak inequality $M_s : L_v^{\Phi_1}(w_1) \to L_u^{\Phi_2,\infty}(w_2)$ is satisfied with the constants $C_1, C_2 > 0$. Let Q be a cube, and $f(\cdot)$ be a locally integrable function whose support is in Q. Since $(M_s f)(x) \geq |Q|^{\frac{s}{n}}(\frac{1}{|Q|}\int_Q |f(y)|dy)1\!\!1_Q(x)$, by taking the real $\lambda = |Q|^{\frac{s}{n}}(\frac{1}{|Q|}\int_Q |f(y)|dy)$ in the above weak inequality we get inequality (11) with the constants C_1 and C_2. Therefore by the first part of Lemma 7, $(u,v) \in A_1(s,\Phi_1,\Phi_2,w_1,w_2)$ with the constants $A_1 = C_1$, $A_2 = C_2$.

To prove the converse, we follow the same lines we used in [12]. Thus, suppose $(u,v) \in A_1(s,\Phi_1,\Phi_2,w_1,w_2)$ with some constants A_1, $A_2 > 0$. Let N be a nonnegative integer and M_s^N the truncated maximal operator defined by

$$(M_s^N f)(x) = \sup\bigg\{|Q|^{\frac{s}{n}-1}\int\limits_Q |f(y)|dy;$$
$$Q \text{ is a cube with } Q \ni x \text{ and } |Q| \leq N\bigg\}.$$

Let $\lambda > 0$, $f(\cdot)$ be a locally integrable function and $\Omega_{\lambda,N} = \{(M_s^N f)(\cdot) > \lambda\}$. Since $\int_{\{(M_s f)(x)>\lambda\}} u(x)dx = \lim_{N\to\infty}\int_{\Omega_{\lambda,N}} u(x)dx$, it is sufficient to prove

$$\int\limits_{\Omega_{\lambda,N}} \Phi_2\big(C_2\lambda w_2(x)\big)u(x)dx \leq \Phi_2\Phi_1^{-1}\bigg[C_1\int\limits_{\mathbb{R}^n}\Phi_1\Big(w_1(x)|f(x)|\Big)v(x)dx\bigg],$$

where C_1, C_2, do not depend on the integer N.

For every $x \in \Omega_{\lambda,N}$, there is a cube $Q(x)$ centered at x such that

$$|Q(x)|^{\frac{s}{n}}\Big(\frac{1}{|Q(x)|}\int\limits_{Q(x)}|f(y)|dy\Big) > c\lambda,$$

where $c = c(s,n) > 0$ depends only on s and n. Since $\sup\{|Q(x)|;\ x \in \Omega_{\lambda,N}\} < \infty$, by the classical Besicovitch covering theorem we can choose from the set $\{Q(x);\ x \in \Omega_{\lambda,N}\}$ a sequence of cubes $(Q_k)_k$ satisfying

$$\Omega_{\lambda,N} \subset \bigcup_k Q_k;\quad \sum_k 1\!\!\mathrm{I}_{Q_k} \le K(n) 1\!\!\mathrm{I}_{\bigcup\limits_k Q_k};\quad |Q_k|^{\frac{s}{n}}\left(\frac{1}{|Q_k|}\int\limits_{Q_k}|f(y)|dy\right) > c\lambda.$$

By the second part of Lemma 7, the condition $(u,v) \in A_1(s,\Phi_1,\Phi_2,w_1,w_2)$ implies inequality (11). Using the latter and the fact that $(\Phi_1\Phi_2^{-1})(\cdot)$ is subadditive, we obtain

$$\begin{aligned}
&\int\limits_{\Omega_{\lambda,N}} \Phi_2\big(cA_2\lambda w_2(x)\big)u(x)dx \le \sum_k \int\limits_{Q_k}\Phi_2\big(cA_2\lambda w_2(x)\big)u(x)dx \le\\
&\le \sum_k\int\limits_{Q_k}\Phi_2\left(A_2 w_2(x)|Q_k|^{\frac{s}{n}}\left[\frac{1}{|Q_k|}\int\limits_{|Q_k|}|f(y)|dy\right]\right)u(x)dx \le\\
&\le \sum_k \Phi_2\Phi_1^{-1}\left[A_1\int\limits_{Q_k}\Phi_1\big(w_1(x)|f(x)|\big)v(x)dx\right] \le\\
&\le \Phi_2\Phi_1^{-1}\left[A_1\sum_k\int\limits_{Q_k}\Phi_1\big(w_1(x)|f(x)|\big)v(x)dx\right] \le\\
&\le \Phi_2\Phi_1^{-1}\left[C(n)A_1\int\limits_{R^n}\Phi_1\big(w_1(x)|f(x)|\big)v(x)dx\right],
\end{aligned}$$

and therefore $M_s : L_v^{\Phi_1}(w_1) \to L_u^{\Phi_2,\infty}(w_2)$ holds with the constants $C_1 = C(n)A_1$, $C_2 = c(s,n)A_2$. $\square$

§ 5. Proofs of Theorem 4 and Proposition 5

Proof of the necessity part of Theorem 4. Assume that Φ_1 is an N-function and suppose the strong inequality $M_s : L_v^{\Phi_1}(w_1) \to L_u^{\Phi_2}(w_2)$ is satisfied for some constants $C_1, C_2 > 0$. This implies the weak inequality $M_s : L_v^{\Phi_1}(w_1) \to L_u^{\Phi_2,\infty}(w_2)$ with the same constants $C_1, C_2 > 0$. Thus, as in the proof of Theorem 2, $(u,v) \in A(s,\Phi_1,\Phi_2,w_1,w_2)$ with the constants $A_1 = C_1$, $A_2 = C_2$. $\square$

To prove the converse, to get $M_s : L_v^{\Phi_1}(w_1) \to L_u^{\Phi_2}(w_2)$, we first do some preliminaries so as to discretize the operator M_s.

Preliminaries for the proof of sufficiency part of Theorem 4. Recall that the dyadic version M_s^d of the maximal operator M_s is defined as

$$(M_s^d f)(x) = \sup\left\{|Q|^{\frac{s}{n}-1}\int\limits_Q |f(y)|dy;\ Q \text{ is a dyadic cube with } Q \ni x\right\}.$$

We need

Lemma 8. *Let $f(\cdot)$ be a function with a compact support and $\lambda > 0$. If $\{(M_s f)(\cdot) > \lambda\}$ is a nonempty set then we can find a family of nonoverlapping maximal dyadic cubes $(Q_j)_j$ for which*

$$\{(M_s f)(\cdot) > \lambda\} \subset \underset{j}{\cup}(3Q_j),\quad 2^{s-2n}\lambda < |Q_j|^{\frac{s}{n}}\left(\frac{1}{|Q_j|}\int\limits_{Q_j}|f(y)|dy\right) \le 2^{-n}\lambda,$$
$$\{(M_s^d f)(\cdot) > 2^{s-2n}\lambda\} = \underset{j}{\cup} Q_j.$$

This lemma is the standard one and was proved in [5]. As a consequence we obtain

Lemma 9. *Let $f(\cdot)$ be a function with a compact support. For each integer k let*

$$\Omega_k = \{(M_s f)(\cdot) > a^k\},\quad \Gamma_k = \{(M_s^d f)(\cdot) > 2^{-2n}a^k\},\quad \textit{where}\quad a = 2^{-3n}.$$

There is a family of maximal nonoverlapping dyadic cubes $(Q_{kj})_j$ for which

$$\Omega_k \subset \underset{j}{\cup}(3Q_{kj}),\quad \Gamma_k = \underset{j}{\cup} Q_{kj},$$
$$a^k < |Q_{kj}|^{\frac{s}{n}}\left(\frac{1}{|Q_{kj}|}\int\limits_{Q_{kj}}|f(y)|dy\right) \le 2^{n-s}a^k. \tag{12}$$

Let $E_{k,j} = Q_{kj}\backslash(Q_{kj}\cap\Gamma_{k+1})$. Then $(E_{k,j})_{k,j}$ is a disjoint family of sets for which

$$|Q_{k,j}| < c|E_{k,j}| \quad \textit{with}\quad c = c(s,n) = \frac{2^{2(n+s)}}{2^{2(n+s)}-1}. \tag{13}$$

The first part of this result is an immediate consequence of Lemma 8, for $\lambda = 2^{2n-s}a^k$, while the second one follows from the estimate

$$|Q_{k,j}\cap\Gamma_{k+1}| < 2^{-2(n+s)}|Q_{k,j}|.$$

The latter one can be obtained from (1) as follows,

$$
\begin{aligned}
|Q_{k,j}\cap\Gamma_{k+1}| &= \sum_i |Q_{k,j}\cap Q_{k+1,i}| \le \sum_{i;Q_{k+1,i}\subset Q_{k,j}} |Q_{k+1,i}| \le \\
&\le \frac{1}{a^{k+1}} \sum_{i;Q_{k+1,i}\subset Q_{k,j}} |Q_{k+1,i}|^{\frac{s}{n}} \int\limits_{Q_{k+1,i}} |f(y)|dy \le \\
&\le \frac{2^{-s}}{a^{k+1}} \sum_{i;Q_{k+1,i}\subset Q_{k,j}} |Q_{k,j}|^{\frac{s}{n}} \int\limits_{Q_{k+1,i}} |f(y)|dy \le \\
&\le \frac{2^{-s}}{a^{k+1}} |Q_{k,j}|^{\frac{s}{n}} \int\limits_{Q_{k,j}} |f(y)|dy \le \frac{2^{n-2s}}{a}|Q_{k,j}| = 2^{-2(n+s)}|Q_{k,j}|.
\end{aligned}
$$

Now we can proceed to

Proof of the sufficiency part of Theorem 4. We assume that $\Phi_1^{\frac{1}{r}}(\cdot)$ is an N-function for some $r>1$ and $(u,v)\in A_r(s,\Phi_1,\Phi_2,w_1,w_2)$ for some constants $A_1, A_2>0$. We have to prove $M_s : L_v^{\Phi_1}(w_1)\to L_u^{\Phi_2}(w_2)$, i.e.,

$$
\int\limits_{\mathbb{R}^n} \Phi_2\big(C_2 w_2(x)(M_s f)(x)\big)u(x)dx \le \Phi_2\Phi_1^{-1}\left[C_1 \int\limits_{\mathbb{R}^n} \Phi_1\big(w_1(x)|f(x)|\big)\right]v(x)dx
$$

for all locally integrable functions $f(\cdot)$.

We assume that $f(\cdot)$ is bounded and it has a compact support. We do not lose generality, since the estimates we obtain do not depend on the bound of $f(\cdot)$, and the monotone convergence theorem yields the conclusion. With the notations of Lemma 9, we have

$$
\begin{aligned}
\mathcal{S} &= \int\limits_{\mathbb{R}^n} \Phi_2\big(3^{-n}2^{-3n}A_2 w_2(x)(M_s f)(x)\big)u(x)dx = \\
&= \sum_k \int\limits_{\Omega_k-\Omega_{k+1}} \Phi_2\big(3^{-n}2^{-3n}A_2 w_2(x)(M_s f)(x)\big)u(x)dx \le \\
&\le \sum_{k,j} \int\limits_{3Q_{k,j}} \Phi_2\big(3^{-n}2^{-3n}A_2 a^{k+1} w_2(x)\big)u(x)dx \le \\
&\le \sum_{k,j} \int\limits_{3Q_{k,j}} \Phi_2\left(3^{-n}A_2 w_2(x)|Q_{k,j}|^{\frac{s}{n}}\Big[\frac{1}{|Q_{k,j}|}\int\limits_{Q_{k,j}}|f(y)|dy\Big]\right)u(x)dx \le
\end{aligned}
$$

(see estimate (12) in Lemma 9)

$$\leq \sum_{k,j} \int\limits_{3Q_{k,j}} \Phi_2\Big(A_2 w_2(x)|3Q_{k,j}|^{\frac{s}{n}}\Big[\frac{1}{|3Q_{k,j}|}\int\limits_{3Q_{k,j}} |f(y)|dy\Big]\Big)u(x)dx \leq$$

$$\leq \sum_{k,j} \Phi_2\Phi_1^{-1}\Big[A_1|3Q_{k,j}|\Big(\frac{1}{|3Q_{k,j}|}\int\limits_{3Q_{k,j}} \big[\Phi_1\big(w_1(x)|f(x)|\big)v(x)\big]^{\frac{1}{r}}dx\Big)^r\Big] \leq$$

(by the first part of Lemma 7)

$$\leq \Phi_2\Phi_1^{-1}\Big[A_1\sum_{k,j}|3Q_{k,j}|\Big(\frac{1}{|3Q_{k,j}|}\int\limits_{3Q_{k,j}} \big[\Phi_1\big(w_1(x)|f(x)|\big)v(x)\big]^{\frac{1}{r}}dx\Big)^r\Big] \leq$$

(recall that $\Phi_1^{-1}\Phi_2$ is subadditive)

$$= \Phi_2\Phi_1^{-1}\Big[3^n A_1\sum_{k,j}|Q_{k,j}|\Big(\frac{1}{|3Q_{k,j}|}\int\limits_{3Q_{k,j}} \big[\Phi_1\big(w_1(x)|f(x)|\big)v(x)\big]^{\frac{1}{r}}dx\Big)^r\Big] \leq$$

$$\leq \Phi_2\Phi_1^{-1}\Big[3^n c(s,n)A_1\sum_{k,j}|E_{k,j}| \times$$

$$\times \Big(\frac{1}{|3Q_{k,j}|}\int\limits_{3Q_{k,j}} \big[\Phi_1\big(w_1(x)|f(x)|\big)v(x)\big]^{\frac{1}{r}}dx\Big)^r\Big] \leq$$

(see estimate (13) in Lemma 9)

$$\leq \Phi_2\Phi_1^{-1}\Big[3^n c(s,n)A_1\sum_{k,j}\int\limits_{E_{k,j}} \big(M[\Phi_1(w_1|f|)v]^{\frac{1}{r}}\big)^r(x)dx\Big] \leq$$

($M = M_0$ is the Hardy–Littlewood maximal operator)

$$\leq \Phi_2\Phi_1^{-1}\Big[3^n c(s,n)A_1\int\limits_{\mathbb{R}^n} \big(M\big[\Phi_1(w_1|f|)v\big]^{\frac{1}{r}}\big)^r(x)dx\Big] \leq$$

($E_{k,j}$'s are disjoint sets)

$$\leq \Phi_2\Phi_1^{-1}\Big[3^n c(s,n)A_1\int\limits_{\mathbb{R}^n} \Phi_1\big(w_1(x)|f(x)|\big)v(x)dx\Big]$$

(by the well known maximal theorem ($r > 1$)). Thus the inequality is fulfilled with the constants $C_1 = 3^n\frac{2(n+s)}{2^{2(n+s)}-1}A_1$, $C_2 = 2^{-3n}3^{-n}A_2$. $\square$

Proof of proposition 5. Since this result can be obtained by using a few changes in the above estimates, we will outline the essential arguments. With the above notation, the condition $\omega(\cdot) \in A_\infty$ implies the existence of a constant $c = c(\omega) > 0$ such that

$$|E_{kj}|_\omega \leq c|Q_{kj}|_\omega.$$

Now, as above, by Lemma 7 (recall that $(u,v) \in A_1(s, \Phi_1, \Phi_2)$) we obtain

$$\mathcal{S} = \int_{\mathbb{R}^n} \Phi_2\big(c_2 A_2 (M_s f)(x)\big) u(x) dx \leq$$

$$\leq \Phi_2 \Phi_1^{-1} \Big[c_1 A_1 \sum_{k,j} |E_{kj}|_\omega \Big(\frac{1}{|3Q_{kj}|_\omega} \int_{3Q_{k,j}} \Big[\Phi_1(|f(x)|) v(x) \frac{1}{\omega(x)} \Big] \omega(x) dx \Big) \Big] \leq$$

$$\leq \Phi_2 \Phi_1^{-1} \Big[c_1 A_1 \sum_{k,j} |E_{kj}|_\omega \Big(\frac{1}{|3Q_{kj}|_\omega} \int_{3Q_{k,j}} \Big[\Phi_1(|f(x)|) v(x) \frac{1}{\omega(x)} \Big]^{\frac{1}{r}} \omega(x) dx \Big)^r \Big] \leq$$

$$\leq \Phi_2 \Phi_1^{-1} \Big[c c_1 A_1 \sum_{k,j} \int_{E_{kj}} \Big(M_\omega \Big[\Phi_1(|f|) v \frac{1}{\omega} \Big]^{\frac{1}{r}} \Big)^r (x) \omega(x) dx \Big] \leq$$

$$\leq \Phi_2 \Phi_1^{-1} \Big[c c_1 A_1 \int_{R^n} \Big([M_\omega \Big[\Phi_1(|f|) v \frac{1}{\omega} \Big]^{\frac{1}{r}} \Big)^r (x) \omega(x) dx \Big] \leq$$

$$\leq \Phi_2 \Phi_1^{-1} \Big[c c_1' A_1 \int_{R^n} \Phi_1\big(|f(x)|\big) v(x) dx \Big]. \quad \square$$

§ 6. Proof of Proposition 6

Let $\varepsilon > 0$ and Q be a cube centered at x_0 and having a side with length $R > 0$.

First we suppose $|x_0| \leq 2R$. Then for a constant $c_1 \geq 1$ which depends only on the dimension n we obtain $Q \subset B(0, c_1 R)$, which results in

$$\mathcal{A}[Q, \varepsilon, A_1, A_2] \leq (c_1^n) \frac{1}{(c_1 R)^n} \int_{B(0,c_1 R)} (\Phi^{\frac{1}{r}})^* \times$$

$$\times \left[(c_1 R)^s \frac{(c_1)^{-s} A_2}{\Phi^{-1}\Big[(c_1)^{-n} A_1 \frac{1}{\frac{1}{(c_1 R)^n} \int_{B(0,c_1 R)} \varepsilon u(y) dy} \Big]} \frac{1}{w_1(x) (\varepsilon v(x))^{\frac{1}{r}}} \right] (\varepsilon v(x))^{\frac{1}{r}} dx.$$

Next we consider the case $2R < |x_0|$. Then $|x| \approx |x_0|$ whenever $x \in Q$, and $Q \subset B(0, c_3|x_0|)$ with a constant $c_3 \geq 1$ which depends only on n. Thus, the growth condition on $u(\cdot)$ yields

$$\frac{1}{|Q|}\int\limits_Q u(y)dy \leq \frac{c_4(n,u)}{(c_3|x_0|)^n}\int\limits_{B(0,c_3|x_0|)} u(y)dy.$$

Since the family of weights $(\Phi^{\frac{1}{r}})^*\Big(\lambda\frac{1}{w_1(\cdot)[v(\cdot)]^{\frac{1}{r}}}\Big)[v(\cdot)]^{\frac{1}{r}}$ satisfies uniformly (in λ) the growth condition $(\mathcal{C})$, we obtain

$$\mathcal{A}[Q,\varepsilon,A_1,A_2] \leq \frac{1}{|Q|}\int\limits_Q (\Phi^{\frac{1}{r}})^* \times$$

$$\times\left[(c_3|x_0|)^s\frac{c_2'A_2}{\Phi^{-1}\Big[c_1'A_1\frac{1}{\frac{1}{(c_3|x_0|)^n}\int_{B(0,c_3|x_0|)}\varepsilon u(y)dy}\Big]}\frac{1}{w_1(x)(\varepsilon v(x))^{\frac{1}{r}}}\right](\varepsilon v(x))^{\frac{1}{r}}dx \leq$$

$$\leq \sup_{x\in Q}(\Phi^{\frac{1}{r}})^* \times$$

$$\times\left[(c_3|x_0|)^s\frac{c_2'A_2}{\Phi^{-1}\Big[c_1'A_1\frac{1}{\frac{1}{(c_3|x_0|)^n}\int_{B(0,c_3|x_0|)}\varepsilon u(y)dy}\Big]}\frac{1}{w_1(x)(\varepsilon v(x))^{\frac{1}{r}}}\right](\varepsilon v(x))^{\frac{1}{r}} \leq$$

$$\leq c_5\frac{1}{(c_3|x_0|)^n}\int\limits_{B(0,c_3|x_0|)}(\Phi^{\frac{1}{r}})^* \times$$

$$\times\left[(c_3|x_0|)^s\frac{c_2'A_2}{\Phi^{-1}\Big[c_1'A_1\frac{1}{\frac{1}{(c_3|x_0|)^n}\int_{B(0,c_3|x_0|)}\varepsilon u(y)dy}\Big]}\frac{1}{w_1(x)(\varepsilon v(x))^{\frac{1}{r}}}\right](\varepsilon v(x))^{\frac{1}{r}}dx.$$

Finally, by these estimates we get

$$\sup_{\varepsilon>0}\sup_Q \mathcal{A}[Q,\varepsilon,A_1,A_2] \leq c_0 \sup_{\varepsilon>0}\sup_{R>0}\mathcal{A}[B(0,R),\varepsilon,c_1''A_1,c_2''A_2]. \qquad \square$$

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