GEORGIAN MATHEMATICAL JOURNAL: Vol. 3, No. 6, 1996, 571-582

ON OSCILLATION OF SOLUTIONS OF SECOND-ORDER SYSTEMS OF DEVIATED DIFFERENTIAL EQUATIONS

N. PARTSVANIA

ABSTRACT. Sufficient conditions are found for the oscillation of proper solutions of the system of differential equations

$$u_1'(t) = f_1\big(t, u_1(\tau_1(t)), \dots, u_1(\tau_m(t)), u_2(\sigma_1(t)), \dots, u_2(\sigma_m(t))\big),$$
$$u_2'(t) = f_2\big(t, u_1(\tau_1(t)), \dots, u_1(\tau_m(t)), u_2(\sigma_1(t)), \dots, u_2(\sigma_m(t))\big),$$

where $f_i : R_+ \times R^{2m} \to R$ $(i = 1, 2)$ satisfy the local Carathéodory conditions and $\sigma_i, \tau_i : R_+ \to R$ $(i = 1, \dots, m)$ are continuous functions such that $\sigma_i(t) \le t$ for $t \in R_+$, $\lim\limits_{t\to+\infty} \sigma_i(t) = +\infty$, $\lim\limits_{t\to+\infty} \tau_i(t) = +\infty$ $(i = 1, \dots, m)$.

INTRODUCTION

The problem of oscillation of solutions of second-order ordinary differential equations has been studied well enough. A number of papers were devoted to an analogous problem for deviated differential equations and systems of ordinary differential equations (see [2]–[6]). As to systems of deviated differential equations, for them the problem of oscillation of solutions has not been studied enough. In the present paper, we give the results for systems of differential equations, which generalize some well-known statements for second-order differential equations.

Consider the system of differential equations

$$\begin{aligned} u_1'(t) &= f_1\big(t, u_1(\tau_1(t)), \dots, u_1(\tau_m(t)), u_2(\sigma_1(t)), \dots, u_2(\sigma_m(t))\big), \\ u_2'(t) &= f_2\big(t, u_1(\tau_1(t)), \dots, u_1(\tau_m(t)), u_2(\sigma_1(t)), \dots, u_2(\sigma_m(t))\big), \end{aligned} \tag{0.1}$$

where $f_i : R_+ \times R^{2m} \to R$ $(i = 1, 2)$ satisfy the local Carathéodory conditions and $\sigma_i, \tau_i : R_+ \to R$ $(i = 1, \dots, m)$ are continuous functions such

1991 *Mathematics Subject Classification.* 34K15.

Key words and phrases. System of deviated differential equations, proper solution, oscillatory solution.

1072-947X/96/1100-0571\$09.50/0

that

$$\begin{gathered}\sigma_i(t) \leq t \quad \text{for} \quad t \in R_+, \quad \lim_{t\to+\infty} \sigma_i(t) = +\infty, \\ \lim_{t\to+\infty} \tau_i(t) = +\infty (i = 1, \dots, m).\end{gathered} \tag{0.2}$$

Definition 0.1. Let $t_0 \in R_+$. A continuous vector-function (u_1, u_2) defined on $[a_0, +\infty[$ (where $a_0 = \min[\min\{\inf_{t\geq t_0} \tau_i(t) : i = 1, \dots, m\}, \min\{\inf_{t\geq t_0} \sigma_i(t) : i = 1, \dots, m\}])$ is said to be a *proper* solution of system (0.1) in $[t_0, +\infty[$ if it is absolutely continuous on each finite segment contained in $[t_0, +\infty[$, satisfies (0.1) almost everywhere in $[t_0, +\infty[$, and

$$\sup\Big\{|u_1(s)| + |u_2(s)| : \ s \geq t\Big\} > 0 \quad \text{for} \quad t \in [t_0, +\infty[\,.$$

Definition 0.2. A proper solution (u_1, u_2) of system (0.1) is said to be *weakly oscillatory* if either u_1 or u_2 has a sequence of zeros tending to infinity. This solution is said to be *oscillatory* if both u_1 and u_2 have sequences of zeros tending to infinity. If there exists $t_0 \in R_+$ such that $u_1(t)u_2(t) \neq 0$ for $t \in [t_0, +\infty[$, then (u_1, u_2) is said to be *nonoscillatory*.

Throughout the paper the following notation will be used:

$$\begin{gathered}\tau_*(t) = \min\big\{t, \ \tau_i(t) : \ i = 1, \dots, m\big\}, \quad \sigma_*(t) = \min\big\{\sigma_i(t) : \ i = 1, \dots, \big\}, \\ \widetilde{\sigma}(t) = \inf\big\{\sigma_*(s) : \ s \geq t\big\}, \quad \sigma^*(t) = \max\big\{\sigma_i(t) : \ i = 1, \dots, m\big\}.\end{gathered}$$

§ 1. Auxiliary Statements

In this section, we consider the system of differential inequalities

$$\begin{gathered}u_1'(t)\operatorname{sign} u_2(t) \geq \sum_{i=1}^m p_i(t)\big|u_2(\sigma_i(t))\big|, \\ u_2'(t)\operatorname{sign} u_1(t) \leq -\sum_{i=1}^m q_i(t)\big|u_1(\tau_i(t))\big|,\end{gathered} \tag{1.1}$$

where $p_i, q_i \in L_{loc}(R_+; R_+)$ $(i = 1, \dots, m)$ and $\sigma_i, \tau_i : R_+ \to R$ $(i = 1, \dots, m)$ are continuous functions satisfying (0.2).

Lemma 1.1. *Let* (u_1, u_2) *be a nonoscillatory solution of* (1.1) *and*

$$h(+\infty) = +\infty, \tag{1.2}$$

where

$$h(t) = \int_0^t \widetilde{p}(s)ds, \quad \widetilde{p}(t) = \sum_{i=1}^m p_i(t). \tag{1.3}$$

Then there exists $t^* \in R_+$ *such that*

$$u_1(t)u_2(t) > 0 \quad \textit{for} \quad t \in [t^*, +\infty[\,. \tag{1.4}$$

If, moreover,

$$\int_0^{+\infty} \widetilde{q}(t)h(t)dt = +\infty, \tag{1.5}$$

where H *is defined by* (1.3) *and*

$$\widetilde{q}(t) = \sum_{i=1}^{m} q_i(t), \tag{1.6}$$

then

$$\lim_{t\to+\infty} |u_1(t)| = +\infty. \tag{1.7}$$

Proof. Suppose that (u_1, u_2) does not satisfy (1.4). Then there exists $t_0 \in R_+$ such that

$$u_1(t)u_2(t) < 0 \quad \text{for} \quad t \in [t_0, +\infty[\,. \tag{1.8}$$

By (1.8) we have from (1.1)

$$|u_1(t)|' \le -\sum_{i=1}^{m} p_i(t)\big|u_2(\sigma_i(t))\big|, \tag{1.9}$$

$$|u_2(t)|' \ge \sum_{i=1}^{m} q_i(t)\big|u_1(\tau_i(t))\big| \quad \text{for} \quad t \in [t_0, +\infty[\,. \tag{1.10}$$

Thus by (1.10) from (1.9) we have

$$|u_1(t)| \le |u_1(t_1)| - c\int_{t_1}^{t} \widetilde{p}(s)ds \quad \text{for} \quad t \in [t_1, +\infty[\,,$$

where $t_1 \in\,]t_0, +\infty[$ is a sufficiently large number and

$$c = \min\Big\{ \inf_{t\in[t_1,+\infty[} \big|u_2(\sigma_i(t))\big| : \ i = 1, \dots, m \Big\} > 0.$$

By (1.2) the latter inequality implies $|u_1(t)| \to -\infty$ as $t \to +\infty$. The contradiction obtained proves (1.4).

Now prove that (1.5) implies (1.7). Since (1.4) is satisfied, from (1.1) we find

$$
\begin{aligned}
&|u_1(t)|' \geq \sum_{i=1}^{m} p_i(t)\big|u_2(\sigma_i(t))\big|,\\
&|u_2(t)|' \leq -\sum_{i=1}^{m} q_i(t)\big|u_1(\tau_i(t))\big| \quad \text{for} \quad t\in[t^*,+\infty[\,.
\end{aligned}
\tag{1.11}
$$

Since $|u_1(t)|$ is a nondecreasing function, by virtue of (0.2) there exist $t_2\in[t^*,+\infty[$ and $a\in]0,+\infty[$ such that

$$
\big|u_1(\tau_i(t))\big| \geq a \quad \text{for} \quad t\in[t_2,+\infty[\ \ (i=1,\dots,m),
$$

whence by (1.11) we get

$$
|u_2(t)|' h(t) \leq -a\widetilde{q}(t)h(t) \quad \text{for} \quad t\in[t_2,+\infty[\,.
$$

Integrating from t_2 to t, we obtain

$$
\begin{aligned}
|u_2(t)|h(t) - |u_2(t_2)|h(t_2) + a\int_{t_2}^{t} \widetilde{q}(s)h(s)ds \leq\\
\leq \int_{t_2}^{t} \widetilde{p}(s)|u_2(s)|ds.
\end{aligned}
$$

Taking into account (0.2) and the fact that $|u_2(t)|$ is a decreasing function, from the latter inequality we have

$$
\begin{aligned}
\int_{t_2}^{t}\sum_{i=1}^{m} p_i(s)\big|u_2(\sigma_i(s))\big|ds \geq \int_{t_2}^{t} \widetilde{p}(s)|u_2(s)|ds \geq\\
\geq a\int_{t_2}^{t} \widetilde{q}(s)h(s)ds - |u_2(t_2)|h(t_2).
\end{aligned}
\tag{1.12}
$$

From (1.11) we find

$$
|u_1(t)| \geq |u_1(t_2)| + \int_{t_2}^{t}\sum_{i=1}^{m} p_i(s)\big|u_2(\sigma_i(s))\big|ds \quad \text{for} \quad t\in[t_2,+\infty[\,,
$$

which by virtue of (1.5), (1.12) implies that (1.7) is valid. $\square$

Lemma 1.2. *If*

$$\int_t^{+\infty} \widetilde{p}(s)ds > 0, \quad \int_t^{+\infty} \widetilde{q}(s)ds > 0 \quad \text{for} \quad t \in R_+, \tag{1.13}$$

where $\widetilde{p}$ and $\widetilde{q}$ are defined by (1.3) *and* (1.6), *respectively, then every weakly oscillatory solution of* (1.1) *is oscillatory.*

Proof. Let (u_1, u_2) be a weakly oscillatory solution of (1.1), and suppose that this solution is not oscillatory. Without loss of generality it can be assumed that there exists an increasing sequence of points $\{t_k\}$ tending to $+\infty$ such that

$$\begin{gathered} u_1(t_k) = 0 \quad (k = 1, 2, \dots), \\ u_2(\sigma_i(t)) > 0 \quad \text{for} \quad t \in [t_1, +\infty[\ \ (i = 1, \dots, m). \end{gathered} \tag{1.14}$$

On account of (1.13), (1.14), there is $k \in N$ such that

$$\int_{t_1}^{t_k} \sum_{i=1}^{m} p_i(s) u_2(\sigma_i(s))ds > 0.$$

On the other hand, by (1.14) from (1.1) we have

$$0 \geq \int_{t_1}^{t_k} \sum_{i=1}^{m} p_i(s) u_2(\sigma_i(s))ds > 0.$$

The contradiction obtained proves the validity of the lemma. □

§ 2. Oscillatory Solutions

Theorem 2.1. *Let*

$$\begin{gathered} f_1(t, x_1, \dots, x_m, y_1, \dots, y_m) \operatorname{sign} y_1 \geq \sum_{i=1}^{m} p_i(t)|y_i|, \\ f_2(t, x_1, \dots, x_m, y_1, \dots, y_m) \operatorname{sign} x_1 \leq -\sum_{i=1}^{m} q_i(t)|x_i| \\ \text{for} \quad t \in R_+, \quad x_1 x_i > 0, \quad y_1 y_i > 0 \quad (i = 1, \dots, m), \end{gathered} \tag{2.1}$$

where $p_i, q_i \in L_{loc}(R_+; R_+)$ $(i = 1, \dots, m)$, and conditions (1.2), (1.5), *and*

$$\int_0^{+\infty} \sum_{i=1}^{m} q_i(t) h(\tau_i(t))dt = +\infty \tag{2.2}$$

be fulfilled. Let, moreover, there exist $\varepsilon_0 > 0$ and a nondecreasing function $\delta : R_+ \to R$ such that $\delta(t) \geq \sigma_*(t)$, and for any $\lambda \in [0,1[$

$$\int_0^{\tau_*(\widetilde{\sigma}(t))} h^{\varepsilon_0}(\delta(s)) \sum_{i=1}^m p_i(s) \int_{\sigma_i(s)}^{+\infty} \sum_{i=1}^m q_i(\xi) h^\lambda(\tau_i(\xi)) d\xi ds >$$
$$> h^\lambda(\tau_*(\widetilde{\sigma}(t))) h^{\varepsilon_0}(\delta(t)) \quad \textit{for} \quad t \in [t_0, +\infty[\,, \tag{2.3}$$

where $t_0 \in R_+$ and h is defined by (1.3). *Then every proper solution of* (0.1) *is oscillatory.*

Proof. Let (u_1, u_2) be a proper solution of (0.1). By virtue of (2.1) it will be a solution of (1.1) as well. Suppose that this solution is not oscillatory. By (1.2) and (2.2) it is obvious that all the conditions of Lemma 1.2 are fulfilled. Thus (u_1, u_2) is nonoscillatory. Therefore by virtue of condition (1.5) and Lemma 1.1 one can find $t_0 \in R_+$ such that

$$u_1(t) u_2(t) > 0 \quad \text{for} \quad t \in [t_0, +\infty[\,, \tag{2.4}$$

and

$$\lim_{t \to +\infty} |u_1(t)| = +\infty. \tag{2.5}$$

By (2.4) we have from (1.1)

$$\begin{aligned} |u_1(t)|' &\geq \sum_{i=1}^m p_i(t) \big| u_2(\sigma_i(t)) \big|, \\ |u_2(t)|' &\leq -\sum_{i=1}^m q_i(t) \big| u_1(\tau_i(t)) \big| \quad \text{for} \quad t \in [t_0, +\infty[\,. \end{aligned} \tag{2.6}$$

From these inequalities we get

$$|u_1(t)| \geq \int_{t_1}^t \sum_{i=1}^m p_i(s) \int_{\sigma_i(s)}^{+\infty} \sum_{i=1}^m q_i(\xi) \big| u_1(\tau_i(\xi)) \big| d\xi ds \qquad \text{for} \quad t \in [t_1, +\infty[\,, \tag{2.7}$$

where $t_1 \in [t_0, +\infty[$ is a sufficiently large number.

Denote by S the set of all $\lambda \in R_+$ satisfying

$$\frac{|u_1(t)|}{h^\lambda(t)} \to +\infty \quad \text{as} \quad t \to +\infty. \tag{2.8}$$

By (2.5), it is obvious that $0 \in S$. Let $\lambda_0 = \sup S$. Suppose that $\lambda_0 > 1$. By virtue of (2.8) there exists $\widetilde{t} \in [t_0, +\infty[$ such that

$$\big|u_1(\tau_i(t))\big| \geq h(\tau_i(t)) \quad \text{for} \quad t \in [\widetilde{t}, +\infty[\;\; (i = 1, \dots, m).$$

Thus on account of (2.2) we find from (2.6)

$$|u_2(t)| \leq |u_2(\widetilde{t})| - \int\limits_{\widetilde{t}}^{t} \sum_{i=1}^{m} q_i(s) h(\tau_i(s)) ds \to -\infty \quad \text{as} \quad t \to +\infty.$$

The contradiction obtained proves that $\lambda_0 \in [0, 1]$. Then by (2.3) there exist $\lambda_* \in [0, \lambda_0] \cap [0, 1[$ and $t_0 \in R_+$ such that

$$\int\limits_{0}^{\tau_*(\widetilde{\sigma}(t))} h^{\varepsilon_0}(\delta(s)) \sum_{i=1}^{m} p_i(s) \int\limits_{\sigma_i(s)}^{+\infty} \sum_{i=1}^{m} q_i(\xi) h^{\lambda_*}(\tau_i(\xi)) d\xi ds >$$
$$> h^{\lambda_*}(\tau_*(\widetilde{\sigma}(t))) h^{\varepsilon_0}(\delta(t)) \quad \text{for} \quad t \in [t_0, +\infty[\,, \tag{2.9}$$

and

$$\lim_{t \to +\infty} \frac{|u_1(t)|}{h^{\lambda_*}(t)} = +\infty, \quad \varliminf_{t \to +\infty} \frac{|u_1(t)|}{h^{\lambda_* + \varepsilon_0}(t)} = 0. \tag{2.10}$$

Introduce the notation

$$\varphi(t) = \inf \Big\{ h^{-\lambda_*}(\tau_*(s)) |u_1(\tau_*(s))| : \; s \geq t \Big\}. \tag{2.11}$$

By (0.2) and (2.10) it is clear that

$$\varphi(t) \uparrow +\infty \quad \text{as} \quad t \uparrow +\infty, \tag{2.12}$$
$$\varliminf_{t \to +\infty} \varphi(\widetilde{\sigma}(t)) h^{-\varepsilon_0}(\delta(t)) = 0. \tag{2.13}$$

Define the sets S_i $(i = 1, 2)$ in the following manner:

$$t \in S_1 \iff \varphi(\widetilde{\sigma}(t)) h^{-\varepsilon_0}(\delta(t)) \leq \varphi(\widetilde{\sigma}(s)) h^{-\varepsilon_0}(\delta(s)) \quad \text{for} \quad s \in [t_0, t],$$
$$t \in S_2 \iff \varphi(\widetilde{\sigma}(t)) = \frac{|u_1(\tau_*(\widetilde{\sigma}(t)))|}{h^{\lambda_*}(\tau_*(\widetilde{\sigma}(t)))}\,.$$

It is clear that by (2.12) and (2.13), $\sup S_i = +\infty$ $(i = 1, 2)$. Show that

$$\sup S_1 \cap S_2 = +\infty^{1}. \tag{2.14}$$

[1] Analogous discussions for nth order equations are given in [4], [5].

Indeed, if we assume that $t_2 \in S_1$ and $t_2 \notin S_2$, by (2.11) and (2.12) there exists $t_3 > t_2$ such that

$$\begin{gathered}\varphi(\widetilde{\sigma}(t_3)) = \frac{|u_1(\tau_*(\widetilde{\sigma}(t_3)))|}{h^{\lambda_*}(\tau_*(\widetilde{\sigma}(t_3)))}\,, \\ \varphi(\widetilde{\sigma}(t)) = \varphi(\widetilde{\sigma}(t_2)) \quad \text{for} \quad t \in [t_2, t_3].\end{gathered} \tag{2.15}$$

On the other hand, since $t_2 \in S_1$ and δ is nondecreasing, we have

$$\varphi(\widetilde{\sigma}(t_3))h^{-\varepsilon_0}(\delta(t_3)) \le \varphi(\widetilde{\sigma}(s))h^{-\varepsilon_0}(\delta(s)) \quad \text{for} \quad s \in [t_0, t_3]. \tag{2.16}$$

Therefore from (2.15), (2.16) it follows that $t_3 \in S_1 \cap S_2$. By the above reasoning we easily ascertain that (2.14) is fulfilled. Thus there exists an increasing sequence of points $\{t_k\}$ such that

$$\begin{gathered}\lim_{k\to+\infty} t_k = +\infty, \\ \varphi(\widetilde{\sigma}(t_k))h^{-\varepsilon_0}(\delta(t_k)) \le \varphi(\widetilde{\sigma}(s))h^{-\varepsilon_0}(\delta(s)) \quad \text{for} \quad s \in [t_0, t_k], \\ \varphi(\widetilde{\sigma}(t_k)) = \frac{|u_1(\tau_*(\widetilde{\sigma}(t_k)))|}{h^{\lambda_*}(\tau_*(\widetilde{\sigma}(t_k)))} \quad (k = 1, 2, \dots).\end{gathered} \tag{2.17}$$

On the other hand, it is obvious that

$$\varphi(t) \le \frac{|u_1(\tau_i(t))|}{h^{\lambda_*}(\tau_i(t))} \quad \text{for} \quad t \in [t_0, +\infty[\quad (i = 1, \dots, m).$$

Thus on account of (0.2), (2.12), (2.17), for sufficiently large k we have from (2.7)

$$\begin{gathered}\big|u_1(\tau_*(\widetilde{\sigma}(t_k)))\big| \ge \\ \ge \int\limits_{t_1}^{\tau_*(\widetilde{\sigma}(t_k))} \sum_{i=1}^m p_i(s) \int\limits_{\sigma_i(s)}^{+\infty} \sum_{i=1}^m q_i(\xi)\frac{|u_1(\tau_i(\xi))|}{h^{\lambda_*}(\tau_i(\xi))} h^{\lambda_*}(\tau_i(\xi))d\xi ds \ge \\ \ge \int\limits_{t_1}^{\tau_*(\widetilde{\sigma}(t_k))} \sum_{i=1}^m p_i(s)\varphi(\widetilde{\sigma}(s)) \int\limits_{\sigma_i(s)}^{+\infty} \sum_{i=1}^m q_i(\xi)h^{\lambda_*}(\tau_i(\xi))d\xi ds \ge \\ \ge \varphi(\widetilde{\sigma}(t_k))h^{-\varepsilon_0}(\delta(t_k)) \times \\ \times \int\limits_{t_1}^{\tau_*(\widetilde{\sigma}(t_k))} h^{\varepsilon_0}(\delta(s)) \sum_{i=1}^m p_i(s) \int\limits_{\sigma_i(s)}^{+\infty} \sum_{i=1}^m q_i(\xi)h^{\lambda_*}(\tau_i(\xi))d\xi ds,\end{gathered}$$

whence we obtain

$$\int\limits_{t_1}^{\tau_*(\widetilde{\sigma}(t_k))} h^{\varepsilon_0}(\delta(s)) \sum_{i=1}^{m} p_i(s) \int\limits_{\sigma_i(s)}^{+\infty} \sum_{i=1}^{m} q_i(\xi) h^{\lambda_*}(\tau_i(\xi)) d\xi ds \leq$$
$$\leq h^{\lambda_*}(\tau_*(\widetilde{\sigma}(t_k))) h^{\varepsilon_0}(\delta(t_k)) \quad \text{for} \quad k \geq k_0,$$

where k_0 is a sufficiently large number. But the latter inequality contradicts (2.9). The contradiction obtained proves the validity of the theorem. □

Theorem 2.2. *Let conditions* (1.2), (1.5), (2.1), *and* (2.2) *be fulfilled. Let, moreover,*

$$\varliminf_{t\to+\infty} \frac{h(\tau_*(\widetilde{\sigma}(t)))}{h(t)} > 0 \tag{2.18}$$

and there exist $\varepsilon \in]0,1[$ *such that for any* $\lambda \in [0,1[$,

$$h^{1-\lambda}(t) \int\limits_{\sigma^*(t)}^{+\infty} \sum_{i=1}^{m} q_i(\xi) h^{\lambda}(\tau_i(\xi)) d\xi \geq \lambda + \varepsilon \tag{2.19}$$
$$\text{for} \quad t \in [t_0, +\infty[\,,$$

where $t_0 \in R_+$ *and* h *is defined by* (1.3). *Then every proper solution of* (0.1) *is oscillatory.*

Proof. By virtue of Theorem 2.1, to prove the theorem it is sufficient to show that (2.19) implies (2.3) with $\delta(t) \equiv t$.

Indeed, choose $\varepsilon_0 > 0$ such that

$$\frac{\lambda+\varepsilon}{\lambda+\varepsilon_0}\, \gamma^{\varepsilon_0} > 1 \quad \text{for} \quad \lambda \in [0,1[\,, \tag{2.20}$$

where $\gamma = \varliminf\limits_{t\to+\infty} \frac{h(\tau_*(\widetilde{\sigma}(t)))}{h(t)}$.

On account of (1.2) and (2.18)–(2.20) we obtain

$$h^{-\lambda}(\tau_*(\widetilde{\sigma}(t))) h^{-\varepsilon_0}(t) \times$$
$$\times \int\limits_{0}^{\tau_*(\widetilde{\sigma}(t))} h^{\varepsilon_0}(s) \sum_{i=1}^{m} p_i(s) \int\limits_{\sigma_i(s)}^{+\infty} \sum_{i=1}^{m} q_i(\xi) h^{\lambda}(\tau_i(\xi)) d\xi ds \geq$$
$$\geq (\lambda+\varepsilon) h^{-\lambda}(\tau_*(\widetilde{\sigma}(t))) h^{-\varepsilon_0}(t) \int\limits_{0}^{\tau_*(\widetilde{\sigma}(t))} \widetilde{p}(s) h^{\varepsilon_0+\lambda-1}(s) ds =$$

$$= \frac{\lambda+\varepsilon}{\lambda+\varepsilon_0} h^{-\lambda}(\tau_*(\widetilde{\sigma}(t))) h^{-\varepsilon_0}(t) h^{\lambda+\varepsilon}(\tau_*(\widetilde{\sigma}(t))) \geq$$
$$\geq \frac{\lambda+\varepsilon}{\lambda+\varepsilon_0} \gamma^{\varepsilon_0} > 1 \quad \text{for} \quad t \in [t^*, +\infty[\,,$$

where $t^* \in]t_0, +\infty[$ is a sufficiently large number. Therefore (2.3) is fulfilled. □

Remark. In a certain sense, condition (2.19) is optimal. If we assume $\varepsilon = 0$, then, in general, Theorem 2.2 is not valid.

Theorem 2.3. *Let conditions* (1.2), (1.5), (2.1), (2.2), *and* (2.18) *be fulfilled. Let, moreover,*

$$\varliminf_{t\to+\infty} \frac{h(t)}{h(\sigma^*(t))} = \alpha > 0 \tag{2.21}$$

and there exist $\varepsilon \in]0,1[$ *such that for any* $\lambda \in [0,1[$,

$$h(t) \int\limits_t^{+\infty} \sum_{i=1}^m q_i(\xi) \Big[\frac{h(\tau_i(\xi))}{h(\xi)}\Big]^\lambda d\xi \geq \alpha^{\lambda-1}(\lambda(1-\lambda)+\varepsilon) \tag{2.22}$$
$$\text{for} \quad t \in [t_0, +\infty[\,,$$

where $t_0 \in R_+$ *and* h *is defined by* (1.3). *Then every proper solution of* (0.1) *is oscillatory.*

Proof. By virtue of Theorem 2.2, to prove the theorem it is sufficient to show that (2.21) and (2.22) imply (2.19).

Indeed, on account of (1.2), (2.21), and (2.22) we have

$$h^{1-\lambda}(t) \int\limits_{\sigma^*(t)}^{+\infty} \sum_{i=1}^m q_i(s) h^\lambda(\tau_i(s)) ds =$$
$$= -h^{1-\lambda}(t) \int\limits_{\sigma^*(t)}^{+\infty} h^\lambda(s) d \int\limits_s^{+\infty} \sum_{i=1}^m q_i(\xi) \Big[\frac{h(\tau_i(\xi))}{h(\xi)}\Big]^\lambda d\xi =$$
$$= h^{1-\lambda}(t) h^\lambda(\sigma^*(t)) \int\limits_{\sigma^*(t)}^{+\infty} \sum_{i=1}^m q_i(\xi) \Big[\frac{h(\tau_i(\xi))}{h(\xi)}\Big]^\lambda d\xi +$$
$$+ \lambda h^{1-\lambda}(t) \int\limits_{\sigma^*(t)}^{+\infty} \widetilde{p}(s) h^{\lambda-1}(s) \int\limits_s^{+\infty} \sum_{i=1}^m q_i(\xi) \Big[\frac{h(\tau_i(\xi))}{h(\xi)}\Big]^\lambda d\xi ds \geq$$

$$\geq \alpha^{\lambda-1}(\lambda(1-\lambda)+\varepsilon)\Big(\Big[\frac{h(t)}{h(\sigma^*(t))}\Big]^{1-\lambda}+\lambda h^{1-\lambda}(t)\int\limits_{\sigma^*(t)}^{+\infty}\widetilde{p}(s)h^{\lambda-2}(s)ds\Big)\geq$$

$$\geq \alpha^{\lambda-1}(\lambda(1-\lambda)+\varepsilon)\Big(\alpha^{1-\lambda}+\frac{\lambda}{1-\lambda}\Big[\frac{h(t)}{h(\sigma^*(t))}\Big]^{1-\lambda}\Big)\geq$$

$$\geq (\lambda(1-\lambda)+\varepsilon)\big(1+\frac{\lambda}{1-\lambda}\big)=\lambda+\frac{1}{1-\lambda}\,\varepsilon\geq\lambda+\varepsilon$$

$$\text{for}\quad t\in[t_*,+\infty[\ ,$$

where $t_* \in]t_0,+\infty[$ is a sufficiently large number, and $\widetilde{p}$ is defined by (1.3). Therefore (2.19) is fulfilled. □

Corollary 2.1. *Let conditions* (1.2), (2.1), (2.2), (2.18), *and* (2.21) *be fulfilled. Let, moreover,*

$$\varliminf_{t\to+\infty}\frac{h(\tau_i(t))}{h(t)}=\beta_i>0, \tag{2.23}$$

and

$$\varliminf_{t\to+\infty} h(t)\int\limits_t^{+\infty}\widetilde{q}(s)ds>\alpha^{-1}\max_{\lambda\in[0,1]}\Big\{\Big(\frac{\alpha}{\beta}\Big)^{\lambda}\lambda(1-\lambda)\Big\},$$

where h, $\widetilde{q}$, and α are defined by (1.3), (1.6), *and* (2.21), *respectively, and*

$$\beta=\min\{\beta_i:\ i=1,\dots,m\}. \tag{2.24}$$

Then every proper solution of (0.1) *is oscillatory.*

Corollary 2.2. *Let conditions* (1.2), (2.1), (2.2), (2.18), (2.21), *and* (2.23) *be fulfilled. Let, moreover, $\alpha=\beta$ and*

$$\varliminf_{t\to+\infty} h(t)\int\limits_t^{+\infty}\widetilde{q}(s)ds>\frac{1}{4\alpha}\ ,$$

where h, $\widetilde{q}$, α, and β are defined by (1.3), (1.6), (2.21), *and* (2.23), (2.24), *respectively. Then every proper solution of* (0.1) *is oscillatory.*

Corollary 2.3. *Let $p,q\in L_{loc}(R_+;R_+)$,*

$$h(+\infty)=+\infty,$$

and

$$\varliminf_{t\to+\infty} h(t)\int\limits_t^{+\infty}q(s)ds>\frac{1}{4}\ ,$$

where

$$h(t) = \int_0^t p(s)ds.$$

Then every proper solution of the system

$$u_1'(t) = p(t)u_2(t),$$
$$u_2'(t) = -q(t)u_1(t)$$

is oscillatory.

Corollary 2.3 immediately implies Hille's well-known theorem for second-order ordinary linear differential equations (see [1]).

References

1. E. Hille, Non-oscillation theorems. *Trans. Amer. Math. Soc.* **64**(1948), 234–252.

2. R. Koplatadze, On oscillation of solutions of second-order retarded differential inequalities and equations. (Russian) *Mathematica Balkanica* **5:29**(1975), 163–172.

3. R. Koplatadze, Criteria of oscillation of solutions of second-order retarded differential inequalities and equations. (Russian) *Proc. Vekua Inst. Appl. Math.* (*Tbiliss. Gos. Univ. Inst. Prikl. Mat. Trudy*) **17**(1986), 104–120.

4. R. Koplatadze, On differential equations with deviating arguments having properties A and B. (Russian) *Differentsial'nye Uravneniya* **25** (1989), No. 11, 1897–1909.

5. R. Koplatadze, On oscillatory properties of solutions of functional differential equations. *Memoirs on Differential Equations and Mathematical Physics* **3**(1994), 1–179, *A. Razmadze Mathematical Institute of the Georgian Academy of Sciences, Tbilisi.*

6. J. D. Mirzov, Asymptotic behavior of solutions of systems of nonlinear non-autonomous ordinary differential equations. (Russian) *Maikop*, 1993.

(Received 19.06.1995)

Author's address:

A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Alexidze St., Tbilisi 380093
Republic of Georgia