GEORGIAN MATHEMATICAL JOURNAL: Vol. 3, No. 5, 1996, 485-500

ON SOME ENTIRE MODULAR FORMS OF WEIGHTS $\frac{7}{2}$ AND $\frac{9}{2}$ FOR THE CONGRUENCE GROUP $\Gamma_0(4N)$

G. LOMADZE

ABSTRACT. Entire modular forms of weights $\frac{7}{2}$ and $\frac{9}{2}$ for the congruence group $\Gamma_0(4N)$ are constructed, which will be useful for revealing the arithmetical sense of additional terms in formulas for the number of representations of positive integers by quadratic forms in 7 and 9 variables.

1.

In this paper N, a, k, q, r, t denote positive integers; u, s are odd positive integers; $H, c, g, h, j, m, n, \alpha, \beta, \gamma, \delta, \xi$ are integers; A, B, C, D, G are complex numbers; z, τ ($\operatorname{Im}\tau > 0$) are complex variables, and $\mathcal{H} = \{\tau \in \mathbb{C} | \operatorname{Im}\tau > 0\}$. Further, $(\frac{h}{u})$ is the generalized Jacobi symbol; $\binom{n}{t}$ is a binomial coefficient; $\varphi(k)$ is Euler's function; $e(z) = \exp 2\pi i z$; $\eta(\gamma) = 1$ if $\gamma \geq 0$ and $\eta(\gamma) = -1$ if $\gamma < 0$.

Let

$$\vartheta_{gh}(z|\tau; c, N) = \sum_{m \equiv c \pmod N} (-1)^{h(m-c)/N} \times \\ \times e\Big(\frac{1}{2N}\big(m + \frac{g}{2}\big)^2 \tau\Big) e\Big(\big(m + \frac{g}{2}\big) z\Big); \tag{1.1}$$

hence

$$\frac{\partial^n}{\partial z^n} \vartheta_{gh}(z|\tau; c, N) = (\pi i)^n \sum_{m \equiv c \pmod N} (-1)^{h(m-c)/N} (2m + g)^n \times \\ \times e\Big(\frac{1}{2N}\big(m + \frac{g}{2}\big)^2 \tau\Big) e\Big(\big(m + \frac{g}{2}\big) z\Big) \quad (n \geq 0). \tag{1.2}$$

1991 *Mathematics Subject Classification.* 11F37, 11F27, 11F11.

Key words and phrases. Entire modular form, congruence subgroup, multiplier system, theta-function, half integral weight.

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Put

$$\vartheta_{gh}^{(n)}(\tau;c,N)=\frac{\partial^n}{\partial z^n}\,\vartheta_{gh}(z|\tau;c,N)\big|_{z=0}\quad (n\geq 0). \tag{1.3}$$

It is known (see, for e.g., [1], formulas (11) and (12)) that

$$\vartheta_{g,h+2j}^{(n)}(\tau;c,N)=\vartheta_{gh}^{(n)}(\tau;c,N)\quad (n\geq 0), \tag{1.4}$$

$$\begin{aligned}\vartheta_{gh}^{(n)}(\tau+\beta;c,N)&=e\Big(\frac{\beta}{2N}\big(c+\frac{g}{2}\big)^2\Big)\vartheta_{g,h+\beta g+\beta N}^{(n)}(\tau;c,N)\ \ (n\geq 0),\\ \vartheta_{gh}^{(n)}(\tau-\beta;c,N)&=(-1)^n e\Big(-\frac{\beta}{2N}\big(c+\frac{g}{2}\big)^2\Big)\times\\ &\quad\times\vartheta_{-g,-h+\beta g-\beta N}^{(n)}(\tau;-c,N)\quad (n\geq 0).\end{aligned} \tag{1.5}$$

From (1.1) and (1.2), according to notation (1.3) it follows, in particular, that

$$\begin{aligned}\vartheta_{gh}^{(n)}(\tau;0,N)=(\pi i)^n\sum_{m=-\infty}^{\infty}&(-1)^{hm}(2Nm+g)^n\times\\ &\times e\Big(\frac{1}{2N}\big(Nm+\frac{g}{2}\big)^2\tau\Big)\ \ (n\geq 0).\end{aligned} \tag{1.6}$$

For $\xi_2\neq 0$, ξ_1,g,h,N with $\xi_1 g+\xi_2 h+\xi_1\xi_2 N\equiv 0 \pmod 2$, put

$$S_{gh}\Big(\begin{matrix}\xi_1\\ \xi_2\end{matrix};c,N\Big)=\sum_{\substack{m \bmod N|\xi_2|\\ m\equiv c \pmod N}}(-1)^{h(m-c)/N}e\Big(\frac{\xi_1}{2N\xi_2}\big(m+\frac{g}{2}\big)^2\Big).$$

Finally, let

$$\Gamma=\Big\{\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,\Big|\,\alpha\delta-\beta\gamma=1\Big\},$$

$$\Gamma_0(4N)=\Big\{\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\in\Gamma\,\Big|\,\gamma\equiv 0 \pmod{4N}\Big\}.$$

For $\tau\in\mathcal{H}$ put

$$(\gamma\tau+\delta)^{s/2}=\big((\gamma\tau+\delta)^{1/2}\big)^s,\quad -\frac{\pi}{2}<\arg(\gamma\tau+\delta)^{1/2}\leq\frac{\pi}{2}.$$

Definition. Let M be a matrix of an arbitrary substitution from $\Gamma_0(4N)$, and let $v(M)$ be a multiplier system on $\Gamma_0(4N)$ and of weight $\frac{s}{2}$. We shall say that a function F defined on $\mathcal{H}$ is an entire modular form of weight $\frac{s}{2}$ and of multiplier system $v(M)$ on $\Gamma_0(4N)$, if

(1) F is regular on $\mathcal{H}$;

(2) for all matrices $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ of substitutions from $\Gamma_0(4N)$ and all $\tau \in \mathcal{H}$,

$$F\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) = v(M)(\gamma\tau+\delta)^{s/2}F(\tau);$$

(3) in the neighborhood of the point $\tau = i\infty$,

$$F(\tau) = \sum_{m=0}^{\infty} A_m e(m\tau);$$

(4) for all substitutions from Γ in the neighborhood of each rational point $\tau = -\frac{\delta}{\gamma}$ ($\gamma \neq 0$, $(\gamma,\delta)=1$),

$$(\gamma\tau+\delta)^{s/2}F(\tau) = \sum_{m=0}^{\infty} A'_m e\Big(\frac{m}{4N}\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big).$$

Lemma 1 ([1], p. 58, Lemma 3). *If g is even, then for $n \geq 0$ and all substitutions from $\Gamma_0(4N)$ we have*

$$\vartheta_{gh}^{(n)}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N\Big) =$$
$$= (\operatorname{sgn}\delta)^n i^{(2n+1)\eta(\gamma)(\operatorname{sgn}\delta-1)/2} i^{(1-|\delta|)/2}\Big(\frac{2\beta N\operatorname{sgn}\delta}{|\delta|}\Big)(\gamma\tau+\delta)^{(2n+1)/2}\times$$
$$\times e\Big(-\frac{\alpha\gamma\delta^2h^2}{16N}\Big)e\Big(\frac{\beta\delta g^2}{4}\,\frac{\delta^{2\varphi(2N)-2}}{4N}\Big)\vartheta_{\alpha g,h}^{(n)}(\tau;0,2N).$$

Lemma 2 ([1], p. 61, Lemma 4). *If $\gamma \neq 0$, then for $n \geq 0$*

$$(\gamma\tau+\delta)^{(2n+1)/2}\vartheta_{gh}^{(n)}(\tau;0,2N) =$$
$$= e\big((2n+1)\operatorname{sgn}\gamma/8\big)\big(2N|\gamma|\big)^{-1/2}\big(-i\operatorname{sgn}\gamma\big)^n\times$$
$$\times\sum_{H \bmod 2N}\varphi_{g'gh}(0,H;2N)\Big\{\vartheta_{g'h'}^{(n)}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H,2N\Big)+$$
$$+\sum_{t=1}^{n}\binom{n}{t}\sum_{k=1}^{t}\frac{A_{tk}}{k!}\Big|_{z=0}\cdot\vartheta_{g'h'}^{(n-t)}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H,2N\Big)\Big\},$$

where

$$g' = \delta g - \gamma h - 2\gamma\delta N, \quad h' = -\beta g + \alpha h - 2\alpha\beta N, \tag{1.7}$$
$$\varphi_{g'gh}(0,H;2N) = e\Big(\frac{\alpha\beta}{4N}\big(H+\frac{g'}{2}\big)^2\Big)e\Big(-\frac{\beta g}{4N}\big(H+\frac{g'}{2}\big)\Big)\times$$
$$\times S_{g-\alpha g',h-\beta g'}\Big(\begin{matrix}\delta\\-\gamma\end{matrix};-\alpha H,2N\Big),$$

$$A_{tk}\Big|_{z=0} = \begin{cases} (2k)!(-2N\gamma\pi i(\gamma\tau+\delta))^k & \textit{if } t = 2k \\ 0 & \textit{if } t \neq 2k \end{cases} \tag{1.8}$$

$$(t = 1, 2, \dots, n; \quad k = 1, 2, \dots, t).$$

From (1.8) it follows that

$$\sum_{k=1}^{t} \frac{A_{tk}}{k!}\Big|_{z=0} = \begin{cases} \frac{A_{t,t/2}}{(t/2)!}\Big|_{z=0} & \text{if } 2|t, \\ 0 & \text{if } 2 \nmid t. \end{cases} \tag{1.9}$$

2.

Lemma 3. *For given* N *let*

$$\begin{aligned} &\Omega(\tau; g_l, h_l, c_l, N_l) = \\ &= \Omega(\tau; g_1, g_2, g_3; h_1, h_2, h_3; c_1, c_2, c_3; N_1, N_2, N_3) = \\ &= \Big\{ \frac{1}{N_1} \vartheta''_{g_1 h_1}(\tau; c_1, 2N_1) \vartheta_{g_2 h_2}(\tau; c_2, 2N_2) + \\ &- \frac{1}{N_2} \vartheta''_{g_2 h_2}(\tau; c_2, 2N_2) \vartheta_{g_1 h_1}(\tau; c_1, 2N_1) \Big\} \vartheta_{g_3 h_3}(\tau; c_3, 2N_3), \end{aligned} \tag{2.1}$$

where

$$2|g_k, \quad N_k|N \quad (k = 1, 2, 3), \quad 4|N \sum_{k=1}^{3} \frac{h_k}{N_k}. \tag{2.2}$$

Then for all substitutions from Γ *in the neighborhood of each rational point* $\tau = -\frac{\delta}{\gamma}$ $(\gamma \neq 0,\ (\gamma, \delta) = 1)$*, we have*

$$(\gamma\tau+\delta)^{7/2} \Omega(\tau; g_l, h_l, 0, N_l) = \sum_{n=0}^{\infty} G_n e\Big(\frac{n}{4N} \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big). \tag{2.3}$$

Proof. 1. From Lemma 2 for $n = 2$ and $n = 0$ (respectively with g_1, h_1, N_1, g'_1, h'_1, H_1 and g_2, h_2, N_2, g'_2, h'_2, H_2 instead of g, h, N, g', h', H), according to (1.8)–(1.9), it follows that

$$\begin{aligned} &\frac{1}{N_1} (\gamma\tau+\delta)^3 \vartheta''_{g_1 h_1}(\tau; 0, 2N_1) \vartheta_{g_2 h_2}(\tau; 0, 2N_2) = \\ &= -e\Big(\frac{3}{4} \operatorname{sgn} \gamma\Big) (4N_1 N_2 \gamma^2)^{-1/2} \times \\ &\times \sum_{\substack{H_1 \bmod 2N_1 \\ H_2 \bmod 2N_2}} \varphi_{g'_1 g_1 h_1}(0, H_1; 2N_1) \varphi_{g'_2 g_2 h_2}(0, H_2; 2N_2) \times \end{aligned}$$

$$\times\Big\{\frac{1}{N_1}\vartheta''_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\vartheta_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)+$$
$$-4\gamma\pi i(\gamma\tau+\delta)\vartheta_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\times$$
$$\times\vartheta_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)\Big\}. \tag{2.4}$$

Replacing $N_1, g_1, h_1, H_1, g_1', h_1'$ by $N_2, g_2, h_2, H_2, g_2', h_2'$ in (2.4), and vice versa, we obtain

$$\frac{1}{N_2}(\gamma\tau+\delta)^3\vartheta''_{g_2h_2}(\tau;0,2N_2)\vartheta_{g_1h_1}(\tau;0,2N_1)=$$
$$=-e\Big(\frac{3}{4}\operatorname{sgn}\gamma\Big)(4N_1N_2\gamma^2)^{-1/2}\times$$
$$\times\sum_{\substack{H_1 \bmod 2N_1\\ H_2 \bmod 2N_2}}\varphi_{g_2'g_2h_2}(0,H_2;2N_2)\varphi_{g_1'g_1h_1}(0,H_1;2N_1)\times$$
$$\times\Big\{\frac{1}{N_2}\vartheta''_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)\vartheta_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)+$$
$$-4\gamma\pi i(\gamma\tau+\delta)\vartheta_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)\times$$
$$\times\vartheta_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\Big\}. \tag{2.5}$$

Subtracting (2.5) from (2.4), we obtain

$$(\gamma\tau+\delta)^3\Big\{\frac{1}{N_1}\vartheta''_{g_1h_1}(\tau;0,2N_1)\vartheta_{g_2h_2}(\tau;0,2N_2)+$$
$$-\frac{1}{N_2}\vartheta''_{g_2h_2}(\tau;0,2N_2)\vartheta_{g_1h_1}(\tau;0,2N_1)\Big\}=$$
$$=-e\Big(\frac{3}{4}\operatorname{sgn}\gamma\Big)(4N_1N_2\gamma^2)^{-1/2}\times$$
$$\times\sum_{\substack{H_1 \bmod 2N_1\\ H_2 \bmod 2N_2}}\varphi_{g_1'g_1h_1}(0,H_1;2N_1)\varphi_{g_2'g_2h_2}(0,H_2;2N_2)\times$$
$$\times\Big\{\frac{1}{N_1}\vartheta''_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\vartheta_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)+$$
$$-\frac{1}{N_2}\vartheta''_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)\vartheta_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\Big\}. \tag{2.6}$$

From Lemma 2 for $n=0$, we have

$$(\gamma\tau+\delta)^{1/2}\vartheta_{g_3h_3}(\tau;0,2N_3)=e\big(\operatorname{sgn}\gamma/8\big)(2N_3|\gamma|)^{-1/2}\times$$

$$\times \sum_{H_3 \bmod 2N_3} \varphi_{g_3' g_3 h_3}(0, H_3; 2N_3) \vartheta_{g_3' h_3'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_3, 2N_3\Big). \qquad (2.7)$$

Multiplying (2.6) by (2.7), according to (2.1), we obtain

$$(\gamma\tau+\delta)^{7/2}\Omega(\tau; g_l, h_l, 0, N_l) =$$
$$= -e(7\operatorname{sgn}\gamma/8)(8N_1N_2N_3|\gamma|^3)^{-1/2} \times$$
$$\times \sum_{\substack{H_1 \bmod 2N_1\\ H_2 \bmod 2N_2\\ H_3 \bmod 2N_3}} \prod_{k=1}^{3} \varphi_{g_k' g_k h_k}(0, H_k; 2N_k)\Omega\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; g_l', h_l', H_l, N_l\Big). \qquad (2.8)$$

2. In (2.8) let γ be even. Then, by (1.7), g_k' $(k = 1, 2, 3)$ are also even. Now in (1.2) and (1.3) instead of m let us introduce new letters of summations m_k defined by the equalities $m - H_k = 2N_k m_k$ $(k = r, t, 3)$, respectively, and for brevity we put

$$T_k = \Big(H_k + 2N_k m_k + \frac{g_k'}{2}\Big)^2 \quad (k = r, t, 3).$$

Then, by (2.1), in (2.8) for $r = 1$, $t = 2$ and $r = 2$, $t = 1$, we get

$$\vartheta''_{g_r' h_r'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_r, 2N_r\Big) \prod_{k=t,3} \vartheta_{g_k' h_k'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_k, 2N_k\Big) =$$
$$= (\pi i)^2 \sum_{m_r=-\infty}^{\infty} (-1)^{h_r' m_r}\big(2(H_r + 2N_r m_r) + g_r'\big)^2 e\Big(\frac{N/N_r}{4N}\, T_r\, \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) \times$$
$$\times \sum_{m_t=-\infty}^{\infty} (-1)^{h_t' m_t} e\Big(\frac{N/N_t}{4N}\, T_t\, \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) \times$$
$$\times \sum_{m_3=-\infty}^{\infty} (-1)^{h_3' m_3} e\Big(\frac{N/N_3}{4N}\, T_3\, \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) =$$
$$= \sum_{n_r=0}^{\infty} B_{n_r} e\Big(\frac{n_r}{4N}\, \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) \sum_{n_t=0}^{\infty} B_{n_t} e\Big(\frac{n_t}{4N}\, \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) \times$$
$$\times \sum_{n_3=0}^{\infty} B_{n_3} e\Big(\frac{n_3}{4N}\, \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) = \sum_{n=0}^{\infty} C_n^{(r,t)} e\Big(\frac{n}{4N}\, \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big), \qquad (2.9)$$

since by (2.2) $n_k = \frac{N}{N_k} T_k$ are non-negative integers. Thus, for even γ, (2.3) follows from (2.1),(2.8), and (2.9).

In (2.8) let now γ be odd. If h_1, h_2, h_3 are even, then by (1.7), g_1', g_2', g_3' are also even, and we obtain the same result. But if h_r is odd, then by

(1.7), h'_r and g'_r will also be odd, and in (1.2) we shall have

$$\big(m + g'_r/2\big)^2 = \big(m + (g'_r - 1)/2\big)^2 + \big(m + (g'_r - 1)/2\big) + 1/4.$$

Hence

$$\vartheta''_{g'_r h'_r}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_r, 2N_r\Big) = (\pi i)^2 e\Big(\frac{h'_r}{16N_r}\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) \times$$
$$\times \sum_{m_r=-\infty}^{\infty} (-1)^{h'_r m_r}\big(2(H_r + 2N_r m_r) + g'_r\big)^2 e\Big(\frac{N/N_r}{4N}\, W_r\, \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) \quad (r=1,2).$$

Here by (1.4) we can imply that $h'_r = 1$, since by (1.7), h'_r and h_r have the same parity, and

$$W_r = \big(H_r + 2N_r m_r + (g'_r - 1)/2\big)^2 + \\ + \big(H_r + 2N_r m_r + (g'_r - 1)/2\big) \quad (r = 1,2). \tag{2.10}$$

Analogously,

$$\vartheta_{g'_t h'_t}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_t, 2N_t\Big) = e\Big(\frac{h'_t}{16N_t}\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) \times$$
$$\times \sum_{m_t=-\infty}^{\infty} (-1)^{h'_t m_t} e\Big(\frac{N/N_t}{4N}\, W_t\, \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) \quad (t = 2,1).$$

The same formula is valid for $t = 3$ with the same remarks concerning h'_3 as for h'_t, $t = 2,1$. Further, for $k = t,3$ we have

$$W_k = \begin{cases} T_k & \text{if } 2|g'_k, \\ (H_k + 2N_k m_k + (g'_k - 1)/2)^2 + & \\ (H_k + 2N_k m_k + (g'_k - 1)/2) & \text{if } 2\nmid g'_k. \end{cases} \tag{2.11}$$

Thus, if among h'_1, h'_2, h'_3 at least one is odd, then, as in (2.9), for $r = 1$, $t = 2$ and $r = 2$, $t = 1$ we have

$$\vartheta''_{g'_r h'_r}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_r, 2N_r\Big) \prod_{k=t,3} \vartheta_{g'_k h'_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_k, 2N_k\Big) =$$
$$= e\Big(\frac{1}{4N}\Big(\frac{1}{4}\,N \sum_{k=1}^{3} \frac{h'_k}{N_k}\Big)\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) \sum_{n_r=0}^{\infty} B'_{n_r} e\Big(\frac{n_r}{4N}\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) \times$$
$$\times \sum_{n_t=0}^{\infty} B'_{n_t} e\Big(\frac{n_t}{4N}\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) \sum_{n_3=0}^{\infty} B'_{n_3} e\Big(\frac{n_3}{4N}\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) =$$
$$= e\Big(\frac{1}{4N}\Big(\frac{1}{4}\,N \sum_{k=1}^{3} \frac{h'_k}{N_k}\Big)\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) \sum_{n'=0}^{\infty} D^{(r,t)}_{n'} e\Big(\frac{n'}{4N}\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) =$$

$$= \sum_{n=0}^{\infty} D_n^{(r,t)} e\Big(\frac{n}{4N}\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big), \tag{2.12}$$

since, by (2.2), (2.10), and (2.11), $n_k = \frac{N}{N_k} W_k$ and $n = \frac{1}{4} N \sum\limits_{k=1}^{3} \frac{h_k'}{N_k} + n'$ are non-negative integers. Thus, for odd γ, (2.3) follows from (2.1), (2.8), and (2.12). □

Theorem 1. *For given N the function $\Omega(\tau; g_l, h_l, 0, N_l)$ is an entire modular form of weight $7/2$ and of the multiplier system*

$$v(M) = i^{3\eta(\gamma)(\operatorname{sgn}\delta - 1)/2} i^{3(|\delta|-1)^2/4} \Big(\frac{\beta\Delta\operatorname{sgn}\delta}{|\delta|}\Big) \tag{2.13}$$

($M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ *is a matrix of the substitution from $\Gamma_0(4N)$ and Δ is the determinant of an arbitrary positive quadratic form with integer coefficients in 7 variables*) *on $\Gamma_0(4N)$ if the following conditions hold:*

$$(1) \quad 2|g_k, \;\; N_k|N \;\; (k = 1,2,3), \tag{2.14}$$

$$(2) \quad 4|N \sum_{k=1}^{3} \frac{h_k^2}{N_k} \quad 4|\sum_{k=1}^{s} \frac{g_k^2}{4N_k}, \tag{2.15}$$

3) *for all α and δ with $\alpha\delta \equiv 1 \pmod{4N}$*

$$\Big(\frac{N_1N_2N_3}{|\delta|}\Big)\Omega(\tau; \alpha g_l, h_l, 0, N_l) = \Big(\frac{\Delta}{|\delta|}\Big)\Omega(\tau; g_l, h_l, 0, N_l). \tag{2.16}$$

Proof. 1. It is well known that theta-series (1.1)–(1.2) are regular on $\mathcal{H}$; hence the function $\Omega(\tau; g_l, h_l, 0, N_l)$ satisfies condition (1) of the Definition.

2. It is easily verified that (2.15) implies

$$4|N\delta^2 \sum_{k=1}^{3} h_k^2/N_k, \quad 4|\sum_{k=1}^{3} \delta^{2\varphi(2N_k)-2} g_k^2/4N_k, \tag{2.17}$$

since $2 \nmid \delta$, because $\alpha\delta \equiv 1 \pmod{4N}$.

From (2.14) it follows that

$$\Gamma_0(4N) \subset \Gamma_0(4N_k) \quad (k = 1,2,3). \tag{2.18}$$

By Lemma 1 for $n = 2$ and $n = 0$, according to (2.17) and (2.18), for all substitutions from $\Gamma_0(4N)$, we have

$$\vartheta''_{g_r h_r}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; 0, 2N_r\Big) = i^{5\eta(\gamma)(\operatorname{sgn}\delta-1)/2} i^{(1-|\delta|)/2} \times$$
$$\times \Big(\frac{2\beta N_r \operatorname{sgn}\delta}{|\delta|}\Big)(\gamma\tau+\delta)^{5/2} \vartheta''_{\alpha g_r, h_r}(\tau; 0, 2N_r), \tag{2.19}$$

$$\vartheta_{g_t h_t}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_t\Big) = i^{\eta(\gamma)(\operatorname{sgn}\delta-1)/2} i^{(1-|\delta|)/2}\times$$
$$\times\Big(\frac{2\beta N_t \operatorname{sgn}\delta}{|\delta|}\Big)(\gamma\tau+\delta)^{1/2}\vartheta_{\alpha g_t,h_t}(\tau;0,2N_t), \tag{2.20}$$
$$\vartheta_{g_3 h_3}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_3\Big) = i^{\eta(\gamma)(\operatorname{sgn}\delta-1)/2} i^{(1-|\delta|)/2}\times$$
$$\times\Big(\frac{2\beta N_3 \operatorname{sgn}\delta}{|\delta|}\Big)(\gamma\tau+\delta)^{1/2}\vartheta_{\alpha g_3,h_3}(\tau;0,2N_3). \tag{2.21}$$

Thus for all substitutions from $\Gamma_0(4N)$, if $r=1$, $t=2$ and $r=2$, $t=1$, we have

$$\vartheta''_{g_r h_r}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_r\Big)\prod_{k=t,3}\vartheta_{g_k h_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_k\Big) =$$
$$= i^{7\eta(\gamma)(\operatorname{sgn}\delta-1)/2} i^{3(1-|\delta|)/2}\Big(\frac{2\beta\operatorname{sgn}\delta}{|\delta|}\Big)\Big(\frac{N_r N_t N_3}{|\delta|}\Big)(\gamma\tau+\delta)^{7/2}\times$$
$$\times\vartheta''_{\alpha g_r,h_r}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_r\Big)\prod_{k=t,3}\vartheta_{\alpha g_k,h_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_k\Big). \tag{2.22}$$

It is not difficult to verify that

$$i^{3(1-|\delta|)/2}\Big(\frac{2}{|\delta|}\Big) = i^{3(|\delta|-1)^2/4}. \tag{2.23}$$

Hence, by (2.1), (2.22), (2.23), (2.13), and (2.16) we get

$$\Omega\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};g_l,h_l,0,N_l\Big) = i^{3\eta(\gamma)(\operatorname{sgn}\delta-1)/2} i^{3(|\delta|-1)^2/4}\times$$
$$\times\Big(\frac{\beta\operatorname{sgn}\delta}{|\delta|}\Big)\Big(\frac{N_1N_2N_3}{|\delta|}\Big)(\gamma\tau+\delta)^{7/2}\Omega(\tau;\alpha g_l,h_l,0,N_l) =$$
$$= v(M)(\gamma\tau+\delta)^{7/2}\Omega(\tau;g_l,h_l,0,N_l).$$

Thus, the function $\Omega(\tau;g_l,h_l,0,N_l)$ satisfies condition (2) of the Definition.

3. From (1.6) it follows for $r=1$, $t=2$ and $r=2$, $t=1$ that

$$\vartheta''_{g_r h_r}(\tau;0,2N_r)\vartheta_{g_t h_t}(\tau;0,2N_t)\vartheta_{g_3 h_3}(\tau;0,2N_3) =$$
$$= (\pi i)^2\sum_{m_r,m_t,m_3=-\infty}^{\infty}(-1)^{h_r m_r + h_t m_t + h_3 m_3}(4N_r m_r + g_r)^2\times$$
$$\times e\Big(\sum_{k=r,t,3}\frac{1}{4N_k}(2N_k m_k + g_k/2)^2\tau\Big) = \sum_{n=0}^{\infty}B_n^{(r,t)}e(n\tau), \tag{2.24}$$

where

$$n = \sum_{k=1}^{3} \frac{1}{4N_k}(2N_k m_k + g_k/2)^2 =$$

$$= \sum_{k=1}^{3}(N_k m_k^2 + m_k g_k/2) + \frac{1}{4}\sum_{k=1}^{3} g_k^2/4N_k,$$

by (2.14) and (2.15), is a non-negative integer. Thus, by (2.1) and (2.24), the function $\Omega(\tau; g_l, h_l, 0, N_l)$ satisfies condition (3) of the Definition.

4. By Lemma 3, the function $\Omega(\tau; g_l, h_l, 0, N_l)$ satisfies condition (4) of the Definition. □

3.

Lemma 4. *For given N let*

$$\Psi_1(\tau; g_l, h_l, c_l, N_l) = \Psi_1(\tau; g_1, g_2, g_3; h_1, h_2, h_3; c_1, c_2, c_3; N_1, N_2, N_3) =$$
$$= \Big\{ \frac{1}{N_1} \vartheta''_{g_1 h_1}(\tau; c_1, 2N_1)\vartheta_{g_2 h_2}(\tau; c_2, 2N_2) +$$
$$- \frac{1}{N_2} \vartheta''_{g_2 h_2}(\tau; c_2, 2N_2)\vartheta_{g_1 h_1}(\tau; c_1, 2N_1)\Big\}\vartheta'_{g_3 h_3}(\tau; c_3, 2N_3) \qquad (3.1)$$

and

$$\Psi_2(\tau; g_l, h_l, c_l, N_l) = \Psi_2(\tau; g_1, g_2, g_3; h_1, h_2, h_3; c_1, c_2, c_3; N_1, N_2, N_3) =$$
$$= \Big\{ \frac{1}{N_1} \vartheta'''_{g_1 h_1}(\tau; c_1, 2N_1)\vartheta_{g_2 h_2}(\tau; c_2, 2N_2) +$$
$$- \frac{3}{N_2} \vartheta''_{g_2 h_2}(\tau; c_2, 2N_2)\vartheta'_{g_1 h_1}(\tau; c_1, 2N_1)\Big\}\vartheta_{g_3 h_3}(\tau; c_3, 2N_3), \qquad (3.2)$$

where

$$2|g_k, \quad N_k|N \quad (k = 1, 2, 3), \quad 4|N \sum_{k=1}^{3} \frac{h_k}{N_k}. \qquad (3.3)$$

Then for all substitutions from Γ, in the neighborhood of each rational point $\tau = -\frac{\delta}{\gamma}$ ($\gamma \neq 0$, $(\gamma, \delta) = 1$), we have

$$(\gamma\tau + \delta)^{9/2}\Psi_j(\tau; g_l, h_l, 0, N_l) = \sum_{n=0}^{\infty} G_n^{(j)} e\Big(\frac{n}{4N}\,\frac{\alpha\tau + \beta}{\gamma\tau + \delta}\Big) \ (j = 1, 2). \qquad (3.4)$$

Proof. 1. Formula (2.6) is obtained in Lemma 3. From Lemma 2, for $n=1$, we have

$$(\gamma\tau+\delta)^{3/2}\vartheta'_{g_3h_3}(\tau;0,2N_3) = -e(3\operatorname{sgn}\gamma/8)(2N_3|\gamma|)^{-1/2}(i\operatorname{sgn}\gamma)\times \\ \times \sum_{H_3 \bmod 2N_3} \varphi_{g'_3g_3h_3}(0,H_3;2N_3)\vartheta'_{g'_3h'_3}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_3,2N_3\Big). \tag{3.5}$$

Multiplying (2.6) by (3.5), according to (3.1), we obtain

$$(\gamma\tau+\delta)^{9/2}\Psi_1(\tau;g_l,h_l,0,N_l) = \\ = e(9\operatorname{sgn}\gamma/8)(8N_1N_2N_3|\gamma|^3)^{-1/2}(i\operatorname{sgn}\gamma)\times \\ \times \sum_{\substack{H_1 \bmod 2N_1\\ H_2 \bmod 2N_2\\ H_3 \bmod 2N_3}} \prod_{k=1}^{3} \varphi_{g'_kg_kh_k}(0,H_k;2N_k)\Psi_1\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};g'_l,h'_l,H_l,N_l\Big). \tag{3.6}$$

Reasoning further just as in Lemma 3.2, but taking everywhere $\vartheta'_{g_3h_3}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_3,2N_3\Big)$ instead of $\vartheta_{g_3h_3}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_3,2N_3\Big)$, from (3.1) and (3.6) we obtain (3.4) if $j=1$.

2. From Lemma 2 for $n=3$, $n=0$, and respectively for $n=2$, $n=1$, it follows that

$$\frac{1}{N_1}(\gamma\tau+\delta)^4\vartheta'''_{g_1h_1}(\tau;0,2N_1)\vartheta_{g_2h_2}(\tau;0,2N_2) = \\ = \frac{1}{N_1}e(\operatorname{sgn}\gamma)(4N_1N_2\gamma^2)^{-1/2}(i\operatorname{sgn}\gamma)\times \\ \times \sum_{H_1 \bmod 2N_1} \varphi_{g'_1g_1h_1}(0,H_1;2N_1)\Big\{\vartheta'''_{g'_1h'_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big) + \\ + \sum_{t=1}^{3}\binom{3}{t}\sum_{k=1}^{t}\frac{A_{tk}}{k!}\Big|_{z=0}\vartheta^{(3-t)}_{g'_1h'_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\Big\}\times \\ \times \sum_{H_2 \bmod 2N_2} \varphi_{g'_2g_2h_2}(0,H_2;2N_2)\vartheta_{g'_2h'_2}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big) = \\ = e(\operatorname{sgn}\gamma)(i\operatorname{sgn}\gamma)(4N_1N_2\gamma^2)^{-1/2}\sum_{\substack{H_1 \bmod 2N_1\\ H_2 \bmod 2N_2}}\varphi_{g'_1g_1h_1}(0,H_1;2N_1)\times \\ \times\varphi_{g'_2g_2h_2}(0,H_2;2N_2)\Big\{\frac{1}{N_1}\vartheta'''_{g'_1h'_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\vartheta_{g'_2h'_2}(\tau;0,2N_2) + \\ -12\gamma\pi i(\gamma\tau+\delta)\vartheta'_{g'_1h'_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\vartheta_{g'_2h'_2}(\tau;0,2N_2)\Big\} \tag{3.7}$$

and

$$
\begin{aligned}
&\frac{3}{N_2}(\gamma\tau+\delta)^4\vartheta''_{g_2h_2}(\tau;0,2N_2)\vartheta'_{g_1h_1}(\tau;0,2N_1)=\\
&=e(\operatorname{sgn}\gamma)(i\operatorname{sgn}\gamma)(4N_1N_2\gamma^2)^{-1/2}\sum_{\substack{H_1 \bmod 2N_1\\ H_2 \bmod 2N_2}}\varphi_{g_1'g_1h_1}(0,H_1;2N_1)\times\\
&\times\varphi_{g_2'g_2h_2}(0,H_2;2N_2)\Big\{\frac{3}{N_2}\vartheta''_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)+\\
&-12\gamma\pi i(\gamma\tau+\delta)\vartheta'_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)\Big\}\vartheta'_{g_1'h_1'}(\tau;0,2N_1). \qquad (3.8)
\end{aligned}
$$

Subtracting (3.8) from (3.7) and multiplying the obtained result by (2.7), according to (3.2), we get

$$
\begin{aligned}
&(\gamma\tau+\delta)^{9/2}\Psi_2(\tau;g_l,h_l,0,N_l)=\\
&=e(9\operatorname{sgn}\gamma/8)(8N_1N_2N_3|\gamma|^3)^{-1/2}(i\operatorname{sgn}\gamma)\times\\
&\times\sum_{\substack{H_1 \bmod 2N_1\\ H_2 \bmod 2N_2\\ H_3 \bmod 2N_3}}\prod_{k=1}^{3}\varphi_{g_k'g_kh_k}(0,H_k;2N_k)\Psi_2\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};g_l',h_l',H_l,N_l\Big). \qquad (3.9)
\end{aligned}
$$

In (3.9) let γ be even. Then, reasoning as in Lemma 3.2, by (3.2), from (3.9) we obtain

$$
\begin{aligned}
&\Big\{\frac{1}{N_1}\vartheta'''_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\vartheta_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)+\\
&-\frac{3}{N_2}\vartheta''_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)\vartheta'_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\Big\}\times\\
&\times\vartheta_{g_3'h_3'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_3,2N_3\Big)=\Big\{\frac{(\pi i)^3}{N_1}\times\\
&\times\sum_{m_1=-\infty}^{\infty}(-1)^{h_1'm_1}\big(2(H_1+2N_1m_1)+g_1'\big)^3e\Big(\frac{N/N_1}{4N}\,T_1\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big)\times\\
&\times\sum_{m_2=-\infty}^{\infty}(-1)^{h_2'm_2}e\Big(\frac{N/N_2}{4N}\,T_2\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big)+\\
&-\frac{3(\pi i)^3}{N_1}\sum_{m_2=-\infty}^{\infty}(-1)^{h_2'm_2}\big(2(H_2+2N_2m_2)+g_2'\big)^2e\Big(\frac{N/N_2}{4N}\,T_2\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big)\times\\
&\times\sum_{m_1=-\infty}^{\infty}(-1)^{h_1'm_1}\big(2(H_1+2N_1m_1)+g_1'\big)e\Big(\frac{N/N_1}{4N}\,T_1\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big)\Big\}\times
\end{aligned}
$$

$$\times \sum_{m_3=-\infty}^{\infty}(-1)^{h_3' m_3} e\Big(\frac{N/N_3}{4N}\, T_3\, \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) =$$
$$= \Big\{ \sum_{n_1=0}^{\infty} B_{n_1} e\Big(\frac{n_1}{4N}\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) \sum_{n_2=0}^{\infty} B_{n_2} e\Big(\frac{n_2}{4N}\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) +$$
$$- \sum_{n_2=0}^{\infty} B'_{n_2} e\Big(\frac{n_2}{4N}\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) \sum_{n_1=0}^{\infty} B'_{n_1} e\Big(\frac{n_1}{4N}\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big)\Big\} \times$$
$$\times \sum_{n_3=0}^{\infty} B_{n_3} e\Big(\frac{n_3}{4N}\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) = \sum_{n=0}^{\infty} C_n e\Big(\frac{n}{4N}\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big). \tag{3.10}$$

Here T_k and n_k have the same meaning as in Lemma 3.2.

Thus, for even γ, (3.4) follows from (3.2), (3.9) and (3.10) if $j = 2$.

Further, reasoning as in Lemma 3.2, we obtain (3.4) for odd γ if $j = 2$. □

Theorem 2. *For given N the functions $\Psi_1(\tau; g_l, h_l, 0, N_l)$ and $\Psi_2(\tau; g_l, h_l, 0, N_l)$ are entire modular forms of weight $9/2$ and of the multiplier system*

$$v(M) = i^{\eta(\gamma)(\operatorname{sgn}\delta-1)/2} i^{(|\delta|-1)^2/4}\Big(\frac{\beta\Delta\operatorname{sgn}\delta}{|\delta|}\Big) \tag{3.11}$$

($M = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ *is the matrix of the substitution from $\Gamma_0(4N)$ and Δ is the determinant of an arbitrary positive quadratic form with integer coefficients in 9 variables*) *on $\Gamma_0(4N)$ if the following conditions hold:*

$$(1)\quad 2|g_k,\ \ N_k|N\ \ (k = 1, 2, 3), \tag{3.12}$$

$$(2)\quad 4|N\sum_{k=1}^{3}\frac{h_k^2}{N_k}\quad 4|\sum_{k=1}^{s}\frac{g_k^2}{4N_k}, \tag{3.13}$$

(3) *for all α and δ with $\alpha\delta \equiv 1 \pmod{4N}$*

$$\operatorname{sgn}\delta\Big(\frac{N_1N_2N_3}{|\delta|}\Big)\Psi_j(\tau; \alpha g_l, h_l, 0, N_l) =$$
$$= \Big(\frac{-\Delta}{|\delta|}\Big)\Psi_j(\tau; g_l, h_l, 0, N_l)\quad (j = 1, 2). \tag{3.14}$$

Proof. 1. As in the case of Theorem 1, the functions $\Psi_1(\tau; g_l, h_l, 0, N_l)$ and $\Psi_2(\tau; g_l, h_l, 0, N_l)$ satisfy condition (1) and, by Lemma 4, also condition (4) of the Definition.

2. By Lemma 1 for $n = 3$ and $n = 1$, according to (2.17) and (2.18), for all substitutions from $\Gamma_0(4N)$, we have

$$\vartheta'''_{g_1h_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; 0, 2N_1\Big) = \operatorname{sgn}\delta\, i^{7\eta(\gamma)(\operatorname{sgn}\delta-1)/2} i^{(1-|\delta|)/2}\times$$

$$\times\Big(\frac{2\beta N_1 \operatorname{sgn}\delta}{|\delta|}\Big)(\gamma\tau+\delta)^{7/2}\vartheta'''_{\alpha g_1,h_1}(\tau;0,2N_1), \tag{3.15}$$

$$\vartheta'_{g_1h_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_1\Big)=\operatorname{sgn}\delta\, i^{3\eta(\gamma)(\operatorname{sgn}\delta-1)/2} i^{(1-|\delta|)/2}\times$$

$$\times\Big(\frac{2\beta N_1 \operatorname{sgn}\delta}{|\delta|}\Big)(\gamma\tau+\delta)^{3/2}\vartheta'_{\alpha g_1,h_1}(\tau;0,2N_1), \tag{3.16}$$

$$\vartheta'_{g_3h_3}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_3\Big)=\operatorname{sgn}\delta\, i^{3\eta(\gamma)(\operatorname{sgn}\delta-1)/2} i^{(1-|\delta|)/2}\times$$

$$\times\Big(\frac{2\beta N_3 \operatorname{sgn}\delta}{|\delta|}\Big)(\gamma\tau+\delta)^{3/2}\vartheta'_{\alpha g_3,h_3}(\tau;0,2N_3). \tag{3.17}$$

Thus, for all substitutions from $\Gamma_0(4N)$ we respectively get:
(1) by (2.19), (2.20), and (3.17)

$$\vartheta''_{g_rh_r}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_r\Big)\vartheta_{g_th_t}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_t\Big)\vartheta'_{g_3h_3}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_3\Big)=$$

$$=\operatorname{sgn}\delta\, i^{\eta(\gamma)(\operatorname{sgn}\delta-1)/2} i^{3(1-|\delta|)/2}\Big(\frac{2\beta\operatorname{sgn}\delta}{|\delta|}\Big)\Big(\frac{N_1N_2N_3}{|\delta|}\Big)\times$$

$$\times(\gamma\tau+\delta)^{9/2}\vartheta''_{\alpha g_r,h_r}(\tau;0,2N_r)\vartheta_{\alpha g_t,h_t}(\tau;0,2N_t)\vartheta'_{\alpha g_3,h_3}(\tau;0,2N_3) \tag{3.18}$$

for $r=1$, $t=2$ and $r=2$, $t=1$;
(2) by (3.15), (2.20) for $t=2$ and (2.21)

$$\vartheta'''_{g_1h_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_1\Big)\vartheta_{g_2h_2}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_2\Big)\vartheta_{g_3h_3}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_3\Big)=$$

$$=\operatorname{sgn}\delta\, i^{\eta(\gamma)(\operatorname{sgn}\delta-1)/2} i^{3(1-|\delta|)/2}\Big(\frac{2\beta\operatorname{sgn}\delta}{|\delta|}\Big)\Big(\frac{N_1N_2N_3}{|\delta|}\Big)\times$$

$$\times(\gamma\tau+\delta)^{9/2}\vartheta'''_{\alpha g_1,h_1}(\tau;0,2N_1)\vartheta_{\alpha g_2,h_2}(\tau;0,2N_2)\vartheta_{\alpha g_3,h_3}(\tau;0,2N_3); \tag{3.19}$$

(3) by (2.19) for $r=2$, (3.16), and (2.21)

$$\vartheta''_{g_2h_2}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_2\Big)\vartheta'_{g_1h_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_1\Big)\vartheta_{g_3h_3}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_3\Big)=$$

$$=\operatorname{sgn}\delta\, i^{\eta(\gamma)(\operatorname{sgn}\delta-1)/2} i^{3(1-|\delta|)/2}\Big(\frac{2\beta\operatorname{sgn}\delta}{|\delta|}\Big)\Big(\frac{N_1N_2N_3}{|\delta|}\Big)\times$$

$$\times(\gamma\tau+\delta)^{9/2}\vartheta''_{\alpha g_2,h_2}(\tau;0,2N_2)\vartheta'_{\alpha g_1,h_1}(\tau;0,2N_1)\vartheta_{\alpha g_3,h_3}(\tau;0,2N_3). \tag{3.20}$$

It is not difficult to verify that

$$i^{3(1-|\delta|)/2}\Big(\frac{-2}{|\delta|}\Big)=i^{(|\delta|-1)^2/4}. \tag{3.21}$$

Hence, by (3.1), (3.18), (3.21), (3.11), (3.14) (if $j=1$) and (3.2), (3.19)–(3.21), (3.11), (3.14) (if $j=2$) we get

$$\begin{aligned}\Psi_j\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};g_l,h_l,0,N_l\Big)&=\operatorname{sgn}\delta\, i^{\eta(\gamma)(\operatorname{sgn}\delta-1)/2}i^{(|\delta|-1)^2/4}\times\\ \times\Big(\frac{-\beta\operatorname{sgn}\delta}{|\delta|}\Big)\Big(\frac{N_1N_2N_3}{|\delta|}\Big)&(\gamma\tau+\delta)^{9/2}\Psi_j(\tau;\alpha g_l,h_l,0,N_l)=\\ &=v(M)(\gamma\tau+\delta)^{9/2}\Psi_j(\tau;g_l,h_l,0,N_l)\ \ (j=1,2).\end{aligned}$$

Thus the functions $\Psi_j(\tau;g_l,h_l,0,N_l)$ $(j=1,2)$ satisfy condition (2) of the Definition.

From (1.6) it follows that

$$\begin{aligned}&\vartheta''_{g_rh_r}(\tau;0,2N_r)\vartheta_{g_th_t}(\tau;0,2N_t)\vartheta'_{g_3h_3}(\tau;0,2N_3)=\\ &=(\pi i)^3\sum_{m_r,m_t,m_3=-\infty}^{\infty}(-1)^{\sum_{k=r,t,3}h_km_k}(4N_rm_r+g_r)^2(4N_3m_3+g_3)\times\\ &\times e\Big(\sum_{k=r,t,3}\frac{1}{4N_k}(2N_km_k+g_k/2)^2\tau\Big)=\sum_{n=0}^{\infty}C_n^{(r,t)}e(n\tau)\end{aligned}$$

for $r=1$, $t=2$ and $r=2$, $t=1$,

$$\begin{aligned}&\vartheta'''_{g_1h_1}(\tau;0,2N_1)\vartheta_{g_2h_2}(\tau;0,2N_2)\vartheta_{g_3h_3}(\tau;0,2N_3)=\\ &=(\pi i)^3\sum_{m_1,m_2,m_3=-\infty}^{\infty}(-1)^{\sum_{k=1}^{3}h_km_k}(4N_1m_1+g_1)^3\times\\ &\times e\Big(\sum_{k=1}^{3}\frac{1}{4N_k}(2N_km_k+g_k/2)^2\tau\Big)=\sum_{n=0}^{\infty}C_ne(n\tau)\end{aligned}$$

and

$$\begin{aligned}&\vartheta''_{g_2h_2}(\tau;0,2N_2)\vartheta'_{g_1h_1}(\tau;0,2N_1)\vartheta_{g_3h_3}(\tau;0,2N_3)=\\ &=(\pi i)^3\sum_{m_1,m_2,m_3=-\infty}^{\infty}(-1)^{\sum_{k=1}^{3}h_km_k}(4N_2m_2+g_2)^2(4N_1m_1+g_1)\times\\ &\times e\Big(\sum_{k=1}^{3}\frac{1}{4N_k}(2N_km_k+g_k/2)^2\tau\Big)=\sum_{n=0}^{\infty}D_ne(n\tau),\end{aligned}$$

since in all these expansions

$$n=\sum_{k=1}^{3}\frac{1}{4N_k}(2N_km_k+g_k/2)^2,$$

as has been shown by Theorem 1.3, is a non-negative integer. Therefore, in view of (3.1) and (3.2), the functions $\Psi_j(\tau; g_l, h_l, 0, N_l)$ $(j = 1, 2)$ satisfy condition (3) of the Definition. $\square$

References

1. G. Lomadze, On some entire modular forms of weights 5 and 6 for the congruence group $\Gamma_0(4N)$. *Georgian Math. J.* **1**(1994), No. 1, 53–76.

(Received 24.11.1994)

Author's address:
Faculty of Mechanics and Mathematics
I. Javakhishvili Tbilisi State University
2, University St., Tbilisi 380043
Republic of Georgia