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ON THE CORRECTNESS OF NONLINEAR BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. The concept of a strongly isolated solution of the nonlinear boundary value problem

$$dx(t) = dA(t) \cdot f(t, x(t)), \quad h(x) = 0,$$

is introduced, where $A : [a, b] \to R^{n\times n}$ is a matrix-function of bounded variation, $f : [a, b] \times R^n \to R^n$ is a vector-function belonging to a Carathéodory class, and h is a continuous operator from the space of n-dimensional vector-functions of bounded variation into R^n.

It is stated that the problems with strongly isolated solutions are correct. Sufficient conditions for the correctness of these problems are given.

1. Statement of the Problem and Formulation of the Results

Let $A = (a_{ij})_{i,j=1}^n : [a, b] \to R^{n\times n}$ be a matrix-function of bounded variation, $f = (f_i)_{i=1}^n : [a, b] \times R^n \to R^n$ be a vector-function belonging to the Carathéodory class $K([a, b] \times R^n, R^n; A)$, and let $h : \mathrm{BV}_s([a, b], R^n) \to R^n$ be a continuous operator such that the nonlinear boundary value problem

$$dx(t) = dA(t) \cdot f(t, x(t)), \tag{1.1}$$

$$h(x) = 0 \tag{1.2}$$

has a solution x^0.

Consider a sequence of matrix-functions of bounded variation $A_k : [a, b] \to R^{n\times n}$ $(k = 1, 2, \dots)$, a sequence of vector-functions $f_k : [a, b] \times R^n \to R^n$, $f_k \in K([a, b] \times R^n, R^n; A_k)$ $(k = 1, 2, \dots)$ and a sequence of continuous operators $h_k : \mathrm{BV}_s([a, b], R^n) \to R^n$ $(k = 1, 2, \dots)$.

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In this paper sufficient conditions are given guaranteeing both the solvability of the problem

$$dx(t) = dA_k(t) \cdot f_k(t, x(t)), \tag{1.1$_k$}$$

$$h_k(x) = 0 \tag{1.2$_k$}$$

for any sufficiently large k and the convergence of its solutions as $k \to +\infty$ to the solution of problem (1.1), (1.2).

An analogous question was studied in [1–4] for initial and boundary value problems for nonlinear systems of ordinary differential equations.

The theory of generalized ordinary differential equations enables one to investigate ordinary differential and difference equations from a unified standpoint. Moreover, the convergence conditions for difference schemes corresponding to the boundary value problems for systems of ordinary differential equations can be obtained from the results concerning the correctness of the boundary value problems for systems of generalized ordinary differential equations [5–8].

Throughout the paper the following notation and definitions will be used:

$R =]-\infty, +\infty[$, $R_+ = [0, +\infty[$; $R^{n\times m}$ is a space of all real $n\times m$ matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm $\|X\| = \max_{j=1,\dots,m} \sum_{i=1}^n |x_{ij}|$; $R^n = R^{n\times 1}$.

$$|X| = \big(|x_{ij}|\big)_{i,j=1}^{n,m}, \quad [X]_+ = \Big(\frac{|x_{ij}| + x_{ij}}{2}\Big)_{i,j=1}^{n,m}.$$

$$R_+^{n\times m} = \Big\{(x_{ij})_{i,j=1}^{n,m} \in R^{n\times m} : \ x_{ij} \geq 0 \ (i = 1, \dots, n; \ j = 1, \dots, m)\Big\},$$

$$R_+^n = R_+^{n\times 1}.$$

If $X \in R^{n\times n}$, then $\det(X)$ and X^{-1} are respectively the determinant of X and the matrix inverse to X; I is the identity $n \times n$ matrix.

$V_a^b X$ is the total variation of the matrix-function $X : [a, b] \to R^{n\times m}$, i.e., the sum of total variations of the latter's components; $X(t-)$ and $X(t+)$ $(X(a-) = X(a), X(b+) = X(b))$ are the left and the right limit of the matrix-function X at the point $t \in [a, b]$,

$$d_1X(t) = X(t) - X(t-), \quad d_2X(t) = X(t+) - x(t);$$

$$\|X\|_s = \sup\big\{\|X(t)\| : \ t \in [a, b]\big\}.$$

$\mathrm{BV}([a, b], D)$ is a set of all $X : [a, b] \to D \subset R^{n\times m}$ such that $V_a^b X < +\infty$.

$\mathrm{BV}_s([a, b], R^n)$ is a normed space $(\mathrm{BV}([a, b], R^n); \|\cdot\|_s)$.

If $y \in \mathrm{BV}_s([a, b], R^n)$, then $U(y; r) = \{x \in \mathrm{BV}_s([a, b], R^n) : \|x - y\|_s < r\}$; $D(y; r)$ is a set of all $x \in R^n$ such that $\inf\{\|x - y(\tau)\| : \tau \in [a, b]\} < r$.

If $J \subset R$ and $D_1 \subset R^n$, then $C(J, D_1)$ is a set of all continuous vector-functions $x : J \to D_1$.

If $g \in \mathrm{BV}([a,b],R)$, $x : [a,b] \to R$ and $a \le s < t \le b$, then $v(g) : [a,b] \to R$ is defined by $v(g)(a) = 0$ and $v(g)(t) = V_a^t g$ for $t \in (a,b]$;

$$\int_s^t x(\tau)dg(\tau) = \int_{]s,t[} x(\tau)dg(\tau) + x(t)d_1 g(t) + x(s)d_2 g(s),$$

where $\int_{]s,t[} x(\tau)dg(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ (if $s = t$, then $\int_s^t x(\tau)dg(\tau) = 0$).

If $G = (g_{ij})_{i,j=1}^{l,n} \in \mathrm{BV}([a,b],R^{l\times n})$, $X = (x_{jk})_{j,k=1}^{n,m} : [a,b] \to D_2 \subset R^{n\times m}$, and $a \le s \le t \le b$, then

$$V(X) = \big(v(x_{jk})\big)_{j,k=1}^{n,m},$$
$$\int_s^t dG(\tau)\cdot X(\tau) = \bigg(\sum_{j=1}^n \int_s^t x_{jk}(\tau)dg_{ij}(\tau)\bigg)_{i,k=1}^{l,m}.$$

$L([a,b],D_2;G)$ is a set of all matrix-functions $(x_{jk})_{j,k=1}^{n,m} : [a,b] \to D_2$ such that x_{jk} is integrable with respect to g_{ij} $(i = 1,\dots,l)$.

$K([a,b]\times D_1,D_2;G)$ is a Carathéodory class, i.e., the set of all mappings $F = (f_{jk})_{j,k=1}^{n,m} : [a,b]\times D_1 \to D_2$ such that (a) $f_{ik}(\cdot,x)$ is measurable with respect to the measures $V(g_{ij})$ and $V(g_{ij}) - g_{ij}$ for $x \in D_1$ $(i = 1,\dots,l)$, (b) $F(t,\cdot) : D_1 \to D_2$ is continuous for $t \in [a,b]$ and

$$\sup\big\{|F(\cdot,x)| : \ x \in D_0\big\} \in L([a,b],R^{n\times m};G)$$

for every compact $D_0 \subset D_1$.

$K^0([a,b]\times D_1,D_2;G)$ is a set of all mappings $F = (f_{jk})_{j,k=1}^{n,m} : [a,b]\times D_1 \to D_2$ such that for every vector-function of bounded variation $x : [a,b] \to D_1$ the function $f_{jk}(\cdot,x(\cdot))$ is measurable with respect to the measures $V(g_{ij})$ and $V(g_{ij}) - g_{ij}$ $(i = 1,\dots,l)$.

If $B \in \mathrm{BV}([a,b],R^n)$, then $M([a,b]\times R_+,R_+^n;B)$ is a set of all vector-functions $\omega \in K([a,b]\times R_+,R_+^n;B)$ such that $\omega(t,\cdot)$ is nondecreasing and $\omega(t,\cdot) = 0$ for every $t \in [a,b]$.

The inequalities between both the vectors and the matrices are understood to be componentwise.

If B_1 and B_2 are normed spaces, then an operator $g : B_1 \to B_2$ is called positive homogeneous if $g(\lambda x) = \lambda g(x)$ for every $\lambda \in R_+$ and $x \in B_1$.

A vector-function $x \in \mathrm{BV}([a,b],R^n)$ is said to be a solution of system (1.1) if

$$x(t) = x(s) + \int_s^t dA(\tau)\cdot f(\tau,x(\tau)) \quad \text{for} \quad a \le s < t \le b.$$

By a solution of the system of generalized ordinary differential inequalities $dx(t) \le dA(t)\cdot f(t,x(t))$ $(\ge)$ we mean a vector-function $x \in \mathrm{BV}([a,b],R^n)$

such that

$$x(t) \le x(s) + \int_s^t dA(\tau) \cdot f(\tau, x(\tau)) \quad (\ge) \quad a \le s < t \le b.$$

Definition 1.1. Let $l : \mathrm{BV}_s([a,b], R^n) \to R^n$ be a linear continuous operator and let $l_0 : \mathrm{BV}_s([a,b], R^n) \to R^n_+$ be a positive homogeneous continuous operator. We shall say that a matrix-function $P : [a,b] \times R^n \to R^{n\times m}$ satisfies the Opial condition with respect to the triplet $(l, l_0; A)$ if:

(a) $P \in K([a,b] \times R^n, R^{n\times n}; A)$ and there exists a matrix-function $\Phi \in L([a,b], R^{n\times n}_+; A)$ such that

$$|P(t,x)| \le \Phi(t) \quad \text{on} \quad [a,b] \times R^n;$$

(b) for every $B \in \mathrm{BV}([a,b], R^{n\times n})$

$$\det\big(I + (-1)^j d_j B(t)\big) \ne 0 \quad \text{for} \quad t \in [a,b] \quad (j=1,2) \tag{1.3}$$

and the problem

$$dx(t) = dB(t) \cdot x(t), \quad |l(x)| \le l_0(x) \tag{1.4}$$

has only the trivial solution provided there exists a sequence

$$y_k \in \mathrm{BV}([a,b], R^n) \quad (k = 1, 2, \dots)$$

such that

$$\lim_{k\to+\infty} \int_a^t dA(\tau) \cdot P(\tau, y_k(\tau)) = B(t) \quad \text{uniformly on} \quad [a,b].$$

Let r be a positive number.

Definition 1.2. x^0 is said to be strongly isolated in the radius r if there exist $P \in K([a,b]\times R^n, R^{n\times n}; A)$, $q \in K([a,b]\times R^n, R^n; A)$, a linear continuous operator $l : \mathrm{BV}_s([a,b], R^n) \to R^n$, and a positive homogeneous operator $\widetilde{l} : \mathrm{BV}_s([a,b], R^n) \to R^n$ such that:

(a) $f(t,x) = P(t,x)x + q(t,x)$ for $t \in [a,b]$, $\|x - x^0(t)\| < r$ and the equality $h(x) = l(x) + \widetilde{l}(x)$ is fulfilled on $U(x^0; r)$;

(b) the vector-functions $\alpha(t,\rho) = \max\{|q(t,x)| : \|x\| \le \rho\}$ and $\beta(\rho) = \sup\{[|\widetilde{l}(x)| - l_0(x)]_+ : \|x\|_s \le \rho\}$ satisfy the conditions

$$\lim_{\rho\to+\infty} \frac{1}{\rho} \int_a^b dV(A)(t) \cdot \alpha(t,\rho) = 0, \quad \lim_{\rho\to+\infty} \frac{\beta(\rho)}{\rho} = 0\ ;$$

(c) the problem

$$dx(t) = dA(t) \cdot \big[P(t, x(t))x(t) + q(t, x(t))\big], \tag{1.5}$$

$$l(x) + \widetilde{l}(x) = 0 \tag{1.6}$$

has no solution different from x^0;

(d) the matrix-function P satisfies the Opial condition with respect to the triplet $(l, l_0; A)$.

The notation

$$\big((A_k, f_k, h_k)\big)_{k=1}^{+\infty} \in W_r(A, f, h; x^0)$$

means that:

(a) for every $x \in D(x^0; r)$

$$\lim_{k\to+\infty} \int_a^t dA_k(\tau)\cdot f_k(\tau, x) = \int_a^t dA(\tau)\cdot f(\tau, x) \tag{1.7}$$

uniformly on $[a, b]$;

(b)

$$\lim_{k\to+\infty} h_k(x) = h(x) \quad \text{uniformly on} \quad U(x^0; r); \tag{1.8}$$

(c) there exists a sequence $\omega_k \in M([a, b]\times R_+, R_+^n; A_k)$ $(k = 1, 2, \dots)$ such that

$$\sup\left\{\Big\| \int_a^b dV(A_k)(\tau)\cdot\omega_k(\tau, r)\Big\| : \ k = 1, 2, \dots\right\} < +\infty, \tag{1.9}$$

$$\lim_{s\to 0+} \sup\left\{\Big\| \int_a^b dV(A_k)(\tau)\cdot\omega_k(\tau, s)\Big\| : \ k = 1, 2, \dots\right\} = 0, \tag{1.10}$$

$$\big|f_k(t, x) - f_k(t, y)\big| \le \omega_k(t, \|x - y\|) \tag{1.11}$$

on $[a, b]\times D(x^0; r)$ $(k = 1, 2, \dots)$.

Remark 1.1. If for every natural m there exists a positive number μ_m such that

$$\omega_k(t, m\sigma) \le \mu_m \omega_k(t, \sigma) \quad \text{for} \quad \sigma > 0, \quad t\in[a, b] \quad (k = 1, 2, \dots),$$

then (1.9) follows from (1.10).

In particular, the sequence of functions

$$\omega_k(t, \sigma) = \max\Big\{|f_k(t, x) - f_k(t, y)| : \ \|x\| \le \|x^0\|_s + r, \\ \|y\| \le \|x^0\|_s + r, \ \|x - y\| \le \sigma\Big\}$$

$(k = 1, 2, \dots)$ has the latter property.

Definition 1.3. Problem (1.1), (1.2) is said to be $(x^0;r)$-correct if for every $\varepsilon\in]0,r[$ and $((A_k,f_k,h_k))_{k=1}^{+\infty}\in W_r(A,f,h;x^0)$ there exists a natural number k_0 such that problem (1.1_k), (1.2_k) has at least one solution contained in $U(x^0;r)$, and any such solution belongs to the ball $U(x^0;r)$ for any $k\geq k_0$.

Definition 1.4. Problem (1.1), (1.2) is said to be correct if it has the unique solution x^0 and for every $r>0$ it is $(x^0;r)$-correct.

Theorem 1.1. *If problem* (1.1), (1.2) *has a solution x^0 which is strongly isolated in the radius r, then it is $(x^0;r)$-correct.*

Corollary 1.1. *Let the inequality*

$$\big|f(t,x)-P(t,x)x\big|\leq\alpha(t,\|x\|)\tag{1.12}$$

be fulfilled on $[a,b]\times R^n$ and

$$|h(x)-l(x)|\leq l_0(x)+l_1(\|x\|_s)\quad\text{for}\quad x\in \mathrm{BV}([a,b],R^n),\tag{1.13}$$

where $l:\mathrm{BV}_s([a,b],R^n)\to R^n$ and $l_0:\mathrm{BV}_s([a,b],R^n)\to R^n_+$ are respectively a linear continuous and a positive homogeneous continuous operator, the matrix-function P satisfies the Opial condition with respect to the triplet $(l,l_0;A)$, $\alpha\in K([a,b]\times R_+,R^n_+;A)$ is nondecreasing in the second variable, $l_1\in C(R_+,R^n_+)$, and

$$\lim_{\rho\to+\infty}\frac{1}{\rho}\int_a^b dV(A)(t)\cdot\alpha(t,\rho)=\lim_{\rho\to+\infty}\frac{l_1(\rho)}{\rho}=0.\tag{1.14}$$

If problem (1.1), (1.2) *has at most one solution, then it is correct.*

Corollary 1.2. *Let inequalities* (1.12) *and*

$$P_1(t)\leq P_1(t,x)\leq P_2(t)\tag{1.15}$$

be fulfilled on $[a,b]\times R^n$ and let (1.13) *hold, where $l:\mathrm{BV}_s([a,b],R^n)\to R^n$ and $l_0:\mathrm{BV}_s([a,b],R^n)\to R^n$ are respectively a linear continuous and a positive homogeneous continuous operator, $P\in K^0([a,b]\times R^n,R^{n\times n};A)$, $P_k\in L([a,b],R^{n\times n};A)$ $(k=1,2)$, $\alpha\in K([a,b]\times R_+,R^n_+;A)$ is nondecreasing in the second variable, $l_1\in C(R_+,R^n_+)$ and* (1.14) *holds. Let, moreover,* (1.3) *hold and let problem* (1.4) *have only a trivial solution for every $B\in\mathrm{BV}([a,b],R^{n\times n})$ satisfying the inequality*

$$\begin{aligned}\bigg|B(t)-B(s)-\frac{1}{2}\int_s^t dA(\tau)\cdot\big[P_1(\tau)+P_2(\tau)\big]\bigg|\leq\\ \leq\frac{1}{2}\int_s^t dV(A)(\tau)\cdot\big[P_2(\tau)-P_1(\tau)\big]\end{aligned}\tag{1.16}$$

for $a \le s \le t \le b$. *If problem* (1.1), (1.2) *has at most one solution, then it is correct.*

Remark 1.2. Corollary 1.2 is of interest only in the case of $P \notin K([a,b] \times R^n, R^{n\times n}; A)$, since for $P \in K([a,b] \times R^n, R^{n\times n}; A)$ it follows immediately from Corollary 1.1.

Corollary 1.3. *Let the conditions*

$$\begin{gathered}\big|f(t,x) - P_0(t)x\big| \le Q(t)|x| + q(t,\|x\|),\\ \det\big(I + (-1)^j d_jA(t)\cdot P_0(t)\big) \neq 0 \quad (j=1,2)\end{gathered} \tag{1.17}$$

and

$$\big\||d_jA(t)|\cdot Q(t)\big\|\big\|\big(I+(-1)^j d_jA(t)\cdot P_0(t)\big)^{-1}\big\| < 1 \quad (j=1,2) \tag{1.18}$$

be fulfilled on $[a,b]\times R^n$ *and let* (1.13) *hold, where* $l : \mathrm{BV}_s([a,b],R^n) \to R^n$ *and* $l_0 : \mathrm{BV}_s([a,b],R^n) \to R^n_+$ *are respectively a linear continuous and a positive homogeneous continuous operator,* $P_0 \in L([a,b], R^{n\times n}; A)$, $Q \in L([a,b], R^{n\times n}_+; A)$, $q \in K([a,b]\times R_+, R^n_+; A)$ *is nondecreasing in the second variable,* $l_1 \in C(R_+, R^n_+)$, *and*

$$\lim_{\rho\to+\infty} \frac{1}{\rho}\int_a^b dV(A)(t)\cdot q(t,\rho) = \lim_{\rho\to+\infty}\frac{l_1(\rho)}{\rho} = 0.$$

Moreover, let the problem

$$\big|dx(t) - dA(t)\cdot P_0(t)x(t)\big| \le dV(A)(t)\cdot Q(t)|x(t)|, \quad |l(x)| \le l_0(x) \tag{1.19}$$

have only the trivial solution. If problem (1.1), (1.2) *has at most one solution, then it is correct.*

Corollary 1.4. *Let the conditions*

$$\big|f(t,x) - P(t)x\big| \le \alpha(t,\|x\|)$$

and

$$\det\big(I + (-1)^j d_jA(t)\cdot P(t)\big) \neq 0 \quad (j=1,2) \tag{1.20}$$

be fulfilled on $[a,b]\times R^n$ *and*

$$|h(x) - l(x)| \le l_1(\|x\|_s) \quad \textit{for} \quad x \in \mathrm{BV}([a,b],R^n),$$

where $l : \mathrm{BV}_s([a,b],R^n) \to R^n$ *is a linear continuous operator,* $P \in L([a,b], R^{n\times n}; A)$, $\alpha \in K([a,b]\times R_+, R^n_+; A)$ *is nondecreasing in the second variable,* $l_1 \in C(R_+, R^n_+)$, *and* (1.14) *holds. Moreover, let the problem*

$$dx(t) = dA(t)\cdot P(t)x(t), \quad l(x) = 0$$

have only the trivial solution. If problem (1.1), (1.2) *has at most one solution, then it is correct.*

Corollary 1.5. *Let conditions* (1.17), (1.18) *and*

$$\big|f(t,x)-f(t,y)-P_0(t)(x-y)\big|\le Q(t)|x-y|$$

be fulfilled on $[a,b]\times R^n$ *and*

$$|h(x)-h(y)-l(x-y)|\le l_0(x-y) \quad on \quad \mathrm{BV}([a,b],R^n),$$

where $l:\mathrm{BV}_s([a,b],R^n)\to R^n$ *and* $l_0:\mathrm{BV}_s([a,b],R^n)\to R^n_+$ *are respectively a linear continuous and a positive homogeneous continuous operator,* $P_0\in L([a,b],R^{n\times n};A)$, $Q\in L([a,b],R^{n\times n}_+;A)$, *and problem* (1.19) *has only the trivial solution. Then problem* (1.1), (1.2) *is correct.*

Let $t_0\in[a,b]$. For every $j\in\{1,2\}$, $(-1)^j(t-t_0)<0$, $c\in R^n$ and $\rho\ge\|c\|$ by $X_j(t,c,\rho)$ we denote a set of all solutions z of system (1.1) such that $z(t_0)=c$ and $\|z(s)\|<\rho$ for $(t-s)(s-t_0)>0$.

Corollary 1.6. *Let* $h(x)=x(t_0)-g(x)$, $t_0\in[a,b]$, *then the set*

$$G=\big\{g(x):\ x\in\mathrm{BV}([a,b],R^n)\big\}$$

is bounded and for any $c\in G$ *every local solution of system* (1.1) *satisfying the condition*

$$x(t_0)=c \tag{1.21}$$

can be continued to the whole segment $[a,b]$. *Let, moreover, the matrix-function* A *be continuous at the point* t_0, *and for every* $j\in\{1,2\}$

$$\lim_{\rho\to+\infty}\sup\bigg(\inf\Big\{\|x+(-1)^j d_jA(t)\cdot f(t,y)\|-\sigma_{3-j}(t,z):\ \|x\|\ge\rho,$$
$$\|y\|=\rho,\ z\in X_j(t,c,\rho),\ c\in G\Big\}\bigg)>0 \tag{1.22}$$

uniformly on $\{t\in[a,b]:\ (-1)^j(t-t_0)<0\}$, *where* $\sigma_j(t,z)=\max\{\|z(t)\|,$ $\|z(t)+(-1)^jd_jz(t)\|\}$. *Then the unique solvability of problem* (1.1), (1.2) *guarantees its correctness.*

Corollary 1.7. *Let* $t_0\in[a,b]$, $c\in R^n$, *and let system* (1.1) *have the unique solution* x_0 *defined on the whole segment* $[a,b]$, *satisfying the initial condition* (1.21). *Moreover, let the matrix-function* A *be continuous at the point* t_0 *and for every* $j\in\{1,2\}$

$$\lim_{\rho\to+\infty}\sup\inf\Big\{\|x+(-1)^jd_jA(t)\cdot f(t,y)\|:\ |x\|\ge\rho,$$
$$\|y\|=\rho\Big\}>\|x^0\|_s \tag{1.23}$$

uniformly on $\{t \in [a,b] : (-1)^j(t-t_0) < 0\}$. *Then problem* (1.1), (1.2) *is correct.*

Remark 1.3. If the matrix-function A is continuous on $[a,b]$, then condition (1.23) is fulfilled. In the case of discontinuous A in (1.23) the strict inequality cannot be replaced by the nonstrict one. Below we shall give the corresponding example.

Example 1.1. Let $n = 1$ and let $m > 2$ be a fixed natural number, $\tau_i = a + \frac{i}{m}(b-a)$ $(i = 0,\dots,m)$; $h(x) = x(t_0) - c_0$, $h_k(x) = x(t_0) - c_k$, where $c_0 = 0$, $c_k = \frac{1}{k}$ $(k = 1,2,\dots)$; $A(t) = i$ for $t \in [\tau_i, \tau_{i+1}[$ $(i = 0,\dots,m-1)$, $A(b) = m$, $A_k(t) \equiv A(t)$ $(k = 1,2,\dots)$; $f(t,x) = f_k(t,x) = 0$ for $t \in [a,\tau_1[\cup]\tau_1,b]$, $x \in R^n$ $(k = 1,2,\dots)$;

$$f(\tau_1,x) = \begin{cases} 0 & \text{for } \ x \in]-\infty, 0[\,, \\ (1 + c_{j+1} - c_j)(x-j) + j + c_j & \text{for } \ x \in [j, j+1[\\ & \qquad (j = 0,1,\dots); \end{cases}$$

$$f_k(\tau_1,x) = \begin{cases} f(\tau_1,x) & \text{for } \ x \in]-\infty, k-1[\cup]k+1, +\infty[\,, \\ (1 - c_{k-1} - c_k)(x-k)+ & \\ +k - c_k & \text{for } \ x \in [k-1, k[\,, \\ (1 + c_{k+1} + c_k)(x-k)+ & \\ +k - c_k & \text{for } \ x \in [k, k+1[\end{cases}$$
$$(k = 1,2,\dots).$$

Then $x^0(t) \equiv 0$, $((A_k, f_k, h_k))_{k=1}^{+\infty} \in W_r(A,f,h;x^0)$ for every positive number r and for every natural k problem (1.1_k), (1.2_k) has the unique solution $x_k(t) \equiv k$. As for condition (1.23), it is transformed into the equality only for $t = \tau_1$.

Corollary 1.8. *Let there exist a solution* x^0 *of problem* (1.1), (1.2) *and a positive number* r *such that*

$$\bigl|f(t,x) - f(t,x^0(t)) - P(t)(x - x^0(t))\bigr| \le Q(t)|x - x^0(t)|$$
$$\text{for } \ t \in [a,b], \quad \|x - x^0(t)\| \le r \tag{1.24}$$

and

$$\bigl|h(x) - l(x - x^0)\bigr| \le l^*\bigl(|x - x^0|\bigr) \quad \text{for } \ x \in U(x^0; r), \tag{1.25}$$

where $l : \mathrm{BV}_s([a,b],R^n) \to R^n$ *and* $l^* : \mathrm{BV}_s([a,b];R^n_+) \to R^n_+$ *are respectively a linear continuous and a positive homogeneous nondecreasing* *continuous operator,* $P \in L([a,b], R^{n\times n}; A)$*,* $Q \in L([a,b], R_+^{n\times n}; A)$*. Let, moreover,* (1.20) *be fulfilled on* $[a,b]$ *and let the problem*

$$\begin{gathered} \big|dx(t) - dA(t)\cdot P(t)x(t)\big| \le dV(A)(t)\cdot Q(t)|x(t)|, \\ |l(x)| \le l^*(|x|) \end{gathered} \tag{1.26}$$

have only the trivial solution. Then problem (1.1), (1.2) *is* $(x^0;r)$*-correct.*

Corollary 1.9. *Let every component* f_j $(j=1,\dots,n)$ *of the vector-function* f *have partial derivatives in the last* n *variables belonging to* $K([a,b] \times R^n, R; a_{ij})$ $(i=1,\dots,n)$ *and let there exist a solution* x^0 *of problem* (1.1), (1.2) *such that the operator* h *has the Frechet derivative* l *in* x^0*. Let, moreover,*

$$\det\big(I + (-1)^j d_j A(t)\cdot F(t, x^0(t))\big) \neq 0 \quad \textit{for} \quad t\in[a,b] \quad (j=1,2)$$

and the problem

$$dx(t) = dA(t)\cdot F(t,x^0(t))x(t), \quad l(x)=0, \tag{1.27}$$

where $F(t,x) = \frac{\partial f(t,x)}{\partial x}$ *have only the trivial solution. Then problem* (1.1), (1.2) *is* $(x^0;r)$*-correct for any sufficiently small* r*.*

2. Auxiliary Propositions

For every positive number ξ and a nondecreasing vector-function $g: [a,b]\to R^n$ we put

$$D_j(a,b,\xi;g) = \big\{t\in[a,b] : \ \|d_j g(t)\| \ge \xi\big\} \quad (j=1,2).$$

Let $R(a,b,\xi;g)$ be a set of all subdivisions $\{\alpha_0,\tau_1,\alpha_1,\dots,\tau_m,\alpha_m\}$ of the segment $[a,b]$ such that

(a) $a=\alpha_0<\alpha_1<\dots<\alpha_m=b$, $\alpha_0\le\tau_1\le\alpha_1\le\dots\le\tau_m\le\alpha_m$;

(b) if $\tau_i\notin D_1(a,b,\xi;g)$, then $\|g(\tau_i)-g(\alpha_{i-1})\|<\xi$, and if $\tau_i\in D_1(a,b,\xi;g)$, then $\alpha_{i-1}<\tau_i$ and $\|g(\tau_i-)-g(\alpha_{i-1})\|<\xi$ $(i=1,\dots,n)$;

(c) if $\tau_i\notin D_2(a,b,\xi;g)$, then $\|g(\alpha_i)-g(\tau_i)\|<\xi$, and if $\tau_i\in D_2(a,b,\xi;g)$, then $\tau_i<\alpha_i$ and $\|g(\alpha_i)-g(\tau_i+)\|<\xi$ $(i=1,\dots,n)$.

Lemma 2.1. *The set* $R(a,b,\xi;g)$ *is not empty.*

We omit the proof of this lemma for it is analogous to that of Lemma 1.1.1 from [5].

Lemma 2.2. *Let* $D\subset R^n$ *and* $A_k\in \mathrm{BV}([a,b],R^{n\times n})$ $(k=1,2,\dots)$*,* $f_k\in K([a,b]\times D, R^n; A_k)$ $(k=1,2,\dots)$*,* $\omega_k\in M([a,b]\times R_+, R_+^n; A_k)$ $(k=0,1,\dots)$*,* $A_0(t)\equiv A(t)$*, and* $y_k\in \mathrm{BV}([a,b],D)$ $(k=1,2,\dots)$ *be sequences such that* (1.7) *is fulfilled uniformly on* $[a,b]\times D$ *and* (1.11) *is* *fulfilled on* $[a,b]\times D$ *for* $k\in\{0,1,\dots\}$, *where* $f_0(t,x)\equiv f(t,x)$. *Let, moreover, conditions* (1.10) *and*

$$\|y_k(t)-y_k(s)\|\le l_k+\|g(t)-g(s)\| \quad \text{for} \quad a\le s<t\le b \quad (k=1,2,\dots) \tag{2.1}$$

hold, where $l_k\ge 0$, $l_k\to 0$ *as* $k\to+\infty$ *and* $g:[a,b]\to R^n$ *is a nondecreasing vector-function. Then*

$$\lim_{k\to+\infty}\left[\int_a^t dA_k(\tau)\cdot f_k(\tau,y_k(\tau))-\int_a^t dA(\tau)\cdot f(\tau,y_k(\tau))\right]=0 \quad \text{uniformly on} \quad [a,b]. \tag{2.2}$$

Proof. Let for every natural k and $t\in[a,b]$

$$x_k(t)=\int_a^t dA_k(\tau)\cdot f_k(\tau,y_k(\tau))-\int_a^t dA(\tau)\cdot f(\tau,y_k(\tau)).$$

We are to show that

$$\lim_{k\to+\infty}\|x_k\|_s=0. \tag{2.3}$$

Let ε be an arbitrary positive number. In view of (1.10) there exists a positive number δ such that

$$\left\|\int_a^b dV(A_k)(\tau)\cdot\omega_k(\tau,2\delta)\right\|<\varepsilon \quad (k=1,2,\dots). \tag{2.4}$$

According to Lemma 2.1 the set $R(a,b,\delta;g)$ is not empty.

Let $\{\alpha_0,\tau_1,\alpha_1,\dots,\tau_m,\alpha_m\}\in R(a,b,\delta;g)$ be fixed. For every natural k we assume that

$$\widetilde{y}_k(t)=\begin{cases} y_k(t) & \text{for} \quad t\in\{\alpha_0,\tau_1,\alpha_1,\dots,\tau_m,\alpha_m\};\\ y_k(\tau_i-) & \text{for} \quad t\in]\alpha_{i-1},\tau_i[\,, \quad \tau_i\in D_1(a,b,\delta;g);\\ y_k(\tau_i) & \text{for} \quad t\in]\alpha_{i-1},\tau_i[\,, \quad \tau_i\notin D_1(a,b,\delta;g);\\ & \text{or} \quad t\in]\tau_i,\alpha_i[\,, \quad \tau_i\notin D_2(a,b,\delta;g);\\ y_k(\tau_i+) & \text{for} \quad t\in]\tau_i,\alpha_i[\,, \quad \tau_i\in D_2(a,b,\delta;g);\\ & \qquad (i=1,\dots,m).\end{cases}$$

It is not difficult to see that

$$\left\|y_k(t)-\widetilde{y}_k(t)\right\|<l_k+\delta \quad \text{for} \quad t\in[a,b] \quad (k=1,2,\dots).$$

Hence there exists a natural k_0 such that

$$\left\|y_k(t)-\widetilde{y}_k(t)\right\|<2\delta \quad \text{for} \quad t\in[a,b] \quad (k\ge k_0).$$

From this and (1.11) we have

$$\|x_k\|_s \le \left\| \int_a^b dV(A_k)(\tau)\cdot \omega_k(\tau,2\delta)\right\| + \left\| \int_a^b dV(A)(\tau)\cdot \omega_0(\tau,2\delta)\right\| +$$
$$+ \mu_{0k} + \sum_{i=1}^m (\lambda_{ik} + 2\mu_{ik} + 2\nu_{ik}) \quad \text{for} \quad k \ge k_0, \tag{2.5}$$

where

$$\lambda_{ik} = \sup\Big\{ \Big\| \int_{]s,t[} dA_k(\tau)\cdot f_k(\tau,x) - \int_{]s,t[} dA(\tau)\cdot f(\tau,x)\Big\| : \ s,t\in[a,b],$$
$$x \in \big\{y_k(\tau_i-),\, y_k(\tau_i),\, y_k(\tau_i+)\big\}\Big\} \quad (i=1,\dots,m),$$
$$\mu_{ik} = \max\Big\{ \big\| d_jA_k(\alpha_i)\cdot f_k(\alpha_i, y_k(\alpha_i)) -$$
$$- d_jA(\alpha_i)\cdot f(\alpha_i, y_k(\alpha_i))\big\| : \ j\in\{1,2\}\Big\} \quad (i=0,\dots,m),$$
$$\nu_{ik} = \max\Big\{ \big\| d_jA_k(\tau_i)\cdot f_k(\tau_i, y_k(\tau_i)) -$$
$$- d_jA(\tau_i)\cdot f(\tau_i, y_k(\tau_i))\big\| : \ j\in\{1,2\}\Big\} \quad (i=1,\dots,m).$$

It follows from the conditions of the lemma and (1.7) that

$$\lim_{k\to+\infty} d_jA_k(t)\cdot f_k(t,x) = d_jA(t)\cdot f(t,x) \quad (j=1,2)$$

and

$$\lim_{k\to+\infty} \int_{]s,t[} dA_k(\tau)\cdot f_k(\tau,x) = \int_{]s,t[} dA(\tau)\cdot f(\tau,x)$$

uniformly on $[a,b]\times D$. Therefore

$$\lim_{k\to+\infty} \mu_{0k} = 0 \quad \text{and}$$
$$\lim_{k\to+\infty} (\lambda_{ik} + 2\mu_{ik} + 2\nu_{ik}) = 0 \quad (i=1,\dots,m). \tag{2.6}$$

On the other hand, we may assume without loss of generality that

$$\left\| \int_a^b dV(A)(\tau)\cdot \omega_0(\tau,2\delta)\right\| < \varepsilon.$$

Taking into account this fact, (2.4) and (2.6), it follows from (2.5) that (2.3) is valid. □

Remark 2.1. If the set D is bounded and continuous, (1.10) and (1.11) hold, then condition (1.7) is fulfilled uniformly on $[a,b]\times D$ if and only if for every $x\in D$ it is fulfilled uniformly on $[a,b]$.

Lemma 2.3. *Let $y, y_k \in \mathrm{BV}([a,b], R^n)$ $(k = 1, 2, \dots)$ be vector-functions such that*

$$\lim_{k\to+\infty} y_k(t) = y(t) \quad \textit{for} \quad t \in [a,b].$$

Let condition (2.1) *hold, where $l_k \geq 0, l_k \to 0$ as $k \to +\infty$, and let $g : [a,b] \to R^n$ be a nondecreasing vector-function. Then*

$$\lim_{k\to+\infty} \|y_k - y\|_s = 0.$$

Proof. Let ε be an arbitrary positive number, $\{\alpha_0, \tau_1, \alpha_1, \dots, \tau_m, \alpha_m\} \in R(a, b, \frac{\xi}{5}; g)$, and let n_ε be a natural number such that

$$l_i < \frac{\varepsilon}{5} \quad \text{and} \quad \|y_i(\tau) - y_k(\tau)\| < \frac{\varepsilon}{5}$$
$$\text{for} \quad \tau \in \{\alpha_0, \tau_1, \alpha_1, \dots, \tau_m, \alpha_m\} \quad (i, k \geq n_\varepsilon).$$

Assume that $\alpha_{j-1} < t < \tau_j$ $(j = 1, \dots, m)$. Then in view of (2.1) we have

$$\begin{gathered}
\|y_i(t) - y_k(t)\| \leq \\
\leq \|y_i(t) - y_i(\tau_j)\| + \|y_i(\tau_j) - y_k(\tau_j)\| + \|y_k(\tau_j) - y_k(t)\| \leq \\
\leq l_i + l_k + 2\|g(t) - g(\tau_j)\| + \|y_i(\tau_j) - y_k(\tau_j)\| < \\
< \frac{3\varepsilon}{5} + 2\|g(\tau_j) - g(\alpha_{j-1})\| < \varepsilon \\
\text{for} \quad \tau_j \notin D_1\Big(a, b, \frac{\varepsilon}{5}; g\Big) \quad (i, k \geq n_\varepsilon) \quad \text{and} \\
\|y_i(t) - y_k(t)\| \leq \\
\leq \|y_i(t) - y_i(\alpha_{j-1})\| + \|y_i(\alpha_{j-1}) - y_k(\alpha_{j-1})\| + \|y_k(\alpha_{j-1}) - y_k(t)\| \leq \\
\leq l_i + l_k + 2\|g(t) - g(\alpha_{j-1})\| + \|y_i(\alpha_{j-1}) - y_k(\alpha_{j-1})\| < \\
< \frac{3\varepsilon}{5} + 2\|g(\tau_j-) - g(\alpha_{j-1})\| < \varepsilon \\
\text{for} \quad \tau_j \in D_1\Big(a, b, \frac{\varepsilon}{5}; g\Big) \quad (i, k \geq n_\varepsilon).
\end{gathered}$$

The case $\tau_j < t < \alpha_j$ $(j = 1, \dots, m)$ is considered analogously. □

Lemma 2.4. *Let condition* (1.13) *hold and inequality* (1.12) *be fulfilled on $[a,b] \times R^n$, where $l : \mathrm{BV}_s([a,b], R^n) \to R^n$ and $l_0 : \mathrm{BV}_s([a,b], R^n) \to R^n_+$ are respectively a linear continuous and a positive homogeneous continuous operator, let the matrix-function P satisfy the Opial condition with respect to the triplet $(l, l_0; A)$, $\alpha \in K([a,b] \times R_+, R^n_+; A)$ be nondecreasing in the* *second variable, $l_1 \in C(R_+, R_+^n)$, and let (1.14) hold. Moreover, let $g \in K([a,b] \times R^n, R^n; G)$ be a vector-function such that*

$$\lim_{\rho \to +\infty} \frac{1}{\rho} \int_a^b dV(G)(t) \cdot \gamma(t,\rho) = 0,$$

where $\gamma(t,\rho) = \max\{|g(t,x)| : \|x\| \le \rho\}$. Then the boundary value problem

$$dx(t) = dA(t) \cdot f(t, x(t)) + dG(t) \cdot g(t, x(t)),{}^{1}, \quad h(x) = 0 \tag{2.7}$$

is solvable.

Proof. Let problem (2.7) have a solution x. Put

$$z(t) = \int_a^t dG(\tau) \cdot g(\tau, x(\tau)) \quad \text{and} \quad y(t) = x(t) - z(t) \quad \text{for} \quad t \in [a,b].$$

Then the vector-function $\widetilde{x} = \binom{y}{z}$ will be a solution of the problem

$$dx(t) = d\widetilde{A}(t) \cdot \widetilde{f}(t, x(t)), \quad \widetilde{h}(x) = 0, \tag{2.8}$$

where

$$\widetilde{A}(t) = \begin{pmatrix} A(t), & 0 \\ 0, & G(t) \end{pmatrix},$$
$$\widetilde{f}(t,x) = \begin{pmatrix} f(t, x_1 + x_{n+1}, \dots, x_n + x_{2n}) \\ g(t, x_1 + x_{n+1}, \dots, x_n + x_{2n}) \end{pmatrix} \quad \text{on} \quad [a,b] \times R^{2n},$$

and

$$\widetilde{h}(x) = \begin{pmatrix} h(x_1 + x_{n+1}, \dots, x_n + x_{2n}) \\ (x_{n+i}(a))_{i=1}^n \end{pmatrix} \quad \text{for} \quad x = (x_i)_{i=1}^{2n} \in \mathrm{BV}([a,b], R^{2n}).$$

Conversely, if $\widetilde{x} = (x_i)_{i=1}^{2n}$ is a solution of problem (2.8), then $x = (x_i + x_{n+i})_{i=1}^n$ will be a solution of problem (2.7). It is not difficult to show that the conditions of the existence theorem (see [9, Theorem 1]) are fulfilled for problem (2.8). Hence it is solvable as problem (2.7). $\square$

[1] A vector-function $x \in \mathrm{BV}([a,b], R^n)$ is said to be a solution of this system if it satisfies the corresponding integral equality.

3. Proof of the Main Results

Proof of Theorem 1.1. Let us assume that the theorem does not hold. Then there exist $\varepsilon \in]0, r[$ and $((A_k, f_k, h_k))_{k=1}^{+\infty} \in W_r(A, f, h; x^0)$ such that for every natural number k problem (1.1_k), (1.2_k) has either no solution belonging to the ball $U(x^0; r)$ or at least one solution belonging to $U(x^0; r) \backslash U(x^0; \varepsilon)$.

Put

$$\chi(t, x) = \begin{cases} x & \text{for } \ \|x - x^0(t)\| \le r, \\ x^0(t) + \frac{r}{\|x - x^0(t)\|}(x - x^0(t)) & \text{for } \ \|x - x^0(t)\| > r; \end{cases} \tag{3.1}$$

$$\widetilde{\chi}(x)(t) = \chi(t, x(t)),$$

$$\widetilde{f}_k(t, x) = f_k\big(t, \chi(t, x)\big) \quad (k = 1, 2, \dots)$$

and

$$\widetilde{h}_k(x) = h_k(\widetilde{\chi}(x)) \quad (k = 1, 2 \dots).$$

By virtue of Lemma 2.4 the problem

$$dx(t) = dA(t) \cdot \big[P(t, x(t))x(t) + q(t, x(t))\big] + d\eta_k(\widetilde{\chi}(x))(t),$$

$$l(x) + \widetilde{l}(x) + \gamma_k(\widetilde{\chi}(x)) = 0,$$

where

$$\gamma_k(y) = h_k(y) - h(y) \quad \text{for} \quad y \in \mathrm{BV}([a, b], R^n)$$

and

$$\eta_k(y)(t) = \int_a^t dA_k(\tau) \cdot f_k(\tau, y(\tau)) - \int_a^t dA(\tau) \cdot f(\tau, y(\tau))$$
$$\text{for} \quad y \in \mathrm{BV}([a, b], R^n),$$

is solvable for every natural k. From the above it is evident that it has a solution x_k satisfying the inequality

$$\|x_k - x^0\|_s \ge \varepsilon. \tag{3.2}$$

It is clear that

$$x_k(t) = x_k(a) + z_k(t) + \eta_k(x^0)(t) \quad \text{for} \quad t \in [a, b] \quad (k = 1, 2, \dots), \tag{3.3}$$

where

$$z_k(t) = \int_a^t dA(\tau) \cdot \big[P(\tau, x_k(\tau))x_k(\tau) + q(\tau, x_k(\tau))\big] +$$
$$+ \int_a^t dA_k(\tau) \cdot \big[f_k(\tau, y_k(\tau)) - f_k(\tau, x^0(\tau))\big] +$$
$$+ \int_a^t dA(\tau) \cdot \big[f(\tau, x^0(\tau)) - f(\tau, y_k(\tau))\big]$$

and

$$y_k(t) = \chi(t, x_k(t)).$$

In view of (1.11), (3.1) and Lemma 2.2

$$\big|\eta_k(y_k)(t) - \eta_k(y_k)(s)\big| \leq \left|\int_s^t dV(A_k)(\tau)\cdot\omega_k(\tau,r)\right| +$$
$$+\left|\int_s^t dV(A)(\tau)\cdot\omega_0(\tau,r)\right| + \delta_k \quad \text{for} \quad a\leq s\leq t\leq b \quad (k=1,2,\dots) \tag{3.4}$$

and

$$\lim_{k\to+\infty}\delta_k = 0, \tag{3.5}$$

where

$$\omega_0(\tau,\sigma) =$$
$$= \max\Big\{|f(\tau,x)-f(\tau,y)| : \|x\|\leq\|x^0\|_s + r, \|y\|\leq\|x^0\|_s + r, \|x-y\|\leq\sigma\Big\}$$

and

$$\delta_k = \sup\Big\{|\eta_k(x^0)(t) - \eta_k(x^0)(s)| : \ a\leq s\leq t\leq b\Big\}.$$

On the other hand, according to Lemma 1 from [9], there exists a positive number ρ_0 such that

$$\|x_k\|_s < \rho_0\left[\big\|\beta(\|x_k\|_s)\big\| + \left\|\int_a^b dV(A)(t)\cdot\alpha\big(t,\|x_k\|_s\big)\right\| + \zeta_k\right]$$
$$(k=1,2,\dots),$$

where

$$\zeta_k = \sup\Big\{\|\eta_k(y_k)(t)\| : \ t\in[a,b]\Big\} + \|\gamma_k(y_k)\|.$$

From this in view of (1.9), (3.4), (3.5) and the condition (b) of Definition 1.2 we have

$$\rho_1 = \sup\Big\{\|x_k\|_s : \ k=1,2,\dots\Big\} < +\infty \tag{3.6}$$

and

$$|z_k(t) - z_k(s)| \leq g(t) - g(s) \quad \text{for} \ \ a\leq s\leq t\leq b \ \ (k=1,2,\dots), \tag{3.7}$$

where

$$g(t) = \int_a^t dV(A)(\tau)\cdot\psi(\tau) + \sup\left\{\int_a^t dV(A_k)(\tau)\cdot\omega_k(\tau,r) : \ k=1,2,\dots\right\},$$
$$\psi(\tau) = \rho_1\max\Big\{|P(\tau,x)| : \ \|x\|\leq\rho_1\Big\} + \alpha_1(\tau,\rho_1) + \omega_0(\tau,r).$$

Hence the sequence $(z_k)_{k=1}^{+\infty}$ satisfies the conditions of Helly's choice theorem and condition (2.1), where $l_k = 0$ $(k = 1, 2, \dots)$. Therefore with regard to Lemma 2.3 and (3.6) we may assume without loss of generality that

$$\lim_{k\to+\infty} \|z_k - z^*\|_s = 0 \quad \text{and} \quad \lim_{k\to+\infty} x_k(a) = c^*,$$

where $z^* \in \mathrm{BV}([a,b], R^n)$, $c^* \in R^n$. By this, (3.5) and (3.7) we have from (3.3) that

$$|x_k(t) - x_k(s)| \le \delta_k + g(t) - g(s) \qquad \text{for} \quad a \le s \le t \le b \quad (k = 1, 2, \dots) \tag{3.8}$$

and

$$\lim_{k\to+\infty} \|x_k - x^*\|_s = 0, \tag{3.9}$$

where $x^*(t) \equiv c^* + z^*(t)$. Using (3.8), it is not difficult to show that

$$\|y_k(t) - y_k(s)\| \le \rho_2\big(\|\delta_k\| + \|V(x^0)(t) + g(t) - V(x^0)(s) - g(s)\|\big)$$

for $a \le s \le t \le b$ $(k = 1, 2, \dots)$, where $\rho_2 = 2 + r^{-1}(\rho_1 + \|x^0\|_s)$.

Consequently, by virtue of Lemma 2.2 condition (2.2) is fulfilled. On the other hand, in view of (1.8)

$$\lim_{k\to+\infty} \gamma_k(y_k) = 0. \tag{3.10}$$

By (2.2), (3.9) and (3.10), passing to the limit as $k \to +\infty$ in equalities (3.3) and

$$l(x_k) + \widetilde{l}(x_k) + \gamma_k(y_k) = 0,$$

we find that x^* is the solution of the problem (1.5), (1.6).

Further, from (3.2) we have

$$\|x^* - x^0\| \ge \varepsilon.$$

But this is impossible, since according to the condition (c) of Definition 1.2 the latter problem has no solution differing from x^0. □

Proof of Corollary 1.1. According to Theorem 1 from [9] problem (1.1), (1.2) has the unique solution x^0 which, in view of conditions (1.12)–(1.14), is strongly isolated in every radius $r > 0$. Hence, the corollary follows from Theorem 1.1. □

Proof of Corollary 1.2. According to Theorem 2 from [9], problem (1.1), (1.2) has the unique solution x^0. Let r be an arbitrary positive number. It suffices to show that x^0 is strongly isolated in the radius r.

Let S be a set of all matrix-functions $B \in \mathrm{BV}([a,b], R^{n\times n})$, $B(a)=0$, satisfying (1.16) for $a \le s \le t \le b$. Then by Lemma 1 from [9] there exists a positive number ρ_0 such that

$$\|x\|_s \le \rho_0 \Big[\big\| \big[|l(x)| - l_0(x) \big]_+ \big\| + \\ + \sup \Big\{ \Big\| x(t) - x(a) - \int_a^t dB(\tau)\cdot x(\tau) \Big\| : \ t \in [a,b] \Big\} \Big] \\ \text{for} \quad x \in \mathrm{BV}([a,b],R^n), \quad B \in S. \tag{3.11}$$

Moreover, in view of (1.14) there exists a number $\rho_1 > \|x^0\|_s + r$ such that

$$\rho_0 \left[\|l_1(\rho)\| + \left\| \int_a^b dV(A)(t)\cdot \alpha(t,\rho) \right\| \right] < \rho \quad \text{for} \quad \rho \ge \rho_1. \tag{3.12}$$

It is evident that

$$f(t,x) = \widetilde{P}(t)x + \widetilde{q}(t,x) \quad \text{for} \quad t\in[a,b], \quad \|x - x^0(t)\| < r,$$

where

$$\widetilde{P}(t) = \frac{1}{2}\big[P_1(t) + P_2(t)\big], \\ \widetilde{q}(t,x) = \big[f(t,x) - \widetilde{P}(t)x\big]\chi(\|x\|) \quad \text{on} \quad [a,b]\times R^n$$

and

$$\chi(s) = \begin{cases} 1 & \text{for} \quad 0 \le s < \rho_1, \\ 2 - \frac{\tau}{\rho_1} & \text{for} \quad \rho_1 \le s < 2\rho_1, \\ 0 & \text{for} \quad s \ge 2\rho_1. \end{cases} \tag{3.13}$$

It follows from (1.13) and (1.14) that the conditions (a), (b), and (d) of Definition 1.2 are fulfilled for $P = \widetilde{P}$, $q = \widetilde{q}$, l, l_0, $\widetilde{l} = h - l$ and A.

Let us show that the problem

$$dx(t) = dA(t)\cdot \big[\widetilde{P}(t)x + \widetilde{q}(t,x)\big] \tag{3.14}$$

has no solution differing from x^0 and satisfying (1.6). Let y^0 be an arbitrary solution of problem (3.14), (1.6). Then y^0 will be the solution of the system

$$dx(t) = dB(t)\cdot x(t) + dA(t)\cdot \big[f(t,y^0(t)) - P(t,y^0(t))y^0(t)\big]\chi\big(\|y^0(t)\|\big),$$

where

$$B(t) = \int_a^b dA(\tau)\cdot \big[\widetilde{P}(\tau) + \big(P(\tau,y^0(\tau)) - \widetilde{P}(\tau)\big)\chi\big(\|y^0(\tau)\|\big)\big].$$

On the other hand, it follows from (1.15) and (3.13) that $B \in S$. Hence, using (1.12), (1.13), (3.11) and (3.12), we obtain

$$\|y^0\|_s < \rho_1.$$

But by (3.13) it is clear that every solution of system (3.14), admitting such an estimate, is the solution of problem (1.1), (1.2). Therefore $x^0(t) \equiv y^0(t)$, and the condition (c) of Definition 1.2 is fulfilled, i.e., x^0 is strongly isolated in every radius r. $\square$

Proof of Corollary 1.3. Let

$$f(t,x) = \big(f_i(t,x)\big)_{i=1}^n, \quad q(t,\rho) = \big(q_i(t,\rho)\big)_{i=1}^n,$$
$$P_0(t) = \big(p_{0ij}(t)\big)_{i,j=1}^n, \quad Q(t) = \big(q_{ij}(t)\big)_{i,j=1}^n,$$
$$\eta_i(t,x) = \Big[\sum_{j=1}^n q_{ij}(t)|x_j| + q_i(t,\|x\|) + 1\Big]^{-1} \times$$
$$\times\Big[f_i(t,x) - \sum_{j=1}^n p_{0ij}(t)x_j\Big] \quad (i = 1,\dots,n)$$

and

$$p_{ij}(t,x) = p_{0ij}(t) + q_{ij}(t)\eta_i(t,x)\operatorname{sign} x_j \quad (i,j = 1,\dots,n)$$

for $t \in [a,b]$, $x = (x_i)_{i=1}^n \in R^n$, $\rho \in R_+$. It is not difficult to verify that the matrix-functions

$$P(t,x) = \big(p_{ij}(t,x)\big)_{i,j=1}^n, \quad P_1(t) = P_0(t) - Q(t), \quad P_2(t) = P_0(t) + Q(t)$$

and the vector-function

$$\alpha(t,\rho) = \big(q_i(t,\rho) + 1\big)_{i=1}^n$$

satisfy the condition of Corollary 1.2. Hence Corollary 1.3 follows from Corollary 1.2. $\square$

Corollary 1.4 is a special case of Corollary 1.3, when $Q(t) \equiv 0$ and $l_0(x) \equiv 0$.

Corollary 1.5 follows from Corollary 1.3 and Theorem 4 from [9].

Proof of Corollary 1.6. By (1.22) there exist sequences of positive numbers $(\rho_k)_{k=0}^{+\infty}$ and $(\mu_k)_{k=1}^{+\infty}$ such that $\rho_k \to +\infty$ as $k \to +\infty$,

$$\sup\Big\{\|g(x)\| : \ x \in \mathrm{BV}([a,b],R^n)\Big\} < \rho_k \quad (k = 1,2,\dots) \tag{3.15}$$

and

$$\big\|x + (-1)^j d_j A(t)\cdot f(t,y)\big\| > \mu_k > \sigma_{3-j}(t,z) + \rho_0$$
$$\text{for} \quad \|x\| \ge \rho_k, \quad \|y\| = \rho_k, \quad (-1)^j(t-t_0) < 0,$$

$$z \in X_j(t, c, \rho_k), \quad c \in G \quad (j = 1, 2;\ k = 1, 2, \dots). \tag{3.16}$$

Let x^0 be the unique solution of problem (1.1), (1.2). According to Theorem 1.1 it is sufficient to establish that x^0 is strongly isolated in the radius $r_k = \rho_k - \|x^0\|_s$ for every natural k.

Put

$$P(t, x) = 0, \quad q(t, x) = f(t, \chi_k(x)) \quad \text{on} \quad [a, b] \times R^n$$

and

$$l(x) = x(t_0), \quad l_0(x) = 0, \quad \widetilde{l}(x) = -g(x) \quad \text{for} \quad x \in \mathrm{BV}([a, b], R^n),$$

where

$$\chi_k(x) = \begin{cases} x & \text{for} \quad \|x\| \le \rho_k, \\ \frac{r}{\|x\|} x & \text{for} \quad \|x\| > \rho_k. \end{cases}$$

It is obvious that the conditions (a), (b), and (d) of Definition 1.2 are fulfilled. Let now y be an arbitrary solution of problem (1.5), (1.6), and let $c = g(y)$. By (3.15) and the fact that the matrix-function A is continuous at the point t_0, we have

$$t^* > t_0,$$

where

$$t^* = \sup\Big\{t : \ \|y(s)\| < \rho_k \ \text{ for } \ t_0 \le s \le t\Big\}.$$

Obviously, $y \in X_1(t^*, c, \rho_k)$, there exists $y(t^*-)$, and

$$\|y(t^*-)\| \le \rho_k.$$

From the above argument, due to (3.16) and the equality

$$y(t^*-) = y(t^*) - d_1 A(t^*) \cdot f\big(t^*, \chi_k(y(t^*))\big)$$

we obtain

$$\|y(t^*)\| < \rho_k.$$

Assume $t^* < b$. If there exists a sequence $\tau_m > t^*$ $(m = 1, 2, \dots)$ such that $\tau_m \to t^*$ as $m \to +\infty$ and

$$\|y(\tau_m)\| \ge \rho_k \quad (m = 1, 2, \dots),$$

then by the first inequality of (3.16)

$$\|y(\tau_m-)\| = \big\|y(\tau_m) - d_1 A(\tau_m) \cdot f(\tau_m, \chi_k(y(\tau_m)))\| > \mu_k \quad (m = 1, 2, \dots).$$

Hence,

$$\|y(t^*+)\| = \lim_{m \to +\infty} \|y(\tau_m-)\| \ge \mu_k.$$

On the other hand, by the second inequality of (3.16) and the definition of $\sigma_2(t, z)$ we have

$$\mu_k > \|y(t^*+)\|.$$

The obtained contradiction shows that there exists a positive number δ such that

$$\|y(t)\| < \rho_k \quad \text{for} \quad t \in [t^*, t^* + \delta].$$

But this contradicts the definition of the point t^*. Therefore $t^* = b$, and y is the solution of system (1.1) on $[a, b]$.

Analogously, we can show that y is the solution of system (1.1) on $[a, t_0]$. It is now clear that y is the solution of problem (1.1), (1.2), and $y(t) \equiv x^0(t)$. Thus the condition (c) of Definition 1.2 is likewise fulfilled. $\square$

Corollary 1.7 follows from Corollary 1.6 for $g(x) = \text{const}$.

Proof of Corollary 1.8. Put

$$P(t, x) = P(t), \quad q(t, x) = f(t, \chi(t, x)) - P(t)\chi(t, x),$$
$$\widetilde{l}(x) = h(\widetilde{\chi}(x)) - l(\widetilde{\chi}(x)), \quad l_0(x) = 0,$$

where χ is the function defined by (3.1), and $\widetilde{\chi}(x)(t) = \chi(t, x(t))$. Then the conditions (a), (b), and (d) of Definition 1.2 are fulfilled. According to Theorem 1.1 it remains to show that the condition (c) is fulfilled. Let $\widetilde{x}$ be an arbitrary solution of problem (1.5), (1.6). Assume

$$x(t) = \widetilde{x}(t) - x^0(t).$$

As $|\chi(t, \widetilde{x}(t)) - x^0(t)| \leq |x(t)|$ for $t \in [a, b]$ and the operator l^* is nondecreasing, it follows from (1.24) and (1.25) that x is the solution of the problem (1.26). But the latter problem has only the trivial solution. Thus problem (1.5), (1.6) has no solution differing from x^0. $\square$

Proof of Corollary 1.9. Let

$$P(t) = F(t, x^0(t)).$$

By Theorem 1 from [7] the unique solvability of problem (1.27) guarantees the existence of a positive number r such that problem (1.26) has no nontrivial solution if

$$l^*(|x|) = \alpha \|x\|_s, \tag{3.17}$$
$$\alpha \in R_+^n, \quad Q \in L([a, b], R_+^{n \times n}; A), \quad \|\alpha\| \leq \delta,$$
$$\left\| \int_a^b dV(A)(t) \cdot Q(t) \right\| \leq \delta. \tag{3.18}$$

Choosing a number $r > 0$ such that

$$\|h(x) - l(x - x^0)\| \leq \frac{\delta}{n} \|x - x^0\|_s \quad \text{for} \quad x \in U(x^0; r) \tag{3.19}$$

inequality (3.18) will be fulfilled, where

$$Q(t) = \max\Big\{|F(t,x) - P(t)| : \ \|x - x^0(t)\| \le r\Big\}.$$

From the representation

$$f(t,x) - f(t,x^0(t)) = \int_0^1 F(t, sx + (1-s)x^0(t))ds \cdot (x - x^0(t))$$

and condition (3.19) follow inequalities (1.24) and (1.25), where l^* is the operator defined by (3.17), and α is the n-dimensional vector all of whose components are equal to $\frac{\delta}{n}$. On the other hand, by our choice of Q and l^* problem (1.26) has only the trivial solution. Thus, all conditions of Corollary 1.8 that guarantee the $(x^0;r)$-correctness of the problem (1.1), (1.2) are fulfilled.

4. Application

Let E_k^n and $\widetilde{E}_k^n$ be the spaces of all vector-function $x : N_k \to R^n$ and $x : \widetilde{N}_k \to R^n$ with the norms $\|x\|_k = \max\{\|x(i)\| : i \in N_k\}$ and $\|x\|_{\widetilde{k}} = \max\{\|x(i)\| : i \in \widetilde{N}_k\}$, respectively, where $N_k = \{1,\dots,k\}$, $\widetilde{N}_k = \{0,\dots,k\}$ $(k = 1,2,\dots)$; let Δ be the first order difference operator, i.e., $\Delta x(i-1) = x(i) - x(i-1)$ for $x \in \widetilde{E}_k^n$ and $i \in N_k$ $(k = 1,2,\dots)$.

By $M(N_n \times R_+, R_+^n)$ we denote a set of all vector-functions $\omega : N_n \times R_+ \to R_+^n$ such that $\omega(i,\cdot)$ is continuous and nondecreasing, and $\omega(i,0) = 0$ for $i \in N_n$.

For the system of ordinary differential equations

$$\frac{dx(t)}{dt} = f(t, x(t)) \tag{4.1}$$

consider the boundary value problem

$$h(x) = 0, \tag{4.2}$$

where $f \in K([a,b] \times R^n; R^n)$, and $h : \mathrm{BV}_s([a,b], R^n) \to R^n$ is a continuous operator.

Along with problem (4.1), (4.2) let us consider its difference analogue

$$\Delta y(i-1) = \frac{1}{k} f_k(i, y(i)) \quad (i = 1,\dots,k), \tag{4.1$_k$}$$

$$h_k(y) = 0 \tag{4.2$_k$}$$

$(k = 2,3,\dots)$, where $f_k(i,\cdot) : R^n \to R^n$ is a continuous function for every $i \in N_k$, and $h_k : \widetilde{E}_k^n \to R^n$ is a continuous operator.

Let $\nu_k : [a,b] \to R$ $(k = 1,2,\dots)$ be the functions defined by equations $\nu_k(t) = i$ for $t \in I_{ik} \cap [a,b]$ $(i = 0,\dots,k)$, where $I_{ik} = [a + \frac{2i-1}{2k}(b-a), a + \frac{2i+1}{2k}(b-a)[$ $(i = 0,\dots,k)$.

Theorem 4.1. *Let x^0 be a solution of problem* (4.1), (4.2), *strongly isolated in the radius $r > 0$, and let*

$$\lim_{k\to+\infty} \left\| \frac{1}{k} \sum_{i=1}^{\nu_k(t)} f_k(i,x) - \int_0^t f(\tau,x)d\tau \right\| = 0$$

uniformly on $[a,b]$ for every $x \in D(x^0;r)$. Moreover, let there exist a sequence $\omega_{rk} \in M(N_n \times R_+, R_+^n)$ $(k = 1,2,\dots)$ such that

$$\sup\left\{ \left\| \frac{1}{k} \sum_{i=1}^{k} \omega_{rk}(i,r) \right\| :\ k = 1,2,\dots \right\} < +\infty,$$

$$\lim_{s\to 0+} \sup\left\{ \left\| \frac{1}{k} \sum_{i=1}^{k} \omega_{rk}(i,s) \right\| :\ k = 1,2,\dots \right\} = 0$$

and

$$|f_k(i,x) - f_k(i,y)| \le \omega_{rk}(i, \|x-y\|)$$

on $N_k \times D(x^0;r)$ $(k = 1,2,\dots)$. Then for every $\varepsilon > 0$ there exists a natural number k_0 such that a set Y_{rk} of solutions of problem (4.1_k), (4.2_k), *satisfying the inequality*

$$\max\left\{ \left\| y(i) - x^0\Big(a + \frac{i}{k}(b-a)\Big) \right\| :\ i \in \widetilde{N}_k \right\} < r$$

is not empty, and

$$\max\left\{ \left\| y(i) - x^0\Big(a + \frac{i}{k}(b-a)\Big) \right\| :\ i \in \widetilde{N}_k \right\} < \varepsilon \quad \text{for } y \in Y_{rk} \ \ (k > k_0).$$

This theorem follows from Theorem 1.1, since problem (4.1_k), (4.2_k) can be written in the form of problem (1.1_k), (1.2_k) for every $k \in \{2,3,\dots\}$.

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