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SPATIAL PROBLEM OF DARBOUX TYPE FOR ONE MODEL EQUATION OF THIRD ORDER

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ABSTRACT. For a hyperbolic type model equation of third order a Darboux type problem is investigated in a dihedral angle. It is shown that there exists a real number ρ_0 such that for $\alpha > \rho_0$ the problem under consideration is uniquely solvable in the Frechet space. In the case where the coefficients are constants, Bochner's method is developed in multidimensional domains, and used to prove the uniquely solvability of the problem both in Frechet and in Banach spaces.

§ 1. Statement of the Problem

Let us consider a partial differential equation of hyperbolic type

$$u_{xyz} = F \tag{1.1}$$

in $\mathbb{R}^3$, where F is a given function and u is an unknown real function.

For equation (1.1) the family of planes $x = \text{const}$, $y = \text{const}$, $z = \text{const}$ is characteristic, while the directions determined by the unit vectors e_1, e_2, e_3 of the coordinate axes are bicharacteristic.

In the space $\mathbb{R}^3$ let $S_i^0 : p_i(x,y,z) \equiv \alpha_i^0 x + \beta_i^0 y + \gamma_i^0 z = 0$, $i = 1,2$, be arbitrarily given planes passing through the origin. Assume that $\nu_1^0 \nparallel \nu_2^0$, $|\nu_i^0| \neq 0$, where $\nu_i^0 \equiv (\alpha_i^0, \beta_i^0, \gamma_i^0)$, $i = 1,2$. The space $\mathbb{R}^3$ is partitioned by the planes S_i^0, $i = 1,2$, into four dihedral angles. We consider equation (1.1) in one of these angles D_0 which, without loss of generality, is assumed to be given in the form

$$D_0 \equiv \{(x,y,z) \in \mathbb{R}^3 : \alpha_i^0 x + \beta_i^0 y + \gamma_i^0 z > 0, \ \ i = 1,2\}.$$

For the domain D_0 we make the following assumptions:

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(a) the edge $\Gamma_0 \equiv \{(x,y,z)\in\mathbb{R}^3 : \alpha_i^0 x+\beta_i^0 y+\gamma_i^0 z=0,\ \ i=1,2\}$ of D_0 lies in none of the coordinate planes, which is equivalent to

$$\begin{vmatrix}\alpha_1^0 & \beta_1^0\\ \alpha_2^0 & \beta_2^0\end{vmatrix}\neq 0,\quad \begin{vmatrix}\alpha_1^0 & \gamma_1^0\\ \alpha_2^0 & \gamma_2^0\end{vmatrix}\neq 0,\quad \begin{vmatrix}\beta_1^0 & \gamma_1^0\\ \beta_2^0 & \gamma_2^0\end{vmatrix}\neq 0; \tag{1.2}$$

this implies that Γ_0 has no bicharacteristic direction, i.e., $\nu^0 \nparallel e_j$, $j=1,2,3$, where $\nu^0\equiv\nu_1^0\times\nu_2^0$ is the vector product of the vectors ν_1^0 and ν_2^0;

(b) the bicharacteristics passing through the edge Γ_0 do not pass into the domain D_0, and this is equivalent to the fulfillment of the inequalities $\alpha_1^0\alpha_2^0<0$, $\beta_1^0\beta_2^0<0$, $\gamma_1^0\gamma_2^0<0$.

Let $P_0(x_0,y_0,z_0)$ be an arbitrary point of the domain D_0, and let $S_1\supset\Gamma_0$ and $S_2\supset\Gamma_0$ be the plane sides of the angle D_0, $\partial D_0=S_1\cup S_2$. Let the bicharacteristic beams $L_i(P_0)$, $i=1,2,3$, of equation (1.1) radiate from the point P_0, in the direction of decreasing values of the z-coordinate of the moving points $L_i(P_0)$ to the intersection with one of the sides S_1 or S_2 at the points P_i, $i=1,2,3$. Assume that these three points do not lie on the same side. Without loss of generality the points P_1 and P_2 are assumed to lie on S_1, while the point P_3 lies on S_2.

In the domain D_0 let us consider the following Darboux type problem: Find in D_0 a regular solution $u(x,y,z)$ of equation (1.1) satisfying the boundary conditions (to shorten the formulas here and below we assume that $S_0\equiv S_1$)

$$(M_i u_{xy}+N_i u_{xz}+Q_i u_{yz})|_{S_{i-1}}=f_i,\ \ i=1,2,3. \tag{1.3}$$

For convenience we transform the domain D_0 into the domain $D: z_1-y_1>0$, $z_1+y_1>0$ of the space of variables x_1,y_1,z_1. To this end let us introduce new independent variables defined by the equalities

$$\begin{gathered}x_1=x,\ \ y_1=\frac{1}{2}\big(p_1(x,y,z)-p_2(x,y,z)\big),\\ z_1=\frac{1}{2}\big(p_1(x,y,z)+p_2(x,y,z)\big).\end{gathered} \tag{1.4}$$

Owing to (1.2), the linear transform (1.4) is obviously nondegenerate, it establishes the one-to-one correspondence between the domains D_0 and D.

Retaining, the previous notation for u,F,M_i,N_i,Q_i,f_i and S_j, $i=1,2,3$, $j=1,2$, in the domain D for the variables x_1,y_1,z_1 we rewrite the problem (1.1), (1.2) as

$$\begin{gathered}\frac{\partial^3 u}{\partial\mu_1\partial\mu_2\partial\mu_3}=F,\\ \Big(M_i\frac{\partial^2 u}{\partial\mu_1\partial\mu_2}+N_i\frac{\partial^2 u}{\partial\mu_1\partial\mu_3}+Q_i\frac{\partial^2 u}{\partial\mu_2\partial\mu_3}\Big)\Big|_{S_{i-1}}=f_i,\ \ i=1,2,3,\end{gathered} \tag{1.5}$$

where $\frac{\partial}{\partial \mu_i}, i = 1,2,3$, are well defined derivatives with respect to various directions expressed by the values α_i^0, β_i^0, γ_i^0, $i = 1,2,3$,

$$S_1 = \{(x,y,z) \in \mathbb{R}^3 : x \in \mathbb{R},\ y = z,\ z \in \overline{\mathbb{R}}_+\},$$
$$S_2 = \{(x,y,z) \in \mathbb{R}^3 : x \in \mathbb{R},\ y = -z,\ z \in \overline{\mathbb{R}}_+\},\ \ \overline{\mathbb{R}}_+ \equiv [0,\infty).$$

In the domain D let us consider, instead of the problem (1.5), the following boundary value problem in more general terms. Find in the domain D a regular solution $u(x,y,z)$ of the equation

$$\frac{\partial^3 u}{\partial l_1 \partial l_2 \partial l_3} = F \tag{1.6}$$

satisfying the boundary conditions

$$\Big(M_i \frac{\partial^2 u}{\partial l_1 \partial l_2} + N_i \frac{\partial^2 u}{\partial l_1 \partial l_3} + Q_i \frac{\partial^2 u}{\partial l_2 \partial l_3}\Big)\Big|_{S_{i-1}} = f_i,\ \ i = 1,2,3. \tag{1.7}$$

Here for the variables x_1, y_1, z_1 we use the previous notation: x, y, z; $\dfrac{\partial}{\partial l_i} \equiv \alpha_i \dfrac{\partial}{\partial x} + \beta_i \dfrac{\partial}{\partial y} + \gamma_i \dfrac{\partial}{\partial z}$, $|\alpha_i| + |\beta_i| + |\gamma_i| \neq 0$, $l_i \equiv (\alpha_i, \beta_i, \gamma_i)$, $i = 1,2,3$, is the derivative with respect to the direction, M_i, N_i, Q_i, f_i, F, $i = 1,2,3$, are the given functions, and u is the unknown real function. Moreover, the bicharacteristics of equation (1.6) and the domain D will be assumed to satisfy conditions (a) and (b) formulated above for equation (1.1) in the domain D_0.

A regular solution of equation (1.6) is said to be a function $u(x,y,z)$ which is continuous in D together with its partial derivatives $\dfrac{\partial^{i+j+k} u}{\partial l_1^i \partial l_2^j \partial l_3^k}$, $i,j,k = 0,1$, and satisfying equation (1.6) in D.

It should be noted that the boundary value problem (1.6), (1.7) is a natural continuation of the known classical statements of the Goursat and Darboux problems (see, e.g., [1]–[3]) for linear hyperbolic equations of second order with two independent variables on a plane. The multidimensional analogues of the Goursat and Darboux problems for one hyperbolic equation of second order in a dihedral angle were studied by Beudon and many other authors (see, e.g., [2], [4]–[7]).

Many works are devoted to the initial boundary value and characteristic problems for a wide class of hyperbolic equations of third and higher orders in multidimensional domains with dominating derivatives (see, e.g., [8], [9]).

Remark 1.1. Note that the hyperbolicity of problem (1.6), (1.7) is taken into account in conditions (1.7) because of the presence of dominating derivatives of second order with respect to $\dfrac{\partial^3 u}{\partial l_1 \partial l_2 \partial l_3}$.

Remark 1.2. Since the bicharacteristic beams of equation (1.6) radiating from an arbitrary point of the domain D in the direction of decreasing values of the z-coordinate of moving points of these beams intersect the side S_1 twice, while the side S_2 only once, we take in the boundary conditions (1.7) two conditions on S_1 and one condition on S_2, respectively.

In the domains D and $\Pi_+ \equiv \{(x,z) \in \mathbb{R}^2 : x \in \mathbb{R},\ z \in \mathbb{R}_+\}$, $\mathbb{R}_+ \equiv (0,\infty)$, let us introduce into consideration the functional spaces

$$\overset{0}{C}_\alpha(\overline{D}) \equiv \Big\{v \in C(\overline{D}) : v|_\Gamma = 0,\ \sup_{\substack{(x,y,z)\in\overline{D}\setminus\Gamma\\ z\leq N}} \rho^{-\alpha}|v(x,y,z)| < \infty, \forall N \in \mathbb{N}\Big\},$$

$$\overset{0}{C}_{\alpha,\beta}(\overline{D}) \equiv \Big\{v \in \overset{0}{C}_\alpha(\overline{D}) : \sup_{(x,y,z)\in\overline{D},z>1} \rho^{-\beta}|v(x,y,z)| < \infty\Big\},$$

where $\Gamma \equiv \{(x,y,z) \in \mathbb{R}^3 : x \in \mathbb{R},\ y = z = 0\}$, ρ is the distance from the point $(x,y,z) \in \overline{D}$ to the edge Γ of D, and the parameters $\alpha = \text{const} \geq 0$, $\beta = \text{const} \geq 0$.

Similarly, we introduce the spaces

$$\overset{0}{C}_\alpha(\overline{\Pi}_+) \equiv \Big\{\varphi \in C(\overline{\Pi}_+) : \varphi|_{\Gamma_1} = 0,\ \sup_{\substack{(x,z)\in\Pi_+\\ z\leq N}} z^{-\alpha}|\varphi(x,z)| < \infty, \forall N \in \mathbb{N}\Big\},$$

$$\overset{0}{C}_{\alpha,\beta}(\overline{\Pi}_+) \equiv \Big\{\varphi \in \overset{0}{C}_\alpha(\overline{\Pi}_+) : \sup_{(x,z)\in\Pi_+,z>1} z^{-\beta}|\varphi(x,z)| < \infty\Big\},$$

$\Gamma_1 \equiv \{(x,z) \in \mathbb{R}^2 : x \in \mathbb{R},\ z = 0\}$.

Obviously, for the semi-norms

$$\|v\|_{\overset{0}{C}_\alpha(\overline{D}_N)} = \sup_{(x,y,z)\in\overline{D}_N\setminus\Gamma} \rho^{-\alpha}|v(x,y,z)|,$$

$$\|\varphi\|_{\overset{0}{C}_\alpha(\overline{\Pi}_{+,N})} = \sup_{(x,z)\in\overline{\Pi}_{+,N}\setminus\Gamma_1} z^{-\alpha}|\varphi(x,z)|,$$

where $D_N \equiv D \cap \{z < N\}$, $\Pi_{+,N} \equiv \Pi_+ \cap \{z < N\}$, $N \in \mathbb{N}$, the spaces $\overset{0}{C}_\alpha(\overline{D})$ and $\overset{0}{C}_\alpha(\overline{\Pi}_+)$ are the countable normed Frechet spaces.

The spaces $\overset{0}{C}_{\alpha,\beta}(\overline{D})$ and $\overset{0}{C}_{\alpha,\beta}(\overline{\Pi}_+)$ are the Banach spaces with the norms

$$\|v\|_{\overset{0}{C}_{\alpha,\beta}(\overline{D})} = \\ = \max\Big\{\sup_{(x,y,z)\in\overline{D}\setminus\Gamma,z\leq 1} \rho^{-\alpha}|v(x,y,z)|,\ \sup_{(x,y,z)\in\overline{D},z>1} \rho^{-\beta}|v(x,y,z)|\Big\},$$

$$\|\varphi\|_{\overset{0}{C}_{\alpha,\beta}(\overline{\Pi}_+)} = \\ = \max\Big\{ \sup_{(x,z)\in\Pi_+, z\le 1} z^{-\alpha}|\varphi(x,z)|, \sup_{(x,z)\in\Pi_+, z>1} z^{-\beta}|\varphi(x,z)|\Big\}.$$

Remark 1.3. Because of the uniform estimate $1 \le \dfrac{\rho}{z} \le \sqrt{2}$, $(x,y,z)\in D$, we can replace the value ρ in the definition of the spaces $\overset{0}{C}_\alpha(\overline{D})$, $\overset{0}{C}_\alpha(\overline{D}_N)$, $\overset{0}{C}_{\alpha,\beta}(\overline{D})$ by the variable z which will be used below.

Throughout this paper we denote by c a positive constant whose particular value is not of principal interest for our investigation.

It can be easily seen that the belonging of the functions $v \in \overset{0}{C}(\overline{D})$ and $\varphi \in \overset{0}{C}(\overline{\Pi}_+)$, respectively, to the spaces $\overset{0}{C}_\alpha(\overline{D})$ and $\overset{0}{C}_\alpha(\overline{\Pi}_+)$ is equivalent to the fulfillment of the inequalities

$$\begin{gathered} |v(x,y,z)| \le cz^\alpha, \quad (x,y,z)\in\overline{D}, \quad z\le N, \\ |\varphi(x,z)| \le cz^\alpha, \quad (x,z)\in\overline{\Pi}_+, \quad z\le N, \quad N\in\mathbb{N}. \end{gathered} \tag{1.8}$$

We investigate problem (1.6), (1.7) in the Frechet space

$$\overset{0}{C}{}^l_\alpha(\overline{D}) \equiv \Big\{ u : \frac{\partial^{i+j+k}u}{\partial l_1^i \partial l_2^j \partial l_3^k} \in \overset{0}{C}_\alpha(\overline{D}),\ i,j,k=0,1 \Big\}, \quad l\equiv(l_1,l_2,l_3),$$

with respect to the semi-norms

$$\|u\|_{\overset{0}{C}{}^l_\alpha(\overline{D}_N)} = \sum_{i,j,k=0}^{1} \Big\| \frac{\partial^{i+j+k}u}{\partial l_1^i \partial l_2^j \partial l_3^k} \Big\|_{\overset{0}{C}_\alpha(\overline{D}_N)}, \quad N\in\mathbb{N},$$

and in the Banach space

$$\overset{0}{C}{}^l_{\alpha,\beta}(\overline{D}) \equiv \Big\{ u : \frac{\partial^{i+j+k}u}{\partial l_1^i \partial l_2^j \partial l_3^k} \in \overset{0}{C}_{\alpha,\beta}(\overline{D}),\ i,j,k=0,1 \Big\}$$

with the norm

$$\|u\|_{\overset{0}{C}{}^l_{\alpha,\beta}(\overline{D})} = \sum_{i,j,k=0}^{1} \Big\| \frac{\partial^{i+j+k}u}{\partial l_1^i \partial l_2^j \partial l_3^k} \Big\|_{\overset{0}{C}_{\alpha,\beta}(\overline{D})}.$$

In considering the problem (1.6), (1.7) in the class $\overset{0}{C}{}^l_\alpha(\overline{D})$ ($\overset{0}{C}{}^l_{\alpha,\beta}(\overline{D})$), we require that $F\in\overset{0}{C}_\alpha(\overline{D})$ ($\overset{0}{C}_{\alpha,\beta}(\overline{D})$), $M_i, N_i, Q_i \in C(\overline{\Pi}_+)$ ($M_i\equiv$ const, $N_i\equiv$ const, $Q_i\equiv$ const), $f_i\in\overset{0}{C}_\alpha(\overline{\Pi}_+)$ ($\overset{0}{C}_{\alpha,\beta}(\overline{\Pi}_+)$), $i=1,2,3$.

§ 2. Equivalent Reduction of Problem (1.6), (1.7) to a Functional Equation

Using the notation $\dfrac{\partial^2 u}{\partial l_1 \partial l_2} \equiv v_1$, $\dfrac{\partial^2 u}{\partial l_1 \partial l_3} \equiv v_2$, $\dfrac{\partial^2 u}{\partial l_2 \partial l_3} \equiv v_3$, problem (1.6), (1.7), in the domain D, can be rewritten equivalently as a boundary value problem for a system of partial differential equations of first order with respect to the unknown functions v_1, v_2, v_3:

$$\frac{\partial v_1}{\partial l_3} = F, \quad \frac{\partial v_2}{\partial l_2} = F, \quad \frac{\partial v_3}{\partial l_1} = F, \tag{2.1}$$

$$(M_i v_1 + N_i v_2 + Q_i v_3)\big|_{S_{i-1}} = f_i, \quad i = 1, 2, 3. \tag{2.2}$$

The equivalence of the initial problem (1.6), (1.7) and problem (2.1), (2.2) is an obvious consequence of

Lemma 2.1. *In the closed domain $\overline{D}_0$ there exists a unique function $u \in \{u : D_x^i D_y^j D_z^k u \in C(\overline{D}_0),\ i, j, k = 0, 1\}$, satisfying both the redefined system of partial differential equations of second order*

$$u_{xy} = v_1, \quad u_{xz} = v_2, \quad u_{yz} = v_3 \tag{2.3}$$

and the conditions

$$u(P^0) = c_0, \ \ u_x|_{\Gamma_0} = \omega_1, \ \ u_y|_{\Gamma_0} = \omega_2, \ \ u_z|_{\Gamma_0} = \omega_3. \tag{2.4}$$

Here v_1, v_2, v_3 are given functions such that $v_i, \frac{\partial v_1}{\partial z}, \frac{\partial v_2}{\partial y}, \frac{\partial v_3}{\partial x} \in C(\overline{D}_0)$, $i = 1, 2, 3$; $\frac{\partial v_1}{\partial z}(x, y, z) = \frac{\partial v_2}{\partial y}(x, y, z) = \frac{\partial v_3}{\partial x}(x, y, z)$, $(x, y, z) \in \overline{D}_0$; c_0 and $\omega_i \in C(\Gamma_0)$, $i = 1, 2, 3$, are, respectively, the given constant and functions on Γ_0; $P^0 = P^0(x^0, y^0, z^0)$ is an arbitrarily fixed point of Γ_0.

Proof. Let $P_0(x_0, y_0, z_0)$ be an arbitrary point of $\overline{D}_0$. It is obvious that owing to the requirement (a) on Γ_0 in §1, the plane $x = x_0$ has the unique point of intersection of $P_0^*(x_0, y^*(x_0), z^*(x_0))$ with the edge Γ_0. Since $(u_x(x_0, y, z))_y = v_1(x_0, y, z)$, $(u_x(x_0, y, z))_z = v_2(x_0, y, z)$ and $u_x(P_0^*) = \omega_1(P_0^*)$, the function $u_x(x_0, y, z)$ is defined uniquely at the point $P_0(x_0, y_0, z_0)$ by the formula

$$u_x(P_0) = \omega_1(P_0^*) + \int\limits_{(y^*(x_0), z^*(x_0))}^{(y_0, z_0)} v_1(x_0, y, z)dy + v_2(x_0, y, z)dz. \tag{2.5}$$

Here the curvilinear integral is taken along any simple smooth curve connecting the points $(y^*(x_0), z^*(x_0))$ and (y_0, z_0) of the plane $x = x_0$ and lying wholly in $\overline{D}_0$. Since the point P_0 is chosen arbitrarily, in the closed domain $\overline{D}_0$ formula (2.5) gives the representation of the function $u_x(P)$, $P \equiv P(x, y, z)$, which is written in terms of the given functions v_1 and v_2.

Analogously, the representation formulas for the functions $u_y(P)$ and $u_z(P)$ in $\overline{D}_0$ are given by the known functions v_1, v_3 and v_2, v_3, respectively. It remains only to note that the function $u(P)$ defined by the formula

$$u(P) = c_0 + \int\limits_{P^0P} u_x dx + u_y dy + u_z dz = c_0 + \int\limits_{P^0P} \Big\{\omega_1(x, y^*(x), z^*(x)) +$$
$$+ \int\limits_{P^*P^1} v_1(x,\eta,\zeta)d\eta + v_2(x,\eta,\zeta)d\zeta\Big\}dx + \Big\{\omega_2(x^{**}(y), y, z^{**}(y)) +$$
$$+ \int\limits_{P^{**}P^2} v_1(\xi,y,\zeta)d\xi + v_3(\xi,y,\zeta)d\zeta\Big\}dy + \Big\{\omega_3(x^{***}(z), y^{***}(z), z) +$$
$$+ \int\limits_{P^{***}P^3} v_2(\xi,\eta,z)d\xi + v_3(\xi,\eta,z)d\eta\Big\}dz \tag{2.6}$$

defines actually the unique solution of the problem (2.3), (2.4). Here $P^1 \equiv P^1(y,z)$, $P^2 \equiv P^2(x,z)$, $P^3 \equiv P^3(x,y)$, $P^* \equiv P^*(y^*(x), z^*(x))$, $P^{**} \equiv P^{**}(x^{**}(y), z^{**}(y))$, $P^{***} \equiv P^{***}(x^{***}(z), y^{***}(z))$. □

Remark 2.1. If instead of system (2.3) we consider the system

$$\frac{\partial^2 u}{\partial l_1 \partial l_2} = v_1, \quad \frac{\partial^2 u}{\partial l_1 \partial l_3} = v_2, \quad \frac{\partial^2 u}{\partial l_2 \partial l_3} = v_3, \tag{2.7}$$

in the dihedral angle $\overline{D}$, then similarly to item (a) for Γ_0 one should require that the edge Γ of D lie in none of the three planes passing through the origin and spanned to the pairs of vectors (l_1, l_2), (l_1, l_3) and (l_2, l_3).

Note that system (2.7) reduces to system (2.3) by means of the nondegenerate transform of variables x, y, z,

$$x = \alpha_1\xi + \alpha_2\eta + \alpha_3\zeta, \quad y = \beta_1\xi + \beta_2\eta + \beta_3\zeta, \quad z = \gamma_1\xi + \gamma_2\eta + \gamma_3\zeta,$$

under the assumption that the vectors l_1, l_2 and l_3 are linearly independent.

Let the bicharacteristic beams $L_i(P)$, $i = 1, 2, 3$, of equation (1.6) rediate from an arbitrary point $P(x,y,z) \in D$ in the direction of decreasing values of the z-coordinate of moving points $L_i(P)$ to the intersection with the sides S_1 and S_2 at the points P_i, $i = 1, 2, 3$.

Denoting for $(x,z) \in \overline{\Pi}_+$ $v_1|_{S_2} \equiv \varphi_1(x,z)$, $v_2|_{S_1} \equiv \varphi_2(x,z)$, $v_3|_{S_1} \equiv \varphi_3(x,z)$, and integrating the equations of system (2.1) along the corresponding bicharacteristics, for $(x,y,z) \in D$ we get

$$\begin{cases} v_1(x,y,z) = \varphi_1(\sigma(x, y+z, z; \widetilde{\alpha}_1, \widetilde{\alpha}_2)) + F_1(x,y,z), \\ v_2(x,y,z) = \varphi_2(\sigma(x, y-z, z; \widetilde{\alpha}_3, \widetilde{\alpha}_4)) + F_2(x,y,z), \\ v_3(x,y,z) = \varphi_3(\sigma(x, y-z, z; \widetilde{\alpha}_5, \widetilde{\alpha}_6)) + F_3(x,y,z), \end{cases} \tag{2.8}$$

where $F_i, i = 1,2,3$, are known functions, $\sigma(x,y,z;\lambda_1,\lambda_2) \equiv (x+\lambda_1 y, z+\lambda_2 y)$, $\sigma_1(x,y;\lambda) \equiv x+\lambda y$, and the superscript -1 here and below denotes the inverse value.

Substituting the expressions for v_1, v_2, and v_3 from equalities (2.8) into the boundary conditions (2.2), for $(x,z) \in \overline{\Pi}_+$ we obtain

$$\begin{cases} M_1(x,z)\varphi_1\big(\sigma(x,z,0;2\widetilde{\alpha}_1,\widetilde{\alpha}_7)\big) + N_1(x,z)\varphi_2(x,z) + \\ \qquad + Q_1(x,z)\varphi_3(x,z) = f_4(x,z), & (2.9) \\ M_2(x,z)\varphi_1\big(\sigma(x,z,0;2\widetilde{\alpha}_1,\widetilde{\alpha}_7)\big) + N_2(x,z)\varphi_2(x,z) + \\ \qquad + Q_2(x,z)\varphi_3(x,z) = f_5(x,z), & (2.10) \\ M_3(x,z)\varphi_1(x,z) + N_3(x,z)\varphi_2\big(\sigma(x,z,0;-2\widetilde{\alpha}_3,\widetilde{\alpha}_8)\big) + \\ \qquad + Q_3(x,z)\varphi_3\big(\sigma(x,z,0;-2\widetilde{\alpha}_5,\widetilde{\alpha}_9)\big) = f_6(x,z). & (2.11) \end{cases}$$

In equalities (2.8)–(2.11) the constants $\widetilde{\alpha}_i$, $i = 1,\dots,9$, are well defined and written in terms of α_i, β_i, γ_i, $i = 1,2,3$.

Condition I. The functions M_i, N_j, Q_j, $i = 1,2$, $j = 1,2,3$, are bounded and uniformly continuous in $\overline{\Pi}_+$.

Note that by Condition I the known functions f_i, $i = 4,5,6$, belong to the class $\overset{0}{C}_\alpha(\overline{\Pi}_+)$. We rewrite equations (2.9) and (2.10) as follows:

$$\begin{cases} N_1(x,z)\varphi_2(x,z) + Q_1(x,z)\varphi_3(x,z) = f_4(x,z) - \\ \quad - M_1(x,z)\varphi_1\big(\sigma(x,z,0;2\widetilde{\alpha}_1,\widetilde{\alpha}_7)\big), \\ N_2(x,z)\varphi_2(x,z) + Q_2(x,z)\varphi_3(x,z) = f_5(x,z) - \\ \quad - M_2(x,z)\varphi_1\big(\sigma(x,z,0;2\widetilde{\alpha}_1,\widetilde{\alpha}_7)\big). \end{cases} \quad (x,z) \in \overline{\Pi}_+, \qquad (2.12)$$

Condition II. The inequality

$$|\Delta_0(x,z)| \geq c, \quad (x,z) \in \overline{\Pi}_+, \tag{2.13}$$

holds for the determinant $\Delta_0(x,z) \equiv (N_1Q_2 - N_2Q_1)(x,z)$.

On account of (2.12) and (2.13) we find that for $(x,z) \in \overline{\Pi}_+$

$$\varphi_{i+1}(x,z) = a_i(x,z) - b_i(x,z)\varphi_1\big(\sigma(x,z,0;2\widetilde{\alpha}_1,\widetilde{\alpha}_7)\big), \quad i = 1,2, \tag{2.14}$$

where a_i, b_i, $i = 1,2$, are given functions. Bearing in mind Conditions I and II, we find that $a_i \in \overset{0}{C}_\alpha(\overline{\Pi}_+)$, $i = 1,2$, while the continuous functions b_i, $i = 1,2$, are bounded in $\overline{\Pi}_+$.

Condition III. The function M_3 satisfies the inequality

$$|M_3(x,z)| \geq c, \quad (x,z) \in \overline{\Pi}_+, \tag{2.15}$$

and the function M_3^{-1} is uniformly continuous in $\overline{\Pi}_+$.

Taking into account (2.15) from (2.11) and (2.14), we obtain the functional equation

$$\varphi_1(x,z) - G_2(x,z)\varphi_1(J_2(x,z)) - G_3(x,z)\varphi_1(J_3(x,z)) = g(x,z) \quad (2.16)$$

with respect to $\varphi_1 : \overline{\Pi}_+ \to \mathbb{R}$.

Due to Conditions I–III, the functions G_2, G_3 are known uniformly continuous and bounded in $\overline{\Pi}_+$, while the function g is expressed by the known functions and belongs to the class $\overset{0}{C}_\alpha(\overline{\Pi}_+)$. The functions $J_i : \overline{\Pi}_+ \to \overline{\Pi}_+$, $i = 2, 3$, act by the formulas

$$J_i : (x,z) \to (x + \delta_i z, \tau_i z), \quad (x,z) \in \overline{\Pi}_+, \quad i = 2, 3, \quad (2.17)$$

where δ_i, τ_i, $i = 2, 3$ are well-defined constants written in terms of α_i, β_i, γ_i, $i = 1, 2, 3$.

Remark 2.2. Note that under the assumptions with respect to the coefficients α_i, β_i, γ_i, $i = 1, 2, 3$, we can easily see that $0 < \tau_i < 1$, $i = 2, 3$.

Remark 2.3. It is obvious that when Conditions I–III are fulfilled, problem (1.6), (1.7) in the class $\overset{0}{C}{}^l_\alpha(\overline{D})$ is equivalently reduced to (2.16) with respect to the unknown function φ_1 of the class $\overset{0}{C}_\alpha(\overline{\Pi}_+)$. Furthermore, if $u \in \overset{0}{C}{}^l_\alpha(\overline{D})$, then $\varphi_1 \in \overset{0}{C}_\alpha(\overline{\Pi}_+)$, and vice versa: if $\varphi_1 \in \overset{0}{C}_\alpha(\overline{\Pi}_+)$, then taking into account the inequalities (1.8), we find from equalities (2.14), (2.8), (2.6) that $u \in \overset{0}{C}{}^l_\alpha(\overline{D})$.

§ 3. Investigation of the Functional Equation (2.16)

Let us introduce the notation

$$(T\varphi_1)(x,z) \equiv \varphi_1(x,z) - \sum_{i=2}^{3} G_i(x,z)\varphi_1\big(J_i(x,z)\big), \quad (x,z) \in \overline{\Pi}_+, \quad (3.1)$$

$$\begin{gathered} h(\rho) \equiv \sum_{i=2}^{3} \eta_i \tau_i^\rho, \quad \eta_1 \equiv \max_{2\le i\le 3} \sup_{(x,z)\in\overline{\Pi}_+} |G_i(x,z)|, \\ \eta_i \equiv \sup_{x\in\mathbb{R}} |G_i(x,0)|, \quad i = 2, 3, \quad \rho \in \mathbb{R}. \end{gathered} \quad (3.2)$$

Let for some value of the index i the number η_i be different from zero. In that case, owing to (2.2), the function $h : \mathbb{R} \to \overline{\mathbb{R}}_+$ is continuous and strictly monotonically decreasing on $\mathbb{R}$; moreover, $\lim_{\rho\to-\infty} h(\rho) = +\infty$ and $\lim_{\rho\to+\infty} h(\rho) = 0$. Therefore there exists a unique real number ρ_0 such that $h(\rho_0) = 1$. For $\eta_2^2 + \eta_3^2 = 0$ we assume $\rho_0 = -\infty$.

Lemma 3.1. *If $\alpha > \rho_0$, then (2.16) is uniquely solvable in $\overset{0}{C}_\alpha(\overline{\Pi}_{+,A})$, $\Pi_{+,A} \equiv \mathbb{R} \times (0, A)$, $\forall A > 0$, and for the solution $\varphi_1 = T^{-1}g$ the estimate*

$$|(T^{-1}g)(x,z)| \leq Cz^\alpha \|g\|_{\overset{0}{C}_\alpha(\overline{\Pi}_{+,z})}, \quad x \in \mathbb{R}, \quad 0 \leq z \leq A, \tag{3.3}$$

holds, where C is a positive constant not depending on the function g.

Proof. By the condition $\alpha > \rho_0$ and the definition of the function h it follows from (3.2) that

$$h(\alpha) = \sum_{i=2}^{3} \eta_i \tau_i^\alpha < 1. \tag{3.4}$$

Because of (3.4) and the uniform continuity of the functions G_2, G_3 there are positive numbers ε ($\varepsilon < A$) and δ such that the inequalities

$$|G_i(x,z)| \leq \eta_i + \delta, \quad i = 2, 3, \quad x \in \mathbb{R}, \tag{3.5}$$

$$\sum_{i=2}^{3} (\eta_i + \delta)\tau_i^\alpha \equiv \beta < 1 \tag{3.6}$$

are valid for $0 \leq z \leq \varepsilon$.

According to Remark 2.2, there exists a natural number $q_0(A)$ such that for $q \geq q_0$

$$\tau_{i_q} \tau_{i_{q-1}} \cdots \tau_{i_1} z \leq \varepsilon, \quad 0 \leq z \leq A, \tag{3.7}$$

where $2 \leq i_s \leq 3$, $s = 1, \dots, q$.

Let us introduce into consideration the operators Λ and T^{-1} defined by

$$(\Lambda\varphi_1)(x,z) = \sum_{i=2}^{3} G_i(x,z)\varphi_1(J_i(x,z)), \quad T^{-1} = I + \sum_{q=1}^{\infty} \Lambda^q,$$

where $(x,z) \in \overline{\Pi}_{+,A}$ and I is an identical operator. The operator T^{-1} is still formally inverse to the operator T defined by equality (3.1). To prove that the operator T^{-1} is really inverse to the operator T, it suffices to establish its continuity in the space $\overset{0}{C}_\alpha(\overline{\Pi}_{+,A})$.

Indeed, it is easily seen that the expression $\Lambda^q g$ represents the sum of summands of the type

$$\begin{aligned} I_{i_1 \cdots i_q}(x,z) = G_{i_1}(x,z) G_{i_2}(J_{i_1}(x,z)) G_{i_3}(J_{i_2}(J_{i_1}(x,z))) \cdots \\ \cdots G_{i_q}(J_{i_{q-1}}(J_{i_{q-2}}(\cdots(J_{i_1}(x,z))\cdots))) g(J_{i_q}(J_{i_{q-1}}(\cdots(J_{i_1}(x,z))\cdots))), \end{aligned}$$

where $2 \leq i_s \leq 3$, $s = 1, \dots, q$.

By virtue of (3.2), (3.5), (3.7) and Remark 2.2, for $q > q_0$, $g \in \overset{0}{C}_\alpha(\overline{\Pi}_{+,A})$ we have

$$
\begin{aligned}
|I_{i_1\cdots i_q}(x,z)| &\le |G_{i_1}(x,z)|\cdots|G_{i_{q_0}}(J_{i_{q_0-1}}(J_{i_{q_0-2}}(\cdots(J_{i_1}(x,z))\cdots)))|\times\\
&\times|G_{i_{q_0+1}}(J_{i_{q_0}}(J_{i_{q_0-1}}(\cdots(J_{i_1}(x,z))\cdots)))|\cdots\\
&\cdots|G_{i_q}(J_{i_{q-1}}(J_{i_{q-2}}(\cdots(J_{i_1}(x,z))\cdots)))|\times\\
&\times|g(J_{i_q}(J_{i_{q-1}}(\cdots(J_{i_1}(x,z))\cdots)))| \le \eta_1^{q_0}(\eta_{i_{q_0+1}}+\delta)\cdots(\eta_{i_q}+\delta)\times\\
&\times(\tau_{i_q}\tau_{i_{q-1}}\cdots\tau_{i_1}z)^\alpha\|g\|_{\overset{0}{C}_\alpha(\overline{\Pi}_{+,z})} \le \eta_1^{q_0}\Big(\prod_{s=q_0+1}^{q}(\eta_{i_s}+\delta)\Big)\Big(\prod_{s=q_0+1}^{q}\tau_{i_s}^\alpha\Big)\times\\
&\times z^\alpha\|g\|_{\overset{0}{C}_\alpha(\overline{\Pi}_{+,z})} = \eta_1^{q_0}\Big(\prod_{s=q_0+1}^{q}(\eta_{i_s}+\delta)\tau_{i_s}^\alpha\Big)z^\alpha\|g\|_{\overset{0}{C}_\alpha(\overline{\Pi}_{+,z})}, \qquad (3.8)
\end{aligned}
$$

while for $1 \le q \le q_0$

$$
\begin{aligned}
|I_{i_1\cdots i_q}(x,z)| &\le \eta_1^q(\tau_{i_q}\tau_{i_{q-1}}\cdots\tau_{i_1}z)^\alpha\|g\|_{\overset{0}{C}_\alpha(\overline{\Pi}_{+,z})} \le\\
&\le \eta_1^q z^\alpha\|g\|_{\overset{0}{C}_\alpha(\overline{\Pi}_{+,z})}. \qquad (3.9)
\end{aligned}
$$

Taking into account (3.8), (3.9), and (3.6) for $q > q_0$, we get

$$
\begin{aligned}
\big|(\Lambda^q g)(x,z)\big| &= \Big|\sum_{i_1,\dots,i_q} I_{i_1\cdots i_q}(x,z)\Big| \le \Big(\sum_{i_1,\dots,i_{q_0}} 1\Big)^{q_0}\eta_1^{q_0}\times\\
&\times\Big[\sum_{i=2}^{3}(\eta_i+\delta)\tau_i^\alpha\Big]^{q-q_0} z^\alpha\|g\|_{\overset{0}{C}_\alpha(\overline{\Pi}_{+,z})} \le c_1\beta^q z^\alpha\|g\|_{\overset{0}{C}_\alpha(\overline{\Pi}_{+,z})}, \qquad (3.10)
\end{aligned}
$$

while for $1 \le q \le q_0$

$$
\big|(\Lambda^q g)(x,z)\big| \le c_2 z^\alpha\|g\|_{\overset{0}{C}_\alpha(\overline{\Pi}_{+,z})}, \qquad (3.11)
$$

where $c_1 = \eta_1^{q_0}\beta^{-q_0}\Big(\sum_{i_1,\dots,i_{q_0}} 1\Big)^{q_0}$, $c_2 = \eta_1^q\Big(\sum_{i_1,\dots,i_q} 1\Big)$.

From (3.10) and (3.11) we finally find

$$
\begin{aligned}
\big|(T^{-1}g)(x,z)\big| &\le \big|g(x,z)\big| + \sum_{q=1}^{q_0}\big|(\Lambda^q g)(x,z)\big| + \sum_{q=q_0+1}^{\infty}\big|(\Lambda^q g)(x,z)\big| \le\\
&\le \big(1 + c_2q_0 + c_1\beta^{q_0+1}(1-\beta)^{-1}\big)z^\alpha\|g\|_{\overset{0}{C}_\alpha(\overline{\Pi}_{+,z})} = Cz^\alpha\|g\|_{\overset{0}{C}_\alpha(\overline{\Pi}_{+,z})},
\end{aligned}
$$

where $C \equiv 1 + c_2q_0 + c_1\beta^{q_0+1}(1-\beta)^{-1}$. This implies that the operator T^{-1} is continuous in the space $\overset{0}{C}_\alpha(\overline{\Pi}_{+,A})$ and estimate (3.3) is valid. Thus the unique solvability of equation (2.16) on $\overline{\Pi}_{+,A}$ is proved for any $A > 0$. $\square$

The unique solvability of this equation on the whole $\overline{\Pi}_+$ in the class $\overset{0}{C}_\alpha(\overline{\Pi}_+)$ follows from

Lemma 3.2. *If the equation*

$$(T\varphi_1)(x,z) = g(x,z), \quad (x,z) \in \overline{\Pi}_+, \tag{3.12}$$

is uniquely solvable on $\overline{\Pi}_{+,A}$ *for any* $A > 0$*, then equation* (3.12) *is uniquely solvable on the whole* $\overline{\Pi}_+$.

Proof. Indeed, let $\varphi_{1,n}(x,z)$ be that unique solution of equation (3.12) on $\overline{\Pi}_{+,n}$ whose existence has been proved above. Owing to the above-established uniqueness of the solution, we have $\varphi_{1,n}(x,z) = \varphi_{1,m}(x,z)$ if $(x,z) \in \overline{\Pi}_{+,n}$ and $m > n$. Then it is obvious that $\varphi_1(x,z) = \varphi_{1,n}(x,z)$ is the unique solution of equation (3.12) for $(x,z) \in \overline{\Pi}_{+,n}$. □

Finally, by Lemmas 3.1 and 3.2, the lemma below is valid.

Lemma 3.3. *If* $\alpha > \rho_0$*, then equation* (2.16) *is uniquely solvable in the space* $\overset{0}{C}_\alpha(\overline{\Pi}_+)$ *for any* $g \in \overset{0}{C}_\alpha(\overline{\Pi}_+)$.

Since problem (1.6), (1.7) in Conditions I–III in the class $\overset{0}{C}{}^l_\alpha(\overline{D})$ has been equivalently reduced to (2.16) in $\overset{0}{C}_\alpha(\overline{\Pi}_+)$, from Lemmas 3.1–3.3 we have

Theorem 3.1. *If* $\alpha > \rho_0$*, then the boundary value problem* (1.6), (1.7) *is uniquely solvable in the class* $\overset{0}{C}{}^l_\alpha(\overline{D})$.

Consider now the case where coefficients of the problem (1.6), (1.7) are constant, i.e., $M_i \equiv \text{const}$, $N_i \equiv \text{const}$, $Q_i \equiv \text{const}$, $i = 1,2,3$. It is evident that in this case Condition I is fulfilled automatically. As for Conditions II and III, they can respectively be replaced by relatively weaker conditions:

Condition II$'$. $\Delta_0 \neq 0$.

Condition III$'$. $M_3 \neq 0$.

By developing Bochner's method (see, e.g., [11]) in multidimensional domains, we investigate the functional equation (2.16) in the classes $\overset{0}{C}_\alpha(\overline{\Pi}_+)$ and $\overset{0}{C}_{\alpha,\beta}(\overline{\Pi}_+)$.

Let $Z \equiv \{\sigma_0, \sigma_1, \dots, \sigma_i, \dots\}$ be a set of all numbers representable in the form $\sum_{k=2}^{3} n_k \log \tau_k$, where n_k are arbitrary integers, and $\sigma_0 = 0$, $\sigma_i \neq \sigma_j$, for $i \neq j$. Denote by $\mathbb{C}$ a set of all complex numbers. Let us consider the entire function $\Delta(s)$, $s \in \mathbb{C}$, corresponding to the operator T_3 from (3.1):

$$\begin{gathered}\Delta(s)=\sum_{p=1}^{3} \widetilde{G}_p e^{s \log \tau_p}, \quad \widetilde{G}_1 \equiv 1, \\ \widetilde{G}_p \equiv -G_p, \quad p=2,3, \quad \tau_1=1, \quad s \in \mathbb{C}.\end{gathered} \tag{3.13}$$

Obviously, $\Delta(s) \not\equiv 0$, $s \in \mathbb{C}$, since $\widetilde{G}_1 \equiv 1$. Denote by H a set of real parts of all zeros of the entire function $\Delta(s)$, $s \in \mathbb{C}$. The set H is either finite or closed countably-bounded [10].

Since the set H is finite or countably-bounded, the complement H on the real axis consists of a finite or a countable set of intervals two of which are half-lines. Let $H_0=(-\infty, b_0)$ and $H_1=(a, \infty)$ be respectively the left and the right half-line, and let H_i, $i=2,3,\dots$, be the rest of the intervals.

It is shown in [11] and [12] that the analytic almost-periodic function $\frac{1}{\Delta(s)}$ decomposes in the strip $\Pi_i \equiv \{s : \operatorname{Re} s \in H_i\}$, $i \geq 0$, and takes the form of an absolutely convergent series

$$\frac{1}{\Delta(s)}=\sum_{j=0}^{\infty} \gamma_{ij} e^{\sigma_j s}, \ \ \sigma_j \in Z, \ \ s \in \Pi_i, \tag{3.14}$$

with uniquely determined coefficients γ_{ij}, $j=0,1,\dots$.

On acount of (3.13) and (3.14) it can be easily seen that for $\operatorname{Re} s \in H_{i_0}$, $i_0 \geq 0$, we have the equality

$$\sum_{q=0}^{\infty} \sum_{p=1}^{3} \widetilde{G}_p \gamma_{i_0 q} e^{(\log \tau_p+\sigma_q) s}=\sum_{\nu=0}^{\infty}\Big(\sum_{(q,p) \in I_\nu} \widetilde{G}_p \gamma_{i_0 q}\Big) e^{\sigma_\nu s}=1, \tag{3.15}$$

where I_ν is the set of all elements (q,p) for which $\log \tau_p+\sigma_q=\sigma_\nu$.

From (3.15), because of the absolute convergence of the series (3.14) in Π_{i_0} and the uniqueness theorem for analytic almost-periodic functions, we get

$$\sum_{(q,p) \in I_\nu} \widetilde{G}_p \gamma_{i_0 q}=\begin{cases}1, & \text{for} \quad \nu=0, \\ 0, & \text{for} \quad \nu \geq 1.\end{cases} \tag{3.16}$$

When investigating the functional equation (2.16) by Bochner's method, we assume that the displacement operators defined by equality (2.17) are permutational, i.e., $J_2(J_3)(x,z)=J_3(J_2)(x,z)$, $(x,z) \in \overline{\Pi}_+$, which is equivalent to the fulfillment of the equality

$$\frac{\delta_3}{1-\tau_3}-\frac{\delta_2}{1-\tau_2}=0. \tag{3.17}$$

Below equality (3.17) is assumed to be fulfilled.

Lemma 3.4. *The operator T defined by formula* (3.1) *is invertible in the space $\overset{0}{C}_\alpha(\overline{\Pi}_+)$ if $\alpha > \sup H$.*

Proof. Consider the operator $\widetilde{T}_{i_0}$ acting in the space $\overset{0}{C}_\alpha(\overline{\Pi}_+)$ by the formula

$$\big(\widetilde{T}_{i_0}\varphi\big)(x,z)=\sum_{q=0}^{\infty}\gamma_{i_0 q}\big(T_2^{n_q}T_3^{m_q}\varphi\big)(x,z),\ \ i_0=1, \tag{3.18}$$

where $(T_p\varphi)(x,z)\equiv\varphi(x+\delta_p z,\tau_p z)$, $(x,z)\in\overline{\Pi}_+$, $\varphi\in\overset{0}{C}_\alpha(\overline{\Pi}_+)$, $\delta_1=0$, $\sigma_q=n_q\log\tau_2+m_q\log\tau_3$, $p=1,2,3$.

The operator $\widetilde{T}_{i_0}$ acting by formula (3.18) is defined correctly according to

Lemma 3.5. *The operator $T_2^{n_q}T_3^{m_q}$ appearing in formula* (3.18) *and corresponding to the decomposition $\sigma_q=n_q\log\tau_2+m_q\log\tau_3$ is correctly defined, i.e., if for σ_q another decomposition $\sigma_q=n_q'\log\tau_2+m_q'\log\tau_3$ takes place, then the equality $T_2^{n_q}T_3^{m_q}=T_2^{n_q'}T_3^{m_q'}$ is valid.*

Proof. Simple calculations show that

$$\big(T_2^{n_q}T_3^{m_q}\varphi\big)(x,z)=\varphi\Big[x+\Big(\delta_2\frac{1-\tau_2^{n_q}}{1-\tau_2}\tau_3^{m_q}+\delta_3\frac{1-\tau_3^{m_q}}{1-\tau_3}\Big)z,\tau_2^{n_q}\tau_3^{m_q}z\Big],$$

which, because of (3.17), takes the form

$$\big(T_2^{n_q}T_3^{m_q}\varphi\big)(x,z)=\varphi\Big[x+\frac{\delta_2}{1-\tau_2}(1-e^{\sigma_q})z,e^{\sigma_q}z\Big],\ \ (x,z)\in\overline{\Pi}_+. \tag{3.19}$$

Equality (3.19) implies that Lemma 3.5 is valid. □

By the assumption

$$c_3\equiv\sum_{q=0}^{\infty}|\gamma_{1q}|e^{\sigma_q\alpha}<\infty,\quad\text{where}\quad\alpha>\sup H. \tag{3.20}$$

Let $P_z(\varphi)\equiv\|\varphi\|_{\overset{0}{C}_\alpha(\overline{\Pi}_{+,z})}$, $\widetilde{\varphi}(x,z)\equiv\varphi\Big[x+\frac{\delta_2}{1-\tau_2}(1-e^{\sigma_q})z,e^{\sigma_q}z\Big]$, $(x,z)\in\overline{\Pi}_+$. Then

$$P_z(\widetilde{\varphi})=\|\widetilde{\varphi}\|_{\overset{0}{C}_\alpha(\overline{\Pi}_{+,z})}\le\sup_{\substack{x\in\mathbb{R},\\ 0<\tau_1\le z_0 z}}|\tau_1^{-\alpha}e^{\sigma_q\alpha}\varphi(x,\tau_1)|\le e^{\sigma_q\alpha}P_{z_0z}(\varphi), \tag{3.21}$$

where $z_0\equiv e^{\max\limits_q\sigma_q}=1$ because $\max\limits_q\sigma_q=-\max\{0,\log\tau_2,\log\tau_3\}=0$.

Let us now prove that the operator $\widetilde{T}_1$ is continuous. By virtue of (3.18) and (3.21) we have $P_z(\widetilde{T}_1\varphi) \le \sum_{q=0}^{\infty}|\gamma_{1q}|P_z(\widetilde{\varphi}) \le \sum_{q=0}^{\infty}|\gamma_{1q}|e^{\sigma_q\alpha}P_z(\varphi)$, whence, taking into account (3.20), we find $P_z(\widetilde{T}_1\varphi) \le c_3P_z(\varphi)$, which proves that the operator $\widetilde{T}_1$ is continuous.

Let us prove now that $T\widetilde{T}_1 = I$. According to the definition of the operator T, formulas (3.1), (3.18), and (3.16), we have

$$\begin{aligned}
\big((T\widetilde{T}_1)\varphi\big)(x,z) &= \sum_{p=1}^{3}\widetilde{G}_p\big(T_p(\widetilde{T}_1\varphi)\big)(x,z) = \sum_{q=0}^{\infty}\sum_{p=1}^{3}\widetilde{G}_p\gamma_{1q}\times\\
\times\varphi\Big\{x+\Big[\frac{\delta_2}{1-\tau_2}(1-e^{\sigma_q})+\delta_pe^{\sigma_q}\Big]z,\tau_pe^{\sigma_q}z\Big\} &= \sum_{\nu=0}^{\infty}\Big(\sum_{(q,p)\in I_\nu}\widetilde{G}_p\gamma_{1q}\Big)\times\\
\times\varphi\Big\{x+\Big[\frac{\delta_2}{1-\tau_2}(\tau_p-e^{\sigma_\nu})+\delta_pe^{\sigma_q}\Big]\tau_p^{-1}z,e^{\sigma_\nu}z\Big\} &=\\
=\varphi\Big\{x+\Big[\frac{\delta_2}{1-\tau_2}(\tau_p-1)+\delta_p\Big]\tau_p^{-1}z,z\Big\} &= \varphi(x,z),
\end{aligned}$$

since

$$\frac{\delta_2}{1-\tau_2}(\tau_p-1)+\delta_p = 0 \tag{3.22}$$

for condition (3.17) for any $p = 1, 2, 3$.

Similarly, taking into account (3.1), (3.16), (3.18), and (3.22), we can prove that

$$\begin{aligned}
\big((\widetilde{T}_1T)\varphi\big)(x,z) &= \sum_{q=0}^{\infty}\gamma_{1q}\big(T_2^{n_q}T_3^{m_q}(T\varphi)\big)(x,z) =\\
&= \sum_{q=0}^{\infty}\gamma_{1q}\sum_{p=1}^{3}\widetilde{G}_p\big(T_2^{n_q}T_3^{m_q}(T_p\varphi)\big)(x,z) =\\
&= \sum_{p=1}^{3}\sum_{q=0}^{\infty}\gamma_{1q}\widetilde{G}_p\varphi\Big\{x+\Big[\frac{\delta_2}{1-\tau_2}(1-e^{\sigma_q})\tau_p+\delta_p\Big]z,\tau_pe^{\sigma_q}z\Big\} =\\
&= \sum_{\nu=0}^{\infty}\Big(\sum_{(q,p)\in I_\nu}\gamma_{1q}\widetilde{G}_p\Big)\varphi\Big\{x+\Big[\frac{\delta_2}{1-\tau_2}(\tau_p-e^{\sigma_\nu})+\delta_p\Big]z,\tau_pe^{\sigma_\nu}z\Big\} =\\
&= \varphi\Big\{x+\Big[\frac{\delta_2}{1-\tau_2}(\tau_p-1)+\delta_p\Big]z,z\Big\} = \varphi(x,z).
\end{aligned}$$

Thus Lemma 3.4 is proved completely. □

Let now $I_{\alpha,\beta} \equiv [\min(\alpha,\beta),\max(\alpha,\beta)]$. It is obvious that if $H\cap I_{\alpha,\beta} = \varnothing$, then the segment $I_{\alpha,\beta}$ is wholly contained in the strip Π_{i_0}, $i_0 \ge 0$.

Lemma 3.6. *The operator T given by formula* (3.1) *is invertible in the space* $\overset{0}{C}_{\alpha,\beta}(\overline{\Pi}_+)$ *if* $H \cap I_{\alpha,\beta} = \varnothing$ *and* $T^{-1} = \widetilde{T}_{i_0}$ [12].

Proof. Since the function $\frac{1}{\Delta(s)}$ in the strip Π_{i_0}, $i_0 \geq 0$, decomposes into the absolutely convergent series (3.14), we have

$$c_4 \equiv \sum_{q=0}^{\infty} |\gamma_{i_0 q}| e^{\sigma_q \alpha} < \infty, \quad c_5 \equiv \sum_{q=0}^{\infty} |\gamma_{i_0 q}| e^{\sigma_q \beta} < \infty. \tag{3.23}$$

By virtue of $\varphi \in \overset{0}{C}_{\alpha,\beta}(\overline{\Pi}_+)$ we have the inequalities

$$\sup_{\substack{x \in \mathbb{R}, \\ 0<\tau\leq 1}} |\tau^{-\alpha} \varphi(x,\tau)| < \infty, \quad \sup_{\substack{x \in \mathbb{R}, \\ \tau > 1}} |\tau^{-\beta} \varphi(x,\tau)| < \infty.$$

Consider the case $\alpha < \beta$ and assume that

$$\varphi^*(x,z) \equiv \varphi\Big[x + \frac{\delta_2}{1-\tau_2}(1-e^\sigma)z, e^\sigma z,\Big], \quad (x,z) \in \overline{\Pi}_+.$$

I. Let $\sigma \geq 0$. Obviously, $\varphi^* \in \overset{0}{C}_{\alpha,\beta}(\overline{\Pi}_+)$, and we have the estimates

$$\sup_{\substack{x \in \mathbb{R}, \\ 0<\tau\leq 1}} |\tau^{-\alpha} \varphi^*(x,\tau)| \leq \sup_{\substack{x \in \mathbb{R}, \\ 0<\tau_1\leq e^\sigma}} |\tau_1^{-\alpha} e^{\sigma\alpha} \varphi(x,\tau_1)| \leq$$
$$\leq e^{\sigma\alpha} \max\Big\{ \sup_{\substack{x \in \mathbb{R}, \\ 0<\tau_1\leq 1}} |\tau_1^{-\alpha} \varphi(x,\tau_1)|, \sup_{\substack{x \in \mathbb{R}, \\ 1<\tau_1\leq e^\sigma}} |\tau_1^{-\beta} \varphi(x,\tau_1) \tau_1^{\beta-\alpha}| \Big\} \leq$$
$$\leq \max\Big\{ e^{\sigma\alpha} \sup_{\substack{x \in \mathbb{R}, \\ 0<\tau_1\leq 1}} |\tau_1^{-\alpha} \varphi(x,\tau_1)|, e^{\sigma\beta} \sup_{\substack{x \in \mathbb{R}, \\ 1<\tau_1\leq e^\sigma}} |\tau_1^{-\beta} \varphi(x,\tau_1)| \Big\} \leq$$
$$\leq e^{\sigma\beta} \|\varphi\|_{\overset{0}{C}_{\alpha,\beta}(\overline{\Pi}_+)},$$
$$\sup_{\substack{x \in \mathbb{R}, \\ \tau > 1}} |\tau^{-\beta} \varphi^*(x,\tau)| \leq \sup_{\substack{x \in \mathbb{R}, \\ \tau_1 > e^\sigma}} |\tau_1^{-\beta} e^{\sigma\beta} \varphi(x,\tau_1)| \leq e^{\sigma\beta} \|\varphi\|_{\overset{0}{C}_{\alpha,\beta}(\overline{\Pi}_+)}.$$

II. In the case $\sigma < 0$ we have

$$\sup_{\substack{x \in \mathbb{R}, \\ 0<\tau\leq 1}} |\tau^{-\alpha} \varphi^*(x,\tau)| \leq \sup_{\substack{x \in \mathbb{R}, \\ 0<\tau_1\leq e^\sigma}} |\tau_1^{-\alpha} e^{\sigma\alpha} \varphi(x,\tau_1)| \leq e^{\sigma\alpha} \|\varphi\|_{\overset{0}{C}_{\alpha,\beta}(\overline{\Pi}_+)},$$
$$\sup_{\substack{x \in \mathbb{R}, \\ \tau > 1}} |\tau^{-\beta} \varphi^*(x,\tau)| \leq \sup_{\substack{x \in \mathbb{R}, \\ \tau_1 > e^\sigma}} |\tau_1^{-\beta} e^{\sigma\beta} \varphi(x,\tau_1)| \leq$$
$$\leq \max\Big\{ \sup_{\substack{x \in \mathbb{R}, \\ e^\sigma<\tau_1\leq 1}} |\tau_1^{-\alpha} \tau_1^{\alpha-\beta} e^{\sigma\beta} \varphi(x,\tau_1)|, \sup_{\substack{x \in \mathbb{R}, \\ \tau_1 > 1}} |\tau_1^{-\beta} e^{\sigma\beta} \varphi(x,\tau_1)| \Big\} \leq$$

$$\leq \max\big\{e^{\sigma\alpha} \sup_{\substack{x\in\mathbb{R},\\ e^{\sigma}<\tau_1\leq 1}} |\tau_1^{-\alpha}\varphi(x,\tau_1)|, e^{\sigma\beta} \sup_{\substack{x\in\mathbb{R},\\ \tau_1>1}} |\tau_1^{-\beta}\varphi(x,\tau_1)|\big\} \leq$$
$$\leq e^{\sigma\alpha}\|\varphi\|_{\overset{0}{C}_{\alpha,\beta}(\overline{\Pi}_+)}.$$

Hence we obtain $\|\varphi^*\|_{\overset{0}{C}_{\alpha,\beta}(\overline{\Pi}_+)} \leq \max\{e^{\sigma\alpha}, e^{\sigma\beta}\}\|\varphi\|_{\overset{0}{C}_{\alpha,\beta}(\overline{\Pi}_+)}$, whence it follows that the operator $\widetilde{T}_{i_0}$ is continuous. Indeed, by virtue of (3.18), (3.23) we have

$$\|T_{i_0}\varphi\|_{\overset{0}{C}_{\alpha,\beta}(\overline{\Pi}_+)} \leq \sum_{q=0}^{\infty} |\gamma_{i_0 q}|\ \|\varphi^*\|_{\overset{0}{C}_{\alpha,\beta}(\overline{\Pi}_+)} \leq$$
$$\leq \Big(\sum_{\sigma_q<0} |\gamma_{i_0 q}| e^{\sigma_q\alpha} + \sum_{\sigma_q\geq 0} |\gamma_{i_0 q}| e^{\sigma_q\beta}\Big)\|\varphi\|_{\overset{0}{C}_{\alpha,\beta}(\overline{\Pi}_+)} \leq c_6\|\varphi\|_{\overset{0}{C}_{\alpha,\beta}(\overline{\Pi}_+)},$$

where $c_6 \equiv c_4 + c_5$.

As in Lemma 3.4 we prove that $T\widetilde{T}_{i_0} = \widetilde{T}_{i_0}T = I$, and thus $T^{-1} = \widetilde{T}_{i_0}$.

Analogously, we consider the case $\alpha \geq \beta$, which proves Lemma 3.6 completely. $\square$

Since problem (1.6), (1.7) in Conditions I, II in the class $\overset{0}{C}{}^l_\alpha(\overline{D})$ ($\overset{0}{C}{}^l_{\alpha,\beta}(\overline{D})$) has been reduced by equivalent transforms to equation (2.16) in the class $\overset{0}{C}_\alpha(\overline{\Pi}_+)$ ($\overset{0}{C}_{\alpha,\beta}(\overline{\Pi}_+)$), from Lemmas 3.4 and 3.6 we have

Theorem 3.2. *The boundary value problem* (1.6), (1.7) *is uniquely solvable in* $\overset{0}{C}{}^l_\alpha(\overline{D})$ *for* $\alpha > \sup H$ *and in* $\overset{0}{C}{}^l_{\alpha,\beta}(\overline{D})$ *for* $I_{\alpha,\beta} \cap H = \varnothing$.

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