

## ON CONTINUOUS EXTENSIONS

LAWRENCE NARICI AND EDWARD BECKENSTEIN

ABSTRACT. We consider various possibilities concerning the continuous extension of continuous functions taking values in an ultrametric space. In Section 1 we consider Tietze-type extension theorems concerning continuous extendibility of continuous functions from compact and closed subsets to the whole space. In Sections 2 and 3 we consider extending “separated” continuous functions in such a way that certain continuous extensions remain separated. Functions taking values in a complete ultravalued field are dealt with in Section 2, and the real and complex cases in Section 3.

### 1. A TIETZE-TYPE EXTENSION THEOREM

An *ultranormal* topological space  $T$  is a Hausdorff space in which disjoint closed subsets may be separated by clopen sets. As shown by R. Ellis ([1], cf. [2], Th. 8.27, p. 258) the following version of the Tietze extension theorem obtains in the nonarchimedean setting:

(Bounded) continuous functions  $x$  mapping a closed subset  $K$  of the ultranormal space  $T$  into a complete separable metric space have (bounded) continuous extensions to all of  $T$ .

We prove a similar result below—we eliminate the separability requirement on the domain  $T$  but consider an *ultrametric* space  $Y$  as the codomain. Thus, the result below is not more general, just different. For the sake of the proof we define a *clopen  $n$ -partition* of a subset  $M$  of  $T$  with respect to  $x : M \rightarrow Y$  to be a finite pairwise disjoint collection of clopen subsets  $V_i$  of  $T$  such that  $M \subset \bigcup_{i=1}^n V_i$  and for all  $s, t \in V_i \cap M$ ,  $d(x(s), x(t)) \leq 1/n$ . As in many of the classical constructions of functions with certain properties (continuous but nowhere differentiable, for example) we create a sequence of functions each of which almost has the desired property, whose uniform limit does.

---

1991 *Mathematics Subject Classification*. 46S10, 54C45, 54D35.

*Key words and phrases*. Continuous extension, Tietze extension theorem.

**Theorem 1.1.** *Let  $y$  be a point of the complete ultrametric space  $(Y, d)$ . In each part below  $y$  could belong to  $K$ .*

(a) *A continuous map  $x$  of a compact subset  $K$  of the ultraregular space  $T$  into  $Y$  may be continuously extended to  $\bar{x}$  defined on all of  $T$  with  $\bar{x}(T) = x(K) \cup \{y\}$ .*

(b) *A continuous map  $x$  of a closed subset  $K$  of the ultranormal space  $T$  into  $Y$  with relatively compact range may be continuously extended to  $\bar{x}$  defined on all of  $T$  with  $\bar{x}(T) \subset [\text{cl } x(K)] \cup \{y\}$  where “cl” denotes topological closure.*

*Proof.* (a) Let  $C(p, r)$  denote the closed ball of radius  $r > 0$  about  $p \in Y$ . For each  $t \in K$ , let  $B(t, 1/n) = \{s \in K : d(x(s), x(t)) \leq 1/n\} = x^{-1}(C(x(t), 1/n))$ . Since  $K$  is compact, there exist  $t_1, \dots, t_{n_1} \in K$  such that  $K = \bigcup_{i=1}^{n_1} B(t_i, 1)$ . In the standard way, we may rewrite the relatively clopen subsets of  $K$ ,  $B(t_i, 1)$ ,  $1 \leq i \leq n_1$ , to get a pairwise disjoint relatively clopen cover  $\{A_i\}$  of  $K$  with the property that, for any  $1 \leq i \leq n_1$  and  $s, t \in A_i$ ,  $d(x(s), x(t)) < 1$ . We continue to assume that the  $A_i$  are nonempty and that  $t_i \in A_i$  for  $1 \leq i \leq n_1$ ; otherwise, we would reindex the sets and choose some new points  $t_i$  as needed. Since  $T$  is ultraregular and  $K$  is compact, we may choose pairwise disjoint clopen subsets  $U_i$  of  $T$  such that  $U_i \cap K = A_i$  for each  $i = 1, 2, \dots, n_1$ . Evidently the family  $\{U_i\}$  is a clopen 1-partition of  $K$  with respect to  $x$ . We define the continuous “step” function

$$x_1(t) = \begin{cases} x(t_i), & t \in U_i, 1 \leq i \leq n_1, \\ y, & t \notin \bigcup_{i=1}^{n_1} U_i. \end{cases}$$

For any  $t \in K$ , for some  $1 \leq i \leq n_1$ ,  $t \in U_i$  and  $x_1(t) = x(t_i)$ ; hence

$$\begin{aligned} d(x_1(t), x(t)) &\leq \max[d(x_1(t), x(t_i)), d(x(t_i), x(t))] \\ &= \max[d(x(t_i), x(t_i)), d(x(t_i), x(t))] \\ &= d(x(t_i), x(t)) \\ &\leq 1. \end{aligned}$$

It follows that  $d_K(x_1, x) = \sup\{d(x_1(t), x(t)) : t \in K\} \leq 1$ . For  $n = 2$ , we decompose each  $A_i$  into a clopen 2-partition with respect to  $x$ . On  $A_1$ , for example, we get nonempty disjoint clopen subsets  $B_1, \dots, B_{r_1}$  of  $T$  such that each is a subset of  $U_1$  and  $(\bigcup_{i=1}^{r_1} B_i) \cap K = A_1$ . Choose  $s_i \in B_i \cap A_1$  for  $1 \leq i \leq r_1$  and define

$$x_2(t) = \begin{cases} x(s_i), & t \in B_i, 1 \leq i \leq r_1, \\ x_1(t) = x(t_1), & t \notin U_1 - \bigcup_{i=1}^{r_1} B_i, \\ y, & t \notin \bigcup_{i=1}^{n_1} U_i \end{cases}$$

and define  $x_2$  in a similar way on the rest of the  $A_i$ . In so doing, we create a continuous step function  $x_2 : T \rightarrow Y$  such that  $d_K(x_2, x) \leq 1/2$  and  $d_T(x_1, x_2) \leq 1$ . We continue in this fashion to construct a sequence  $(x_n)$

of functions continuous on  $T$  such that, for every  $n$ ,  $d_K(x_n, x) \leq 1/n$  and  $d_T(x_n, x_{n-1}) \leq 1/(n-1)$ . We denote the pointwise limit of  $(x_n)$  as  $\bar{x}$ . As  $x_n \rightarrow \bar{x}$  uniformly, it follows that  $\bar{x}$  is continuous; clearly  $\bar{x} = x$  on  $K$ .

(b) As  $\text{cl } x(K)$  is compact, there exist  $y_1, y_2, \dots, y_{n_1} \in \text{cl } x(K)$  such that the  $C(y_i, 1)$  cover  $\text{cl } x(K)$ . Consequently  $K = \bigcup_{i=1}^{n_1} x^{-1}[C(y_i, 1)]$ . Since  $T$  is ultranormal there are pairwise disjoint clopen sets  $U_i$  such that  $x^{-1}[C(y_i, 1)] \subset U_i$  for  $1 \leq i \leq n_1$ . Choose  $t_i \in x^{-1}[C(y_i, 1)]$  for  $1 \leq i \leq n_1$  and define  $x_1$  as was done in (a) and continue the process to obtain the sequence  $(x_n)$  and the limit  $\bar{x}$ .  $\square$

**Corollary 1.2.** *Let  $T, y$  and  $Y$  be as in the theorem. (a) Let  $x$  and  $w$  be continuous functions defined on the disjoint compact subsets  $K$  and  $L$  of  $T$  taking values in  $Y$ . For any disjoint clopen supersets  $U$  and  $V$  of  $K$  and  $L$ , respectively,  $x$  and  $w$  can be continuously extended to  $\bar{x}$  and  $\bar{w}$ , respectively, defined on  $T$ , with  $\bar{x}(T) \subset x(K) \cup \{y\}$ ,  $\bar{w}(T) \subset w(L) \cup \{y\}$  and  $x(t) = w(t) = y$  on  $\mathbf{C}(U \cup V)$ .*

(b) *If  $x$  and  $w$  are continuous functions with relatively compact range defined on the disjoint closed subsets  $K$  and  $L$  of the ultranormal space  $T$  taking values in  $Y$ , then  $x$  and  $w$  can be continuously extended to  $\bar{x}$  and  $\bar{w}$ , respectively, defined on  $T$ , with  $\bar{x}(T) \subset \text{cl } x(K) \cup \{y\}$ ,  $\bar{w}(T) \subset \text{cl } w(L) \cup \{y\}$  and  $x(t) = w(t) = y$  on  $\mathbf{C}(U \cup V)$ .*

*Proof.* (a) By the theorem,  $x$  and  $w$  may be continuously extended to  $U$  and  $V$ , respectively. It only remains to define those extensions to assume the value  $y$  on  $\mathbf{C}(U \cup V)$  to obtain the desired extensions to all of  $T$ . The result of (b) follows directly from part (b) of the theorem.  $\square$

For any topological space  $T$  and any semigroup  $G$  we say that two functions  $x, y : T \rightarrow G$  are *strongly separated* if  $\text{cl } \text{coz } x \cap \text{cl } \text{coz } y = \emptyset$ , and *separated* if  $\text{coz } x \cap \text{coz } y = \emptyset$ , where  $\text{coz } x$  denotes the cozero set of  $x$ . Note that  $x$  and  $y$  are separated if and only if  $xy = 0$ . As has been developed in a number of papers ([3]–[8]) there is ample reason for interest in additive maps  $H : C(T) \rightarrow C(S)$  between spaces of scalar-valued continuous functions  $C(T)$  and  $C(S)$  which are [*weakly*] *separating* in the sense that [*strongly*] separated functions  $x$  and  $y$  have separated images  $Hx$  and  $Hy$ . For that reason it is of importance to know when continuous functions which are separated or weakly separated on a subset have continuous extensions which remain separated, the subject of the corollary below; the point  $y$  of the theorem is taken to be the zero element of the semigroup.

**Corollary 1.3.** *Let  $T$  be ultraregular and let  $Y$  be a semigroup on which a complete ultrametric is defined (not necessarily a topological semigroup).*

(a) *If let  $x$  and  $w$  are strongly separated continuous functions defined on a compact subset  $K$  of  $T$  then there are separated continuous extensions  $\bar{x}$  and  $\bar{w}$  defined on  $T$  such that  $\bar{x}\bar{w} = 0$ .*

(b) *Two strongly separated continuous functions  $x$  and  $w$  with relatively compact range defined on a closed subset  $K$  of the ultranormal space  $T$  have continuous extensions  $\bar{x}$  and  $\bar{w}$  defined on  $T$  such that  $\bar{x}\bar{w} = 0$ .*

## 2. THE CASE OF COMPLETE VALUED FIELDS

Let  $T$  be ultraregular and let  $C(T, F)$  denote the space of continuous functions taking values in the complete, nonarchimedean nontrivially valued field  $F$ . We show in Theorem 2.2 that pairwise strongly separated functions  $x_1, x_2, \dots, x_n \in C(K, F)$  have pairwise separated extensions  $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n \in C(T, F)$  for two types of subset  $K$ .

**Lemma 2.1.** *Let  $K$  be a closed subset of the compact ultraregular space  $T$  and let  $x_1, x_2, \dots, x_n$  be elements of  $C(T, F)$  which are pairwise separated on  $K$ . Then there exist pairwise separated elements  $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n$  in  $C(T, F)$  such that  $\bar{x}_i \upharpoonright K = x_i \upharpoonright K$  for  $i = 1, \dots, n$ . (If the  $x_i$  were only defined on  $K$  then they could be continuously extended to  $T$  by Corollary 1.3.)*

*Proof.* Assume that  $x, y \in C(T, F)$  are separated on  $K$ . Let  $M$  bound  $x$  and  $y$ . Let  $G(x, i) = x^{-1} \{a \in F : |a| \leq M/i\}$  for each positive integer  $i$ , with  $G(y, i)$  defined analogously. Let  $U(x, i) = G(x, i) - G(x, i + 1)$  and let  $U(y, i)$  be defined similarly. Let  $C(x, i)$  and  $C(y, i)$  denote the intersections of  $U(x, i)$  and  $U(y, i)$ , respectively, with  $K$ . Note that the unions of  $C(x, i)$  and  $C(y, i)$ ,  $i \in N$ , yield  $\text{coz } x$  and  $\text{coz } y$ , respectively. Since  $x$  and  $y$  are separated on  $K$ ,  $C(x, i)$  and  $C(y, j)$  are disjoint for all positive integers  $i$  and  $j$ . Let  $G(i)$  denote the clopen set  $G(x, i) \cap G(y, i)$  for each  $i$ . Since  $x$  and  $y$  are separated on  $K$ ,  $C(x, m)$  and  $C(y, m)$  are disjoint subsets of  $G(i)$  for all  $m \geq i$ ;  $C(x, m)$  and  $C(y, m)$  are subsets of  $CG(i)$ , the complement of  $G(i)$ , for all  $m < i$ . Thus there exist pairwise disjoint clopen sets  $V(x, i)$  and  $V(y, i)$  such that  $C(x, i) \subset V(x, i) \subset U(x, i)$  and  $C(y, i) \subset V(y, i) \subset U(y, i)$ . Let  $k(x, i)$  and  $k(y, i)$  denote the characteristic functions of  $V(x, i)$  and  $V(y, i)$ , respectively. The functions  $x_m = \sum_{i=1}^m k(x, i)x$  and  $y_m = \sum_{i=1}^m k(y, i)y$  are uniform Cauchy sequences on  $T$ . Call their limits  $\bar{x}$  and  $\bar{y}$  in  $C(T, F)$ , respectively. Since  $x_m$  and  $y_m$  are separated for each  $m$ ,  $\bar{x}$  and  $\bar{y}$  are separated as well; the restrictions of  $\bar{x}$  and  $\bar{y}$  to  $K$  are just  $x$  and  $y$ , as is easy to verify. The method for extending this to a finite number of functions—rather than just two—is a simple modification of this idea.  $\square$

In the next result we drop the assumption of compactness and get an extension theorem.

**Theorem 2.2.** *For compact  $K$  and ultraregular  $T$  [or closed  $K$  and ultranormal  $T$ ] and pairwise strongly separated functions  $x_1, x_2, \dots, x_n \in C(K, F)$  [with relatively compact range], there exist pairwise separated extensions  $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n \in C(T, F)$  of  $x_1, x_2, \dots, x_n$ .*

*Proof.* It suffices to consider two separated functions  $x$  and  $y$  in  $C(K, F)$ . Let  $\beta_0 T$  denote the Banaschewski compactification of  $T$ , and continuously extend  $x$  and  $y$  to  $\beta x$  and  $\beta y$  in  $C(\beta_0 T, F)$  by Corollary 1.3; then replace  $\beta x$  and  $\beta y$  by functions  $\bar{x}$  and  $\bar{y}$  as described in the preceding lemma. Finally, restrict  $\bar{x}$  and  $\bar{y}$  to  $T$ .  $\square$

### 3. THE REAL AND COMPLEX CASE

Let  $K$  be a compact subset of the Tihonov space  $T$ , and let  $C(K, F)$  and  $C(T, F)$  denote the spaces of  $F$ -valued functions on  $K$  and  $T$ , where  $F = \mathbf{R}$  or  $\mathbf{C}$ . Let  $H$  be a finite subset of  $C(K, F)$  which is pairwise separated in the sense that  $xy = 0$  for each  $x$  and  $y$  in  $H$ . We show in Theorem 3.3 that each  $x$  in  $H$  has an extension  $\bar{x}$  in  $C(T, F)$  such that the set  $\bar{H}$  of extensions is pairwise separated. We begin with two lemmas about nonnegative functions.

**Lemma 3.1.** *If the functions in  $H$  are nonnegative then each  $x$  in  $H$  has an extension  $\bar{x} \in C(T, \mathbf{R})$  such that the set  $\bar{H}$  of such extensions is pairwise separated.*

*Proof.* We proceed by induction. We begin by considering the case in which  $H$  has two elements  $x$  and  $y$ . Let  $\beta v$  denote the continuous extension of the difference  $v = x - y$  to the Stone-Cech compactification  $\beta T$  of  $T$ . Let  $\beta v^+$  and  $\beta v^-$  denote the positive and negative parts of  $\beta v$ , respectively, so that  $\beta v = \beta v^+ - \beta v^-$ . Since  $x$  and  $y$  are nonnegative and  $xy = 0$ ,  $\beta v^+$  and  $\beta v^-$  are continuous extensions of  $x$  and  $y$ , respectively; clearly  $(\beta v^+)(\beta v^-) = 0$ . It only remains to restrict  $\beta v^+$  and  $\beta v^-$  to  $T$  to complete the argument in this case. Now suppose that  $H$  has  $n + 1 \geq 3$  elements and choose  $x_1, x_2, \dots, x_n$  from  $H$ . By the induction hypothesis, there exist continuous, separated extensions  $w_1, w_2, \dots, w_n$  of  $x_1, x_2, \dots, x_n$  to  $T$ . We now repeatedly pair  $x_{n+1}$  with each of the  $w_i$ ,  $i = 1, \dots, n$ , to get separated continuous extensions (by the first part of the argument)  $z_i$  of  $x_i$  and  $v_i$  of  $x_{n+1}$  for  $i = 1, \dots, n$ . Now let  $\bar{x}_i = \inf(w_i, z_i)$  for  $i = 1, \dots, n$  and let  $\bar{x}_{n+1} = \inf\{v_1, \dots, v_n\}$ . These are the desired extensions of  $x_1, x_2, \dots, x_n$ .  $\square$

In the next result we assume that each  $x$  is separated from each  $y$  but do not suppose that the  $x$ 's are a pairwise separated set, nor are the  $y$ 's.

**Lemma 3.2.** *Let  $x_1, x_2, \dots, x_n$  and  $y_1, y_2, \dots, y_n$  denote families of nonnegative functions in  $C(K)$  such that  $x_i$  is separated from  $y_j$  for all  $i$  and  $j$ . Then there exist continuous extensions  $\bar{x}_i$  and  $\bar{y}_i$ ,  $i = 1, \dots, n$ , of these functions to  $T$  with the same property.*

*Proof.* By Lemma 3.1 there exist separated extensions  $x(i, j)$  and  $y(i, j)$  of  $x_i$  and  $y_j$ , respectively, for each  $i$  each  $j$ . The functions  $\bar{x}_i = \inf\{x(i, j) : 1 \leq j \leq n\}$  and  $\bar{y}_j = \inf\{y(i, j) : 1 \leq i \leq n\}$  are separated on all of  $T$  for all  $i$  and  $j$ .  $\square$

**Theorem 3.3.** *With  $F = \mathbb{R}$  or  $\mathbb{C}$ , and  $H$  as before Lemma 3.1, then each  $x$  in  $H$  has an extension  $\bar{x}$  in  $C(T, F)$  such that the set  $\bar{H}$  of extensions is pairwise separated.*

*Proof.* Let  $F = \mathbb{R}$  and let  $x$  and  $y$  be separated members of  $C(K, \mathbb{R})$ . The positive and negative parts  $x^+, x^-, y^+$  and  $y^-$  of  $x$  and  $y$ , respectively, are pairwise separated. Since they are nonnegative, they can be extended to a pairwise separated collection  $\overline{x^+}, \overline{x^-}, \overline{y^+}$  and  $\overline{y^-}$  on  $T$  as in Lemma 3.1. The desired extensions are now given by  $\bar{x} = \overline{x^+} - \overline{x^-}$  and  $\bar{y} = \overline{y^+} - \overline{y^-}$ . It is straightforward to extend this to the case of more than two functions. To handle the case where the field is  $\mathbb{C}$ , let  $x$  and  $y$  be separated continuous functions on  $K$ . Each of the functions  $(\operatorname{Re} x)^+, (\operatorname{Re} x)^-, (\operatorname{Im} x)^+, (\operatorname{Im} x)^-$  is separated from each of the functions  $(\operatorname{Re} y)^+, (\operatorname{Re} y)^-, (\operatorname{Im} y)^+, (\operatorname{Im} y)^-$  on  $K$  and Lemma 3.2 can be applied to these functions. The separated extensions to  $T$  of those functions can be used to construct separated extensions of  $x$  and  $y$ .  $\square$

#### REFERENCES

1. R. L. Ellis, Extending continuous functions on zero-dimensional spaces. *Math. Annal.* **186**(1970), 114–122.
2. J. B. Prolla, Topics in functional analysis over valued division rings. *North-Holland Math. Studies* **77**, North-Holland, New York, 1982.
3. J. Araujo, E. Beckenstein, and L. J. Narici, Biseparating maps and homeomorphic real compactifications. *J. Math. Anal. Appl.* **192**(1995), 258–265.
4. E. Beckenstein and L. Narici, A non-Archimedean Stone–Banach theorem. *Proc. Amer. Math. Soc.* **100**(1987), 242–246.
5. E. Beckenstein and L. Narici, Automatic continuity of certain linear isomorphisms. *Acad. Roy. Belg. Bull. Cl. Sci.* **73**(1987), 191–200.
6. E. Beckenstein, L. Narici, and A. Todd, Automatic continuity of linear maps on spaces of continuous functions. *Manuscripta Math.* **62**(1988), 257–275.
7. S. Hernandez, E. Beckenstein, and L. Narici, Banach–Stone theorems and separating maps. *Manuscripta Math.* **86**(1995), 409–416.
8. L. Narici, E. Beckenstein, and J. Araujo, Separating maps on rings of continuous functions. *p-Adic functional analysis, ed. by N. de Grande-de Kimpe, S. Navarro, and W. Schikhof, 69–82, Ed. Univ. Santiago de Chile, Santiago, 1994.*

(Received 08.07.1994)

Authors' addresses:

Lawrence Narici  
St. John's University  
Jamaica, NY 11439 USA

Edward Beckenstein  
St. John's University  
Staten Island, NY 10301 USA