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NON-BAIRE UNIONS IN CATEGORY BASES

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Abstract. We consider a partition of a space X consisting of a meager subset of X and obtain a sufficient condition for the existence of a subfamily of this partition which gives a non-Baire subset of X. The condition is formulated in terms of the theory of J. Morgan [1].

All notions concerning category bases come from Morgan's monograph (see [1]).

We establish the following theorem.

Theorem A. *Let $(X, \mathcal{S})$ be an arbitrary category base and $\mathcal{M}(\mathcal{S})$ be the σ-ideal of all meager sets in the base $(X, \mathcal{S})$, satisfying the following conditions:*

(1) *for an arbitrary cardinal number $\alpha < \operatorname{card} X$, the family $\mathcal{M}(\mathcal{S})$ is α-additive, i.e., this family is closed under the unions of arbitrary α-sequences of its elements,*

(2) *there exists a base $\mathcal{P}$ of $\mathcal{M}(\mathcal{S})$ of cardinality not greater than* $\operatorname{card} X$.

Thus, if $X \notin \mathcal{M}(\mathcal{S})$, then, for an arbitrary family $\{X_t\}_{t\in T}$ of meager sets, being a partition of X, there exists a set $T' \subset T$ such that $\cup_{t\in T'} X_t$ is not a Baire set.

The proof of this theorem is based on the following lemmas.

Lemma 1. *If $(X, \mathcal{S})$ is a category base and $\{A_\alpha : \alpha < \lambda\}$, where $\lambda \le \operatorname{card} \mathcal{S}$, is the family of essentially disjoint abundant Baire sets, then there exists a family of disjoint regions $\{B_\alpha : \alpha < \lambda\}$ such that every set A_α is abundant everywhere in B_α for each $\alpha < \lambda$.*

The proof of this lemma is similar to that of Theorem 1.5 in [1].

Lemma 2. *If $(X, \mathcal{S})$ is a category base, $\mathcal{M}(\mathcal{S})$ is the σ-ideal of all meager sets in the base $(X, \mathcal{S})$ and Φ is a family of subsets of X such that*

(1) $\operatorname{card} \Phi > \operatorname{card} X$,

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(2) $\forall_{Z_1,Z_2\in\Phi}(Z_1 \neq Z_2 \Rightarrow Z_1 \cap Z_2 \in \mathcal{M}(\mathcal{S}))$,
(3) $\forall_{Z\in\Phi} Z \notin \mathcal{M}(\mathcal{S})$,
then there exists a member of the family Φ *which is not a Baire set.*

Proof. Let us suppose that all members of the family Φ are Baire sets. First of all, we observe that $\operatorname{card}\Phi \leq \operatorname{card}\mathcal{S}$. Indeed, for each $Z \in \Phi$, by the fundamental theorem [1, Ch. 1], we can consider a region V such that Z is abundant everywhere in V. By Theorem 1.3 in [1], we claim that if $Z_1 \neq Z_2$, then $V_1 \neq V_2$. Hence $\operatorname{card}\Phi \leq \operatorname{card}\mathcal{S}$. Applying Lemma 1, we obtain a family Φ^* of disjoint nonempty regions such that $\operatorname{card}\Phi^* = \operatorname{card}\Phi$. In that case, we see that $\operatorname{card}\Phi \leq \operatorname{card}X$. This contradicts condition (1). □

Lemma 3. *If* X *is an infinite set and* Φ_1 *is a family of subsets of* X *such that*
(1) $\operatorname{card}\Phi_1 \leqq \operatorname{card}X$,
(2) $\forall_{Z\in\Phi_1}(\operatorname{card}Z = \operatorname{card}X)$,
then there exists a family Φ_2 *of subsets of* X *such that*
(a) $\operatorname{card}\Phi_2 > \operatorname{card}X$,
(b) $\forall_{Z_1,Z_2\in\Phi_2}(Z_1 \neq Z_2 \Rightarrow \operatorname{card}(Z_1 \cap Z_2) < \operatorname{card}X)$,
(c) $\forall_{Y\in\Phi_1}\forall_{Z\in\Phi_2}(\operatorname{card}(Z \cap Y) = \operatorname{card}X)$.

This combinatorial lemma is formulated and proved in [2].

Now, we shall prove Theorem A.

We assume that X is not meager and $\{X_t\}_{t\in T}$ is a family of meager sets such that $\cup_{t\in T} X_t = X$. Our goal is to find a set $T' \subset T$ such that $\cup_{t\in T'} X_t$ is not a Baire set. We see that $\operatorname{card}T = \operatorname{card}X$ because, otherwise, X would be meager.

Let $\mathcal{P}$ be a base of $\mathcal{M}(\mathcal{S})$ of cardinality not greater than $\operatorname{card}X$.

Let $\mathcal{Y} = \{Y \subset X : X\setminus Y \in \mathcal{P}\}$, $K(Y) = \{t \in T : X_t \cap Y \neq \varnothing\}$ and $\Phi_1 = (K(Y))_{Y\in\mathcal{Y}}$. Then $\operatorname{card}\Phi_1 \leq \operatorname{card}X$. Moreover, we conclude that each member of the family Φ_1 is of cardinality equal to $\operatorname{card}X$. Indeed, if we have that, for some $Y \subset X$, $\operatorname{card}K(Y) < \operatorname{card}X$, then

$$Y = \bigcup_{t\in T\setminus K(Y)} (X_t \cap Y) \cup \bigcup_{t\in K(Y)} (X_t \cap Y) = \bigcup_{t\in K(Y)} (X_t \cap Y),$$

but, by condition (a), we would get that Y is meager. At the same time, $X\setminus Y$ is meager and, finally, X is meager. This contradicts the fact that X is not meager.

By Lemma 3, there exists a family Φ_2 of subsets of T such that
(a) $\operatorname{card}\Phi_2 > \operatorname{card}T$,
(b) $\forall_{Z_1,Z_2\in\Phi_2}(Z_1 \neq Z_2 \Rightarrow \operatorname{card}(Z_1 \cap Z_2) < \operatorname{card}T)$,
(c) $\forall_{Y\in\Phi_1}\forall_{Z\in\Phi_2}(\operatorname{card}(Z \cap Y) = \operatorname{card}T)$.

Let $X(Z) = \cup_{t\in Z} X_t$ for each $Z \in \Phi_2$.

We observe that $X(Z)$ is not meager because, otherwise, if $X(Z)$ were meager for some $Z \in \Phi_2$, then there would exist a member P of base $\mathcal{P}$, such that $X(Z) \subset P$, and then $K(X\setminus P) \cap Z \neq \varnothing$. Hence there exists $t \in$

$K(X\backslash P)$, $t \in Z$. If $t \in K(X\backslash P)$, then $X_t \cap (X\backslash P) \neq \varnothing$, but, simultaneously, $X_t \subset X(Z) \subset P$. This contradiction ends the proof that $X(Z)$ is not meager. It is clear that, for arbitrary $Z_1, Z_2 \in \Phi_2$, $Z_1 \neq Z_2$, we have $X(Z_1) \cap X(Z_2) = \cup_{t \in Z_1 \cap Z_2} X_t$.

Since $\mathrm{card}(Z_1 \cap Z_2) < \mathrm{card}\, X$, therefore $X(Z_1) \cap X(Z_2)$ is a meager set.

Putting $\Phi = \{X(Z) : Z \in \Phi_2\}$, we see that

(a) $\mathrm{card}\, \Phi > \mathrm{card}\, X$,

(b) $\forall_{X(Z_1), X(Z_2)} (X(Z_1) \neq X(Z_2) \Rightarrow X(Z_1) \cap X(Z_2) \in \mathcal{M}(\mathcal{S}))$,

(c) $\forall_{X(Z) \in \Phi} X(Z) \notin \mathcal{M}(\mathcal{S})$;

hence, by Lemma 2, there exists a set Z such that $X(Z)$ is not Baire. This precisely means that there exists $T' \subset T$ such that $\cup_{t \in T'} X_t$ is not a Baire set.

Now, we need the following well-known auxiliary proposition (see, e.g., [3]).

Lemma 4. *Let* $(X_i)_{i \in I}$ *be a family of subsets of an infinite set* X *such that*

(a) $\mathrm{card}(X_i) = \mathrm{card}\, X$ *for each* $i \in I$,

(b) $\mathrm{card}(I) \leqq \mathrm{card}\, X$.

Then there exist sets Y, Z, *being a decomposition of* X, *such that* $\mathrm{card}(Y \cap X_i) = \mathrm{card}(Z \cap X_i) = \mathrm{card}\, X$ *for each* $i \in I$.

Note that this lemma easily follows from the classical construction of F. Bernstein.

Let $(X, \mathcal{S})$ be a category base.

Definition1. We shall say that a subfamily $\mathcal{S}' \subset \mathcal{S}$ is a π-base if each region $A \in \mathcal{S}$ contains a subregion $B \in \mathcal{S}'$.

Definition 2 (cf. [4]). We shall say that a set $A \subset X$ is not exhausted if, for each meager set $P \subset X$, $\mathrm{card}(A\backslash P) = \mathrm{card}\, X$.

Now, we prove

Theorem B. *Let a category base* $(X, \mathcal{S})$ *satisfy the following conditions:*

(a) *there exists a* π*-base* $\mathcal{S}'$ *such that* $\mathrm{card}\, \mathcal{S}' \leq \mathrm{card}\, X$ *and each set of the family* $\mathcal{S}'$ *is not exhausted,*

(b) *there exists a base* $\mathcal{P}$ *of the family* $\mathcal{M}(\mathcal{S})$ *of all meager sets of cardinality not greater than* $\mathrm{card}\, X$.

Then the following statements are equivalent:

(a) X *is an abundant set,*

(b) X *contains a non-Baire set.*

Proof. Only implication (a) $\Rightarrow$ (b) needs a proof. Let $\mathcal{S} = \{B_r\}_{r \in R}$ be a π-base of X such that $\mathrm{card}\, R \leq \mathrm{card}\, X$ and $\mathcal{P} = \{P_t\}_{t \in T}$ a base of $\mathcal{M}(\mathcal{S})$ such that $\mathrm{card}\, T \leq \mathrm{card}\, X$. Let $\mathcal{A} = \{(B_r \backslash P_t)\}_{r \in R, t \in T}$. We see that $\mathrm{card}\, \mathcal{A} \leq \mathrm{card}\, X$ and $\mathrm{card}(B_r \backslash P_t) = \mathrm{card}\, X$ for any $r \in R$, $t \in T$. By Lemma 4, there exist sets $Y, Z \subset X$ such that $X = Y \cup Z$, $Y \cap Z = \varnothing$

and, moreover, $Z \cap (B_r \backslash P_t) \neq \varnothing \neq Y \cap (B_r \backslash P_t)$ for any $r \in R$, $t \in T$. It is clear that Y or Z is abundant. Let us assume that the set Y is abundant. We prove that it is not a Baire set. Suppose that Y is a Baire set. By the fundamental theorem, there exists a region V such that Y is abundant everywhere in V. Let $V_r \subset X$ be a member of a π-base $\mathcal{S}'$ such that $V_r \subset V$. It is obvious that Y is abundant everywhere in V_r. Since Y is a Baire set, $V_r \backslash Y$ is a meager set [1, Th. 1.3]. Thus there exists $P_t \in \mathcal{P}$ such that $V_r \backslash Y \subset P_t$. Hence we have $V_r \backslash P_t \subset Y$ and $(V_r \backslash P_t) \cap Z = \varnothing$. This contradicts the fact that the last set should not be empty. □

Corollary. *In the case where the category base is the family of sets of positive Lebesgue measure over the real line* $\mathbb{R}$ *or the natural topology of this line, we can obtain the existence of a nonmeasurable set or a set without the Baire property.*

To conclude, let us recall the following theorem which generalizes one result of Kharazishvili (the proof of this theorem is contained in [5]).

Theorem H. *Let* $(X, \mathcal{S})$ *be a category base on an infinite set* X*, such that the following conditions are satisfied:*

1^0 $\mathcal{M}_0 \subset \mathcal{M}(\mathcal{S})$*, where* $\mathcal{M}_0 = \{A \subset X : \operatorname{card} A < \operatorname{card} X\}$ *and* $\mathcal{M}(\mathcal{S})$ *is a* σ*-ideal of meager sets,*

2^0 *there exists a base of the* σ*-ideal* $\mathcal{M}(\mathcal{S})$ *of cardinality not greater than* $\operatorname{card} X$.

Then the following statements are equivalent:

(a) X *is a meager set,* (b) *each subset of* X *is a Baire set.*

For a purely topological version of this theorems see [6].

If we consider Theorems B and H as the criteria of the existence of non-Baire sets in category bases, we discover, by examples 1 and 2 in [4], that they are independent.

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