

ON SOME ENTIRE MODULAR FORMS OF WEIGHTS $\frac{7}{2}$
AND $\frac{9}{2}$ FOR THE CONGRUENCE GROUP $\Gamma_0(4N)$

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ABSTRACT. Entire modular forms of weights $\frac{7}{2}$ and $\frac{9}{2}$ for the congruence group $\Gamma_0(4N)$ are constructed, which will be useful for revealing the arithmetical sense of additional terms in formulas for the number of representations of positive integers by quadratic forms in 7 and 9 variables.

1.

In this paper N, a, k, q, r, t denote positive integers; u, s are odd positive integers; $H, c, g, h, j, m, n, \alpha, \beta, \gamma, \delta, \xi$ are integers; A, B, C, D, G are complex numbers; z, τ ($\text{Im } \tau > 0$) are complex variables, and $\mathcal{H} = \{\tau \in \mathbb{C} \mid \text{Im } \tau > 0\}$. Further, $\left(\frac{h}{u}\right)$ is the generalized Jacobi symbol; $\binom{n}{t}$ is a binomial coefficient; $\varphi(k)$ is Euler's function; $e(z) = \exp 2\pi iz$; $\eta(\gamma) = 1$ if $\gamma \geq 0$ and $\eta(\gamma) = -1$ if $\gamma < 0$.

Let

$$\begin{aligned} \vartheta_{gh}(z|\tau; c, N) = & \sum_{m \equiv c \pmod{N}} (-1)^{h(m-c)/N} \times \\ & \times e\left(\frac{1}{2N}\left(m + \frac{g}{2}\right)^2 \tau\right) e\left(\left(m + \frac{g}{2}\right)z\right); \end{aligned} \quad (1.1)$$

hence

$$\begin{aligned} \frac{\partial^n}{\partial z^n} \vartheta_{gh}(z|\tau; c, N) = & (\pi i)^n \sum_{m \equiv c \pmod{N}} (-1)^{h(m-c)/N} (2m + g)^n \times \\ & \times e\left(\frac{1}{2N}\left(m + \frac{g}{2}\right)^2 \tau\right) e\left(\left(m + \frac{g}{2}\right)z\right) \quad (n \geq 0). \end{aligned} \quad (1.2)$$

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Put

$$\vartheta_{gh}^{(n)}(\tau; c, N) = \frac{\partial^n}{\partial z^n} \vartheta_{gh}(z|\tau; c, N)|_{z=0} \quad (n \geq 0). \tag{1.3}$$

It is known (see, for e.g., [1], formulas (11) and (12)) that

$$\vartheta_{g,h+2j}^{(n)}(\tau; c, N) = \vartheta_{gh}^{(n)}(\tau; c, N) \quad (n \geq 0), \tag{1.4}$$

$$\begin{aligned} \vartheta_{gh}^{(n)}(\tau + \beta; c, N) &= e\left(\frac{\beta}{2N}\left(c + \frac{g}{2}\right)^2\right) \vartheta_{g,h+\beta g+\beta N}^{(n)}(\tau; c, N) \quad (n \geq 0), \\ \vartheta_{gh}^{(n)}(\tau - \beta; c, N) &= (-1)^n e\left(-\frac{\beta}{2N}\left(c + \frac{g}{2}\right)^2\right) \times \\ &\times \vartheta_{-g,-h+\beta g-\beta N}^{(n)}(\tau; -c, N) \quad (n \geq 0). \end{aligned} \tag{1.5}$$

From (1.1) and (1.2), according to notation (1.3) it follows, in particular, that

$$\begin{aligned} \vartheta_{gh}^{(n)}(\tau; 0, N) &= (\pi i)^n \sum_{m=-\infty}^{\infty} (-1)^{hm} (2Nm + g)^n \times \\ &\times e\left(\frac{1}{2N}\left(Nm + \frac{g}{2}\right)^2 \tau\right) \quad (n \geq 0). \end{aligned} \tag{1.6}$$

For $\xi_2 \neq 0$, ξ_1, g, h, N with $\xi_1 g + \xi_2 h + \xi_1 \xi_2 N \equiv 0 \pmod{2}$, put

$$S_{gh}\left(\begin{matrix} \xi_1 \\ \xi_2 \end{matrix}; c, N\right) = \sum_{\substack{m \pmod{N} | \xi_2 \\ m \equiv c \pmod{N}}} (-1)^{h(m-c)/N} e\left(\frac{\xi_1}{2N\xi_2}\left(m + \frac{g}{2}\right)^2\right).$$

Finally, let

$$\begin{aligned} \Gamma &= \left\{ \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \mid \alpha\delta - \beta\gamma = 1 \right\}, \\ \Gamma_0(4N) &= \left\{ \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \in \Gamma \mid \gamma \equiv 0 \pmod{4N} \right\}. \end{aligned}$$

For $\tau \in \mathcal{H}$ put

$$(\gamma\tau + \delta)^{s/2} = ((\gamma\tau + \delta)^{1/2})^s, \quad -\frac{\pi}{2} < \arg(\gamma\tau + \delta)^{1/2} \leq \frac{\pi}{2}.$$

Definition. Let M be a matrix of an arbitrary substitution from $\Gamma_0(4N)$, and let $v(M)$ be a multiplier system on $\Gamma_0(4N)$ and of weight $\frac{s}{2}$. We shall say that a function F defined on \mathcal{H} is an entire modular form of weight $\frac{s}{2}$ and of multiplier system $v(M)$ on $\Gamma_0(4N)$, if

- (1) F is regular on \mathcal{H} ;

(2) for all matrices $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ of substitutions from $\Gamma_0(4N)$ and all $\tau \in \mathcal{H}$,

$$F\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right) = v(M)(\gamma\tau + \delta)^{s/2}F(\tau);$$

(3) in the neighborhood of the point $\tau = i\infty$,

$$F(\tau) = \sum_{m=0}^{\infty} A_m e(m\tau);$$

(4) for all substitutions from Γ in the neighborhood of each rational point $\tau = -\frac{\delta}{\gamma}$ ($\gamma \neq 0$, $(\gamma, \delta) = 1$),

$$(\gamma\tau + \delta)^{s/2}F(\tau) = \sum_{m=0}^{\infty} A'_m e\left(\frac{m}{4N} \frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right).$$

Lemma 1 ([1], p. 58, Lemma 3). *If g is even, then for $n \geq 0$ and all substitutions from $\Gamma_0(4N)$ we have*

$$\begin{aligned} & \vartheta_{gh}^{(n)}\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N\right) = \\ & = (\text{sgn } \delta)^n i^{(2n+1)\eta(\gamma)(\text{sgn } \delta - 1)/2} i^{(1-|\delta|)/2} \left(\frac{2\beta N \text{sgn } \delta}{|\delta|}\right) (\gamma\tau + \delta)^{(2n+1)/2} \times \\ & \quad \times e\left(-\frac{\alpha\gamma\delta^2 h^2}{16N}\right) e\left(\frac{\beta\delta g^2}{4} \frac{\delta^{2\varphi(2N)-2}}{4N}\right) \vartheta_{\alpha g, h}^{(n)}(\tau; 0, 2N). \end{aligned}$$

Lemma 2 ([1], p. 61, Lemma 4). *If $\gamma \neq 0$, then for $n \geq 0$*

$$\begin{aligned} & (\gamma\tau + \delta)^{(2n+1)/2} \vartheta_{gh}^{(n)}(\tau; 0, 2N) = \\ & = e((2n+1) \text{sgn } \gamma/8) (2N|\gamma|)^{-1/2} (-i \text{sgn } \gamma)^n \times \\ & \times \sum_{H \bmod 2N} \varphi_{g'gh}(0, H; 2N) \left\{ \vartheta_{g'h'}^{(n)}\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H, 2N\right) + \right. \\ & \left. + \sum_{t=1}^n \binom{n}{t} \sum_{k=1}^t \frac{A_{tk}}{k!} \Big|_{z=0} \cdot \vartheta_{g'h'}^{(n-t)}\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H, 2N\right) \right\}, \end{aligned}$$

where

$$\begin{aligned} & g' = \delta g - \gamma h - 2\gamma\delta N, \quad h' = -\beta g + \alpha h - 2\alpha\beta N, \quad (1.7) \\ & \varphi_{g'gh}(0, H; 2N) = e\left(\frac{\alpha\beta}{4N} \left(H + \frac{g'}{2}\right)^2\right) e\left(-\frac{\beta g}{4N} \left(H + \frac{g'}{2}\right)\right) \times \\ & \quad \times S_{g-\alpha g', h-\beta g'}\left(\begin{matrix} \delta \\ -\gamma \end{matrix}; -\alpha H, 2N\right), \end{aligned}$$

$$A_{tk} \Big|_{z=0} = \begin{cases} (2k)!(-2N\gamma\pi i(\gamma\tau + \delta))^k & \text{if } t = 2k \\ 0 & \text{if } t \neq 2k \end{cases} \quad (1.8)$$

$$(t = 1, 2, \dots, n; \quad k = 1, 2, \dots, t).$$

From (1.8) it follows that

$$\sum_{k=1}^t \frac{A_{tk}}{k!} \Big|_{z=0} = \begin{cases} \frac{A_{t,t/2}}{(t/2)!} \Big|_{z=0} & \text{if } 2|t, \\ 0 & \text{if } 2 \nmid t. \end{cases} \quad (1.9)$$

2.

Lemma 3. For given N let

$$\begin{aligned} \Omega(\tau; g_l, h_l, c_l, N_l) &= \\ &= \Omega(\tau; g_1, g_2, g_3; h_1, h_2, h_3; c_1, c_2, c_3; N_1, N_2, N_3) = \\ &= \left\{ \frac{1}{N_1} \vartheta''_{g_1 h_1}(\tau; c_1, 2N_1) \vartheta_{g_2 h_2}(\tau; c_2, 2N_2) + \right. \\ &\left. - \frac{1}{N_2} \vartheta''_{g_2 h_2}(\tau; c_2, 2N_2) \vartheta_{g_1 h_1}(\tau; c_1, 2N_1) \right\} \vartheta_{g_3 h_3}(\tau; c_3, 2N_3), \end{aligned} \quad (2.1)$$

where

$$2|g_k, \quad N_k|N \quad (k = 1, 2, 3), \quad 4|N \sum_{k=1}^3 \frac{h_k}{N_k}. \quad (2.2)$$

Then for all substitutions from Γ in the neighborhood of each rational point $\tau = -\frac{\delta}{\gamma}$ ($\gamma \neq 0$, $(\gamma, \delta) = 1$), we have

$$(\gamma\tau + \delta)^{7/2} \Omega(\tau; g_l, h_l, 0, N_l) = \sum_{n=0}^{\infty} G_n e\left(\frac{n}{4N} \frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right). \quad (2.3)$$

Proof. 1. From Lemma 2 for $n = 2$ and $n = 0$ (respectively with $g_1, h_1, N_1, g'_1, h'_1, H_1$ and $g_2, h_2, N_2, g'_2, h'_2, H_2$ instead of g, h, N, g', h', H), according to (1.8)–(1.9), it follows that

$$\begin{aligned} &\frac{1}{N_1} (\gamma\tau + \delta)^3 \vartheta''_{g_1 h_1}(\tau; 0, 2N_1) \vartheta_{g_2 h_2}(\tau; 0, 2N_2) = \\ &= -e\left(\frac{3}{4} \operatorname{sgn} \gamma\right) (4N_1 N_2 \gamma^2)^{-1/2} \times \\ &\times \sum_{\substack{H_1 \bmod 2N_1 \\ H_2 \bmod 2N_2}} \varphi_{g'_1 h_1}(0, H_1; 2N_1) \varphi_{g'_2 h_2}(0, H_2; 2N_2) \times \end{aligned}$$

$$\begin{aligned}
& \times \left\{ \frac{1}{N_1} \vartheta_{g'_1 h'_1}'' \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) \vartheta_{g'_2 h'_2} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) + \right. \\
& \quad - 4\gamma\pi i(\gamma\tau + \delta) \vartheta_{g'_1 h'_1} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) \times \\
& \quad \left. \times \vartheta_{g'_2 h'_2} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) \right\}. \tag{2.4}
\end{aligned}$$

Replacing $N_1, g_1, h_1, H_1, g'_1, h'_1$ by $N_2, g_2, h_2, H_2, g'_2, h'_2$ in (2.4), and vice versa, we obtain

$$\begin{aligned}
& \frac{1}{N_2} (\gamma\tau + \delta)^3 \vartheta_{g_2 h_2}'' (\tau; 0, 2N_2) \vartheta_{g_1 h_1} (\tau; 0, 2N_1) = \\
& \quad = -e\left(\frac{3}{4} \operatorname{sgn} \gamma\right) (4N_1 N_2 \gamma^2)^{-1/2} \times \\
& \quad \times \sum_{\substack{H_1 \bmod 2N_1 \\ H_2 \bmod 2N_2}} \varphi_{g'_2 g_2 h_2} (0, H_2; 2N_2) \varphi_{g'_1 g_1 h_1} (0, H_1; 2N_1) \times \\
& \times \left\{ \frac{1}{N_2} \vartheta_{g'_2 h'_2}'' \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) \vartheta_{g'_1 h'_1} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) + \right. \\
& \quad - 4\gamma\pi i(\gamma\tau + \delta) \vartheta_{g'_2 h'_2} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) \times \\
& \quad \left. \times \vartheta_{g'_1 h'_1} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) \right\}. \tag{2.5}
\end{aligned}$$

Subtracting (2.5) from (2.4), we obtain

$$\begin{aligned}
& (\gamma\tau + \delta)^3 \left\{ \frac{1}{N_1} \vartheta_{g_1 h_1}'' (\tau; 0, 2N_1) \vartheta_{g_2 h_2} (\tau; 0, 2N_2) + \right. \\
& \quad \left. - \frac{1}{N_2} \vartheta_{g_2 h_2}'' (\tau; 0, 2N_2) \vartheta_{g_1 h_1} (\tau; 0, 2N_1) \right\} = \\
& \quad = -e\left(\frac{3}{4} \operatorname{sgn} \gamma\right) (4N_1 N_2 \gamma^2)^{-1/2} \times \\
& \quad \times \sum_{\substack{H_1 \bmod 2N_1 \\ H_2 \bmod 2N_2}} \varphi_{g'_1 g_1 h_1} (0, H_1; 2N_1) \varphi_{g'_2 g_2 h_2} (0, H_2; 2N_2) \times \\
& \times \left\{ \frac{1}{N_1} \vartheta_{g'_1 h'_1}'' \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) \vartheta_{g'_2 h'_2} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) + \right. \\
& \quad \left. - \frac{1}{N_2} \vartheta_{g'_2 h'_2}'' \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) \vartheta_{g'_1 h'_1} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) \right\}. \tag{2.6}
\end{aligned}$$

From Lemma 2 for $n = 0$, we have

$$(\gamma\tau + \delta)^{1/2} \vartheta_{g_3 h_3} (\tau; 0, 2N_3) = e(\operatorname{sgn} \gamma / 8) (2N_3 |\gamma|)^{-1/2} \times$$

$$\times \sum_{H_3 \bmod 2N_3} \varphi_{g'_3 g_3 h_3}(0, H_3; 2N_3) \vartheta_{g'_3 h'_3} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_3, 2N_3 \right). \quad (2.7)$$

Multiplying (2.6) by (2.7), according to (2.1), we obtain

$$\begin{aligned} & (\gamma\tau + \delta)^{7/2} \Omega(\tau; g_l, h_l, 0, N_l) = \\ & = -e(7 \operatorname{sgn} \gamma/8) (8N_1 N_2 N_3 |\gamma|^3)^{-1/2} \times \\ & \times \sum_{\substack{H_1 \bmod 2N_1 \\ H_2 \bmod 2N_2 \\ H_3 \bmod 2N_3}} \prod_{k=1}^3 \varphi_{g'_k g_k h_k}(0, H_k; 2N_k) \Omega \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; g'_l, h'_l, H_l, N_l \right). \end{aligned} \quad (2.8)$$

2. In (2.8) let γ be even. Then, by (1.7), g'_k ($k = 1, 2, 3$) are also even. Now in (1.2) and (1.3) instead of m let us introduce new letters of summations m_k defined by the equalities $m - H_k = 2N_k m_k$ ($k = r, t, 3$), respectively, and for brevity we put

$$T_k = \left(H_k + 2N_k m_k + \frac{g'_k}{2} \right)^2 \quad (k = r, t, 3).$$

Then, by (2.1), in (2.8) for $r = 1, t = 2$ and $r = 2, t = 1$, we get

$$\begin{aligned} & \vartheta''_{g'_r h'_r} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_r, 2N_r \right) \prod_{k=t,3} \vartheta_{g'_k h'_k} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_k, 2N_k \right) = \\ & = (\pi i)^2 \sum_{m_r=-\infty}^{\infty} (-1)^{h'_r m_r} (2(H_r + 2N_r m_r) + g'_r)^2 e \left(\frac{N/N_r}{4N} T_r \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) \times \\ & \quad \times \sum_{m_t=-\infty}^{\infty} (-1)^{h'_t m_t} e \left(\frac{N/N_t}{4N} T_t \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) \times \\ & \quad \times \sum_{m_3=-\infty}^{\infty} (-1)^{h'_3 m_3} e \left(\frac{N/N_3}{4N} T_3 \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) = \\ & = \sum_{n_r=0}^{\infty} B_{n_r} e \left(\frac{n_r}{4N} \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) \sum_{n_t=0}^{\infty} B_{n_t} e \left(\frac{n_t}{4N} \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) \times \\ & \quad \times \sum_{n_3=0}^{\infty} B_{n_3} e \left(\frac{n_3}{4N} \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) = \sum_{n=0}^{\infty} C_n^{(r,t)} e \left(\frac{n}{4N} \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right), \end{aligned} \quad (2.9)$$

since by (2.2) $n_k = \frac{N}{N_k} T_k$ are non-negative integers. Thus, for even γ , (2.3) follows from (2.1), (2.8), and (2.9).

In (2.8) let now γ be odd. If h_1, h_2, h_3 are even, then by (1.7), g'_1, g'_2, g'_3 are also even, and we obtain the same result. But if h_r is odd, then by

(1.7), h'_r and g'_r will also be odd, and in (1.2) we shall have

$$(m + g'_r/2)^2 = (m + (g'_r - 1)/2)^2 + (m + (g'_r - 1)/2) + 1/4.$$

Hence

$$\begin{aligned} & \vartheta''_{g'_r h'_r} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_r, 2N_r \right) = (\pi i)^2 e \left(\frac{h'_r}{16N_r} \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) \times \\ & \times \sum_{m_r=-\infty}^{\infty} (-1)^{h'_r m_r} (2(H_r + 2N_r m_r) + g'_r)^2 e \left(\frac{N/N_r}{4N} W_r \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) \quad (r=1, 2). \end{aligned}$$

Here by (1.4) we can imply that $h'_r = 1$, since by (1.7), h'_r and h_r have the same parity, and

$$\begin{aligned} W_r &= (H_r + 2N_r m_r + (g'_r - 1)/2)^2 + \\ &+ (H_r + 2N_r m_r + (g'_r - 1)/2) \quad (r = 1, 2). \end{aligned} \tag{2.10}$$

Analogously,

$$\begin{aligned} & \vartheta_{g'_t h'_t} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_t, 2N_t \right) = e \left(\frac{h'_t}{16N_t} \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) \times \\ & \times \sum_{m_t=-\infty}^{\infty} (-1)^{h'_t m_t} e \left(\frac{N/N_t}{4N} W_t \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) \quad (t = 2, 1). \end{aligned}$$

The same formula is valid for $t = 3$ with the same remarks concerning h'_3 as for h'_t , $t = 2, 1$. Further, for $k = t, 3$ we have

$$W_k = \begin{cases} T_k & \text{if } 2|g'_k, \\ (H_k + 2N_k m_k + (g'_k - 1)/2)^2 + \\ (H_k + 2N_k m_k + (g'_k - 1)/2) & \text{if } 2 \nmid g'_k. \end{cases} \tag{2.11}$$

Thus, if among h'_1, h'_2, h'_3 at least one is odd, then, as in (2.9), for $r = 1$, $t = 2$ and $r = 2, t = 1$ we have

$$\begin{aligned} & \vartheta''_{g'_r h'_r} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_r, 2N_r \right) \prod_{k=t,3} \vartheta_{g'_k h'_k} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_k, 2N_k \right) = \\ & = e \left(\frac{1}{4N} \left(\frac{1}{4} N \sum_{k=1}^3 \frac{h'_k}{N_k} \right) \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) \sum_{n_r=0}^{\infty} B'_{n_r} e \left(\frac{n_r}{4N} \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) \times \\ & \times \sum_{n_t=0}^{\infty} B'_{n_t} e \left(\frac{n_t}{4N} \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) \sum_{n_3=0}^{\infty} B'_{n_3} e \left(\frac{n_3}{4N} \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) = \\ & = e \left(\frac{1}{4N} \left(\frac{1}{4} N \sum_{k=1}^3 \frac{h'_k}{N_k} \right) \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) \sum_{n'=0}^{\infty} D_{n'}^{(r,t)} e \left(\frac{n'}{4N} \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) = \end{aligned}$$

$$= \sum_{n=0}^{\infty} D_n^{(r,t)} e\left(\frac{n}{4N} \frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right), \tag{2.12}$$

since, by (2.2), (2.10), and (2.11), $n_k = \frac{N}{N_k} W_k$ and $n = \frac{1}{4}N \sum_{k=1}^3 \frac{h'_k}{N_k} + n'$ are non-negative integers. Thus, for odd γ , (2.3) follows from (2.1), (2.8), and (2.12). \square

Theorem 1. *For given N the function $\Omega(\tau; g_l, h_l, 0, N_l)$ is an entire modular form of weight $7/2$ and of the multiplier system*

$$v(M) = i^{3\eta(\gamma)(\text{sgn } \delta - 1)/2} i^{3(|\delta| - 1)^2/4} \left(\frac{\beta\Delta \text{sgn } \delta}{|\delta|}\right) \tag{2.13}$$

($M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is a matrix of the substitution from $\Gamma_0(4N)$ and Δ is the determinant of an arbitrary positive quadratic form with integer coefficients in 7 variables) on $\Gamma_0(4N)$ if the following conditions hold:

$$(1) \quad 2|g_k, N_k|N \quad (k = 1, 2, 3), \tag{2.14}$$

$$(2) \quad 4|N \sum_{k=1}^3 \frac{h_k^2}{N_k} \quad 4| \sum_{k=1}^s \frac{g_k^2}{4N_k}, \tag{2.15}$$

3) for all α and δ with $\alpha\delta \equiv 1 \pmod{4N}$

$$\left(\frac{N_1 N_2 N_3}{|\delta|}\right) \Omega(\tau; \alpha g_l, h_l, 0, N_l) = \left(\frac{\Delta}{|\delta|}\right) \Omega(\tau; g_l, h_l, 0, N_l). \tag{2.16}$$

Proof. 1. It is well known that theta-series (1.1)–(1.2) are regular on \mathcal{H} ; hence the function $\Omega(\tau; g_l, h_l, 0, N_l)$ satisfies condition (1) of the Definition.

2. It is easily verified that (2.15) implies

$$4|N\delta^2 \sum_{k=1}^3 h_k^2/N_k, \quad 4| \sum_{k=1}^3 \delta^{2\varphi(2N_k)-2} g_k^2/4N_k, \tag{2.17}$$

since $2 \nmid \delta$, because $\alpha\delta \equiv 1 \pmod{4N}$.

From (2.14) it follows that

$$\Gamma_0(4N) \subset \Gamma_0(4N_k) \quad (k = 1, 2, 3). \tag{2.18}$$

By Lemma 1 for $n = 2$ and $n = 0$, according to (2.17) and (2.18), for all substitutions from $\Gamma_0(4N)$, we have

$$\begin{aligned} \vartheta''_{g_r h_r} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_r\right) &= i^{5\eta(\gamma)(\text{sgn } \delta - 1)/2} i^{(1-|\delta|)/2} \times \\ &\times \left(\frac{2\beta N_r \text{sgn } \delta}{|\delta|}\right) (\gamma\tau + \delta)^{5/2} \vartheta''_{\alpha g_r, h_r}(\tau; 0, 2N_r), \end{aligned} \tag{2.19}$$

$$\begin{aligned} \vartheta_{g_t h_t} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_t \right) &= i^{\eta(\gamma)(\text{sgn } \delta - 1)/2} i^{(1-|\delta|)/2} \times \\ &\times \left(\frac{2\beta N_t \text{sgn } \delta}{|\delta|} \right) (\gamma\tau + \delta)^{1/2} \vartheta_{\alpha g_t, h_t}(\tau; 0, 2N_t), \end{aligned} \tag{2.20}$$

$$\begin{aligned} \vartheta_{g_3 h_3} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_3 \right) &= i^{\eta(\gamma)(\text{sgn } \delta - 1)/2} i^{(1-|\delta|)/2} \times \\ &\times \left(\frac{2\beta N_3 \text{sgn } \delta}{|\delta|} \right) (\gamma\tau + \delta)^{1/2} \vartheta_{\alpha g_3, h_3}(\tau; 0, 2N_3). \end{aligned} \tag{2.21}$$

Thus for all substitutions from $\Gamma_0(4N)$, if $r = 1, t = 2$ and $r = 2, t = 1$, we have

$$\begin{aligned} &\vartheta''_{g_r h_r} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_r \right) \prod_{k=t,3} \vartheta_{g_k h_k} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_k \right) = \\ &= i^{7\eta(\gamma)(\text{sgn } \delta - 1)/2} i^{3(1-|\delta|)/2} \left(\frac{2\beta \text{sgn } \delta}{|\delta|} \right) \left(\frac{N_r N_t N_3}{|\delta|} \right) (\gamma\tau + \delta)^{7/2} \times \\ &\times \vartheta''_{\alpha g_r, h_r} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_r \right) \prod_{k=t,3} \vartheta_{\alpha g_k, h_k} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_k \right). \end{aligned} \tag{2.22}$$

It is not difficult to verify that

$$i^{3(1-|\delta|)/2} \left(\frac{2}{|\delta|} \right) = i^{3(|\delta|-1)^2/4}. \tag{2.23}$$

Hence, by (2.1), (2.22), (2.23), (2.13), and (2.16) we get

$$\begin{aligned} \Omega \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; g_l, h_l, 0, N_l \right) &= i^{3\eta(\gamma)(\text{sgn } \delta - 1)/2} i^{3(|\delta|-1)^2/4} \times \\ &\times \left(\frac{\beta \text{sgn } \delta}{|\delta|} \right) \left(\frac{N_1 N_2 N_3}{|\delta|} \right) (\gamma\tau + \delta)^{7/2} \Omega(\tau; \alpha g_l, h_l, 0, N_l) = \\ &= v(M) (\gamma\tau + \delta)^{7/2} \Omega(\tau; g_l, h_l, 0, N_l). \end{aligned}$$

Thus, the function $\Omega(\tau; g_l, h_l, 0, N_l)$ satisfies condition (2) of the Definition.

3. From (1.6) it follows for $r = 1, t = 2$ and $r = 2, t = 1$ that

$$\begin{aligned} &\vartheta''_{g_r h_r}(\tau; 0, 2N_r) \vartheta_{g_t h_t}(\tau; 0, 2N_t) \vartheta_{g_3 h_3}(\tau; 0, 2N_3) = \\ &= (\pi i)^2 \sum_{m_r, m_t, m_3 = -\infty}^{\infty} (-1)^{h_r m_r + h_t m_t + h_3 m_3} (4N_r m_r + g_r)^2 \times \\ &\times e \left(\sum_{k=r,t,3} \frac{1}{4N_k} (2N_k m_k + g_k/2)^2 \tau \right) = \sum_{n=0}^{\infty} B_n^{(r,t)} e(n\tau), \end{aligned} \tag{2.24}$$

where

$$\begin{aligned}
 n &= \sum_{k=1}^3 \frac{1}{4N_k} (2N_k m_k + g_k/2)^2 = \\
 &= \sum_{k=1}^3 (N_k m_k^2 + m_k g_k/2) + \frac{1}{4} \sum_{k=1}^3 g_k^2/4N_k,
 \end{aligned}$$

by (2.14) and (2.15), is a non-negative integer. Thus, by (2.1) and (2.24), the function $\Omega(\tau; g_l, h_l, 0, N_l)$ satisfies condition (3) of the Definition.

4. By Lemma 3, the function $\Omega(\tau; g_l, h_l, 0, N_l)$ satisfies condition (4) of the Definition. \square

3.

Lemma 4. *For given N let*

$$\begin{aligned}
 \Psi_1(\tau; g_l, h_l, c_l, N_l) &= \Psi_1(\tau; g_1, g_2, g_3; h_1, h_2, h_3; c_1, c_2, c_3; N_1, N_2, N_3) = \\
 &= \left\{ \frac{1}{N_1} \vartheta''_{g_1 h_1}(\tau; c_1, 2N_1) \vartheta_{g_2 h_2}(\tau; c_2, 2N_2) + \right. \\
 &\quad \left. - \frac{1}{N_2} \vartheta''_{g_2 h_2}(\tau; c_2, 2N_2) \vartheta_{g_1 h_1}(\tau; c_1, 2N_1) \right\} \vartheta'_{g_3 h_3}(\tau; c_3, 2N_3) \quad (3.1)
 \end{aligned}$$

and

$$\begin{aligned}
 \Psi_2(\tau; g_l, h_l, c_l, N_l) &= \Psi_2(\tau; g_1, g_2, g_3; h_1, h_2, h_3; c_1, c_2, c_3; N_1, N_2, N_3) = \\
 &= \left\{ \frac{1}{N_1} \vartheta'''_{g_1 h_1}(\tau; c_1, 2N_1) \vartheta_{g_2 h_2}(\tau; c_2, 2N_2) + \right. \\
 &\quad \left. - \frac{3}{N_2} \vartheta''_{g_2 h_2}(\tau; c_2, 2N_2) \vartheta'_{g_1 h_1}(\tau; c_1, 2N_1) \right\} \vartheta_{g_3 h_3}(\tau; c_3, 2N_3), \quad (3.2)
 \end{aligned}$$

where

$$2|g_k, \quad N_k|N \quad (k = 1, 2, 3), \quad 4|N \sum_{k=1}^3 \frac{h_k}{N_k}. \quad (3.3)$$

Then for all substitutions from Γ , in the neighborhood of each rational point $\tau = -\frac{\delta}{\gamma}$ ($\gamma \neq 0, (\gamma, \delta) = 1$), we have

$$(\gamma\tau + \delta)^{9/2} \Psi_j(\tau; g_l, h_l, 0, N_l) = \sum_{n=0}^{\infty} G_n^{(j)} e\left(\frac{n}{4N} \frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right) \quad (j = 1, 2). \quad (3.4)$$

Proof. 1. Formula (2.6) is obtained in Lemma 3. From Lemma 2, for $n = 1$, we have

$$\begin{aligned}
 (\gamma\tau + \delta)^{3/2} \vartheta'_{g_3 h_3}(\tau; 0, 2N_3) &= -e(3 \operatorname{sgn} \gamma/8)(2N_3|\gamma|)^{-1/2}(i \operatorname{sgn} \gamma) \times \\
 &\times \sum_{H_3 \bmod 2N_3} \varphi_{g'_3 g_3 h_3}(0, H_3; 2N_3) \vartheta'_{g'_3 h'_3} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_3, 2N_3 \right). \quad (3.5)
 \end{aligned}$$

Multiplying (2.6) by (3.5), according to (3.1), we obtain

$$\begin{aligned}
 &(\gamma\tau + \delta)^{9/2} \Psi_1(\tau; g_l, h_l, 0, N_l) = \\
 &= e(9 \operatorname{sgn} \gamma/8)(8N_1 N_2 N_3 |\gamma|^3)^{-1/2}(i \operatorname{sgn} \gamma) \times \\
 &\times \sum_{\substack{H_1 \bmod 2N_1 \\ H_2 \bmod 2N_2 \\ H_3 \bmod 2N_3}} \prod_{k=1}^3 \varphi_{g'_k g_k h_k}(0, H_k; 2N_k) \Psi_1 \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; g'_l, h'_l, H_l, N_l \right). \quad (3.6)
 \end{aligned}$$

Reasoning further just as in Lemma 3.2, but taking everywhere $\vartheta'_{g_3 h_3} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_3, 2N_3 \right)$ instead of $\vartheta_{g_3 h_3} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_3, 2N_3 \right)$, from (3.1) and (3.6) we obtain (3.4) if $j = 1$.

2. From Lemma 2 for $n = 3$, $n = 0$, and respectively for $n = 2$, $n = 1$, it follows that

$$\begin{aligned}
 &\frac{1}{N_1} (\gamma\tau + \delta)^4 \vartheta'''_{g_1 h_1}(\tau; 0, 2N_1) \vartheta_{g_2 h_2}(\tau; 0, 2N_2) = \\
 &= \frac{1}{N_1} e(\operatorname{sgn} \gamma)(4N_1 N_2 \gamma^2)^{-1/2}(i \operatorname{sgn} \gamma) \times \\
 &\times \sum_{H_1 \bmod 2N_1} \varphi_{g'_1 g_1 h_1}(0, H_1; 2N_1) \left\{ \vartheta'''_{g'_1 h'_1} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) + \right. \\
 &\quad \left. + \sum_{t=1}^3 \binom{3}{t} \sum_{k=1}^t \frac{A_{tk}}{k!} \Big|_{z=0} \vartheta^{(3-t)}_{g'_1 h'_1} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) \right\} \times \\
 &\times \sum_{H_2 \bmod 2N_2} \varphi_{g'_2 g_2 h_2}(0, H_2; 2N_2) \vartheta_{g'_2 h'_2} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) = \\
 &= e(\operatorname{sgn} \gamma)(i \operatorname{sgn} \gamma)(4N_1 N_2 \gamma^2)^{-1/2} \sum_{\substack{H_1 \bmod 2N_1 \\ H_2 \bmod 2N_2}} \varphi_{g'_1 g_1 h_1}(0, H_1; 2N_1) \times \\
 &\times \varphi_{g'_2 g_2 h_2}(0, H_2; 2N_2) \left\{ \frac{1}{N_1} \vartheta'''_{g'_1 h'_1} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) \vartheta_{g'_2 h'_2}(\tau; 0, 2N_2) + \right. \\
 &\quad \left. - 12\gamma\pi i(\gamma\tau + \delta) \vartheta'_{g'_1 h'_1} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) \vartheta_{g'_2 h'_2}(\tau; 0, 2N_2) \right\} \quad (3.7)
 \end{aligned}$$

and

$$\begin{aligned}
 & \frac{3}{N_2}(\gamma\tau + \delta)^4 \vartheta''_{g_2 h_2}(\tau; 0, 2N_2) \vartheta'_{g_1 h_1}(\tau; 0, 2N_1) = \\
 & = e(\operatorname{sgn} \gamma)(i \operatorname{sgn} \gamma)(4N_1 N_2 \gamma^2)^{-1/2} \sum_{\substack{H_1 \bmod 2N_1 \\ H_2 \bmod 2N_2}} \varphi_{g'_1 g_1 h_1}(0, H_1; 2N_1) \times \\
 & \quad \times \varphi_{g'_2 g_2 h_2}(0, H_2; 2N_2) \left\{ \frac{3}{N_2} \vartheta''_{g'_2 h'_2} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) + \right. \\
 & \quad \left. - 12\gamma\pi i(\gamma\tau + \delta) \vartheta'_{g'_2 h'_2} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) \right\} \vartheta'_{g'_1 h'_1}(\tau; 0, 2N_1). \quad (3.8)
 \end{aligned}$$

Subtracting (3.8) from (3.7) and multiplying the obtained result by (2.7), according to (3.2), we get

$$\begin{aligned}
 & (\gamma\tau + \delta)^{9/2} \Psi_2(\tau; g_l, h_l, 0, N_l) = \\
 & = e(9 \operatorname{sgn} \gamma/8)(8N_1 N_2 N_3 |\gamma|^3)^{-1/2} (i \operatorname{sgn} \gamma) \times \\
 & \times \sum_{\substack{H_1 \bmod 2N_1 \\ H_2 \bmod 2N_2 \\ H_3 \bmod 2N_3}} \prod_{k=1}^3 \varphi_{g'_k g_k h_k}(0, H_k; 2N_k) \Psi_2 \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; g'_l, h'_l, H_l, N_l \right). \quad (3.9)
 \end{aligned}$$

In (3.9) let γ be even. Then, reasoning as in Lemma 3.2, by (3.2), from (3.9) we obtain

$$\begin{aligned}
 & \left\{ \frac{1}{N_1} \vartheta'''_{g'_1 h'_1} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) \vartheta_{g'_2 h'_2} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) + \right. \\
 & \left. - \frac{3}{N_2} \vartheta''_{g'_2 h'_2} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) \vartheta'_{g'_1 h'_1} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) \right\} \times \\
 & \quad \times \vartheta_{g'_3 h'_3} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_3, 2N_3 \right) = \left\{ \frac{(\pi i)^3}{N_1} \times \right. \\
 & \times \sum_{m_1=-\infty}^{\infty} (-1)^{h'_1 m_1} (2(H_1 + 2N_1 m_1) + g'_1)^3 e \left(\frac{N/N_1}{4N} T_1 \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) \times \\
 & \quad \times \sum_{m_2=-\infty}^{\infty} (-1)^{h'_2 m_2} e \left(\frac{N/N_2}{4N} T_2 \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) + \\
 & \left. - \frac{3(\pi i)^3}{N_1} \sum_{m_2=-\infty}^{\infty} (-1)^{h'_2 m_2} (2(H_2 + 2N_2 m_2) + g'_2)^2 e \left(\frac{N/N_2}{4N} T_2 \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) \times \right. \\
 & \left. \times \sum_{m_1=-\infty}^{\infty} (-1)^{h'_1 m_1} (2(H_1 + 2N_1 m_1) + g'_1) e \left(\frac{N/N_1}{4N} T_1 \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) \right\} \times
 \end{aligned}$$

$$\begin{aligned}
 & \times \sum_{m_3=-\infty}^{\infty} (-1)^{h'_3 m_3} e\left(\frac{N/N_3}{4N} T_3 \frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right) = \\
 & = \left\{ \sum_{n_1=0}^{\infty} B_{n_1} e\left(\frac{n_1}{4N} \frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right) \sum_{n_2=0}^{\infty} B_{n_2} e\left(\frac{n_2}{4N} \frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right) + \right. \\
 & \left. - \sum_{n_2=0}^{\infty} B'_{n_2} e\left(\frac{n_2}{4N} \frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right) \sum_{n_1=0}^{\infty} B'_{n_1} e\left(\frac{n_1}{4N} \frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right) \right\} \times \\
 & \times \sum_{n_3=0}^{\infty} B_{n_3} e\left(\frac{n_3}{4N} \frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right) = \sum_{n=0}^{\infty} C_n e\left(\frac{n}{4N} \frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right). \tag{3.10}
 \end{aligned}$$

Here T_k and n_k have the same meaning as in Lemma 3.2.

Thus, for even γ , (3.4) follows from (3.2), (3.9) and (3.10) if $j = 2$.

Further, reasoning as in Lemma 3.2, we obtain (3.4) for odd γ if $j = 2$. \square

Theorem 2. For given N the functions $\Psi_1(\tau; g_l, h_l, 0, N_l)$ and $\Psi_2(\tau; g_l, h_l, 0, N_l)$ are entire modular forms of weight $9/2$ and of the multiplier system

$$v(M) = i^{\eta(\gamma)(\text{sgn } \delta - 1)/2} i^{(|\delta| - 1)^2/4} \left(\frac{\beta \Delta \text{sgn } \delta}{|\delta|} \right) \tag{3.11}$$

($M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is the matrix of the substitution from $\Gamma_0(4N)$ and Δ is the determinant of an arbitrary positive quadratic form with integer coefficients in 9 variables) on $\Gamma_0(4N)$ if the following conditions hold:

$$(1) \quad 2|g_k, \quad N_k|N \quad (k = 1, 2, 3), \tag{3.12}$$

$$(2) \quad 4|N \sum_{k=1}^3 \frac{h_k^2}{N_k} \quad 4| \sum_{k=1}^s \frac{g_k^2}{4N_k}, \tag{3.13}$$

(3) for all α and δ with $\alpha\delta \equiv 1 \pmod{4N}$

$$\begin{aligned}
 & \text{sgn } \delta \left(\frac{N_1 N_2 N_3}{|\delta|} \right) \Psi_j(\tau; \alpha g_l, h_l, 0, N_l) = \\
 & = \left(\frac{-\Delta}{|\delta|} \right) \Psi_j(\tau; g_l, h_l, 0, N_l) \quad (j = 1, 2). \tag{3.14}
 \end{aligned}$$

Proof. 1. As in the case of Theorem 1, the functions $\Psi_1(\tau; g_l, h_l, 0, N_l)$ and $\Psi_2(\tau; g_l, h_l, 0, N_l)$ satisfy condition (1) and, by Lemma 4, also condition (4) of the Definition.

2. By Lemma 1 for $n = 3$ and $n = 1$, according to (2.17) and (2.18), for all substitutions from $\Gamma_0(4N)$, we have

$$\vartheta_{g_1 h_1}''' \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_1 \right) = \text{sgn } \delta i^{7\eta(\gamma)(\text{sgn } \delta - 1)/2} i^{(1 - |\delta|)/2} \times$$

$$\times \left(\frac{2\beta N_1 \operatorname{sgn} \delta}{|\delta|} \right) (\gamma\tau + \delta)^{7/2} \vartheta'''_{\alpha_{g_1, h_1}}(\tau; 0, 2N_1), \quad (3.15)$$

$$\begin{aligned} \vartheta'_{g_1 h_1} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_1 \right) &= \operatorname{sgn} \delta i^{3\eta(\gamma)(\operatorname{sgn} \delta - 1)/2} i^{(1-|\delta|)/2} \times \\ &\times \left(\frac{2\beta N_1 \operatorname{sgn} \delta}{|\delta|} \right) (\gamma\tau + \delta)^{3/2} \vartheta'_{\alpha_{g_1, h_1}}(\tau; 0, 2N_1), \end{aligned} \quad (3.16)$$

$$\begin{aligned} \vartheta'_{g_3 h_3} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_3 \right) &= \operatorname{sgn} \delta i^{3\eta(\gamma)(\operatorname{sgn} \delta - 1)/2} i^{(1-|\delta|)/2} \times \\ &\times \left(\frac{2\beta N_3 \operatorname{sgn} \delta}{|\delta|} \right) (\gamma\tau + \delta)^{3/2} \vartheta'_{\alpha_{g_3, h_3}}(\tau; 0, 2N_3). \end{aligned} \quad (3.17)$$

Thus, for all substitutions from $\Gamma_0(4N)$ we respectively get:

(1) by (2.19), (2.20), and (3.17)

$$\begin{aligned} \vartheta''_{g_r h_r} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_r \right) \vartheta_{g_t h_t} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_t \right) \vartheta'_{g_3 h_3} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_3 \right) &= \\ &= \operatorname{sgn} \delta i^{\eta(\gamma)(\operatorname{sgn} \delta - 1)/2} i^{3(1-|\delta|)/2} \left(\frac{2\beta \operatorname{sgn} \delta}{|\delta|} \right) \left(\frac{N_1 N_2 N_3}{|\delta|} \right) \times \\ &\times (\gamma\tau + \delta)^{9/2} \vartheta''_{\alpha_{g_r, h_r}}(\tau; 0, 2N_r) \vartheta_{\alpha_{g_t, h_t}}(\tau; 0, 2N_t) \vartheta'_{\alpha_{g_3, h_3}}(\tau; 0, 2N_3) \end{aligned} \quad (3.18)$$

for $r = 1, t = 2$ and $r = 2, t = 1$;

(2) by (3.15), (2.20) for $t = 2$ and (2.21)

$$\begin{aligned} \vartheta'''_{g_1 h_1} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_1 \right) \vartheta_{g_2 h_2} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_2 \right) \vartheta_{g_3 h_3} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_3 \right) &= \\ &= \operatorname{sgn} \delta i^{\eta(\gamma)(\operatorname{sgn} \delta - 1)/2} i^{3(1-|\delta|)/2} \left(\frac{2\beta \operatorname{sgn} \delta}{|\delta|} \right) \left(\frac{N_1 N_2 N_3}{|\delta|} \right) \times \\ &\times (\gamma\tau + \delta)^{9/2} \vartheta'''_{\alpha_{g_1, h_1}}(\tau; 0, 2N_1) \vartheta_{\alpha_{g_2, h_2}}(\tau; 0, 2N_2) \vartheta_{\alpha_{g_3, h_3}}(\tau; 0, 2N_3); \end{aligned} \quad (3.19)$$

(3) by (2.19) for $r = 2$, (3.16), and (2.21)

$$\begin{aligned} \vartheta''_{g_2 h_2} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_2 \right) \vartheta'_{g_1 h_1} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_1 \right) \vartheta_{g_3 h_3} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_3 \right) &= \\ &= \operatorname{sgn} \delta i^{\eta(\gamma)(\operatorname{sgn} \delta - 1)/2} i^{3(1-|\delta|)/2} \left(\frac{2\beta \operatorname{sgn} \delta}{|\delta|} \right) \left(\frac{N_1 N_2 N_3}{|\delta|} \right) \times \\ &\times (\gamma\tau + \delta)^{9/2} \vartheta''_{\alpha_{g_2, h_2}}(\tau; 0, 2N_2) \vartheta'_{\alpha_{g_1, h_1}}(\tau; 0, 2N_1) \vartheta_{\alpha_{g_3, h_3}}(\tau; 0, 2N_3). \end{aligned} \quad (3.20)$$

It is not difficult to verify that

$$i^{3(1-|\delta|)/2} \left(\frac{-2}{|\delta|} \right) = i^{(|\delta|-1)^2/4}. \quad (3.21)$$

Hence, by (3.1), (3.18), (3.21), (3.11), (3.14) (if $j = 1$) and (3.2), (3.19)–(3.21), (3.11), (3.14) (if $j = 2$) we get

$$\begin{aligned} \Psi_j\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; g_l, h_l, 0, N_l\right) &= \operatorname{sgn} \delta i^{\eta(\gamma)(\operatorname{sgn} \delta - 1)/2} i^{(|\delta| - 1)^2/4} \times \\ &\times \left(\frac{-\beta \operatorname{sgn} \delta}{|\delta|}\right) \left(\frac{N_1 N_2 N_3}{|\delta|}\right) (\gamma\tau + \delta)^{9/2} \Psi_j(\tau; \alpha g_l, h_l, 0, N_l) = \\ &= v(M)(\gamma\tau + \delta)^{9/2} \Psi_j(\tau; g_l, h_l, 0, N_l) \quad (j = 1, 2). \end{aligned}$$

Thus the functions $\Psi_j(\tau; g_l, h_l, 0, N_l)$ ($j = 1, 2$) satisfy condition (2) of the Definition.

From (1.6) it follows that

$$\begin{aligned} &\vartheta''_{g_r h_r}(\tau; 0, 2N_r) \vartheta_{g_t h_t}(\tau; 0, 2N_t) \vartheta'_{g_3 h_3}(\tau; 0, 2N_3) = \\ &= (\pi i)^3 \sum_{m_r, m_t, m_3 = -\infty}^{\infty} (-1)^{\sum_{k=r,t,3} h_k m_k} (4N_r m_r + g_r)^2 (4N_3 m_3 + g_3) \times \\ &\times e\left(\sum_{k=r,t,3} \frac{1}{4N_k} (2N_k m_k + g_k/2)^2 \tau\right) = \sum_{n=0}^{\infty} C_n^{(r,t)} e(n\tau) \end{aligned}$$

for $r = 1, t = 2$ and $r = 2, t = 1$,

$$\begin{aligned} &\vartheta'''_{g_1 h_1}(\tau; 0, 2N_1) \vartheta_{g_2 h_2}(\tau; 0, 2N_2) \vartheta_{g_3 h_3}(\tau; 0, 2N_3) = \\ &= (\pi i)^3 \sum_{m_1, m_2, m_3 = -\infty}^{\infty} (-1)^{\sum_{k=1}^3 h_k m_k} (4N_1 m_1 + g_1)^3 \times \\ &\times e\left(\sum_{k=1}^3 \frac{1}{4N_k} (2N_k m_k + g_k/2)^2 \tau\right) = \sum_{n=0}^{\infty} C_n e(n\tau) \end{aligned}$$

and

$$\begin{aligned} &\vartheta''_{g_2 h_2}(\tau; 0, 2N_2) \vartheta'_{g_1 h_1}(\tau; 0, 2N_1) \vartheta_{g_3 h_3}(\tau; 0, 2N_3) = \\ &= (\pi i)^3 \sum_{m_1, m_2, m_3 = -\infty}^{\infty} (-1)^{\sum_{k=1}^3 h_k m_k} (4N_2 m_2 + g_2)^2 (4N_1 m_1 + g_1) \times \\ &\times e\left(\sum_{k=1}^3 \frac{1}{4N_k} (2N_k m_k + g_k/2)^2 \tau\right) = \sum_{n=0}^{\infty} D_n e(n\tau), \end{aligned}$$

since in all these expansions

$$n = \sum_{k=1}^3 \frac{1}{4N_k} (2N_k m_k + g_k/2)^2,$$

as has been shown by Theorem 1.3, is a non-negative integer. Therefore, in view of (3.1) and (3.2), the functions $\Psi_j(\tau; g_l, h_l, 0, N_l)$ ($j = 1, 2$) satisfy condition (3) of the Definition. \square

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