GEORGIAN MATHEMATICAL JOURNAL: Vol. 3, No. 5, 1996, 475-484

ON THE CORRECTNESS OF THE CAUCHY PROBLEM FOR A LINEAR DIFFERENTIAL SYSTEM ON AN INFINITE INTERVAL

I. KIGURADZE

Abstract. The conditions ensuring the correctness of the Cauchy problem

$$\frac{dx}{dt} = \mathcal{P}(t)x + q(t), \quad x(t_0) = c_0$$

on the nonnegative half-axis R_+ are found, where $\mathcal{P} : R_+ \to R^{n\times n}$ and $q : R_+ \to R^n$ are locally summable matrix and vector functions, respectively, $t_0 \in R_+$ and $c_0 \in R^n$.

Consider the differential system

$$\frac{dx}{dt} = \mathcal{P}(t)x + q(t) \tag{1}$$

with the initial condition

$$x(t_0) = c_0 \tag{2}$$

where $\mathcal{P} : R_+ \to R^{n\times n}$ and $q : R_+ \to R^n$ are, respectively, the matrix and the vector functions with the components summable on every finite interval, $t_0 \in R_+$ and $c_0 \in R^n$. It is known [1,2] that problem (1), (2) for arbitrarily fixed $a \in]0, +\infty[$ is correct on the interval $[0, a]$, i.e., its solution on this interval is stable with respect to small, in an integral sense, perturbations $\mathcal{P}$ and q and small perturbations t_0 and c_0. In the paper under consideration we establish sufficient conditions for problem (1), (2) to be correct on R_+.

We shall use the following notation:

R is the set of real numbers, $R_+ = [0, +\infty[$;

1991 *Mathematics Subject Classification.* 34B05, 35D10.

Key words and phrases. Cauchy problem for a linear differential system, correct problem, weakly correct problem.

1072-947X/96/0900-0475$12.50/0

R^n is the space of n-dimensional column vectors $x = (x_i)_{i=1}^n$ with real components x_i $(i = 1, \dots, n)$ and the norm

$$\|x\| = \sum_{i=1}^n |x_i|;$$

$R^{n\times n}$ is the space of matrices $X = (x_{ik})_{i,k=1}^n$ with real components x_{ik} $(i, k = 1, \dots, n)$ and the norm

$$\|X\| = \sum_{i,k=1}^n |x_{ik}|;$$

$L_{loc}(R_+; R^n)$ is the space of vector functions $g : R_+ \to R^n$ whose components are summable on the segment $[0, a]$ for every $a \in]0, +\infty[$;

$L_{loc}(R_+; R^{n\times n})$ is the space of matrix functions $G : R_+ \to R^{n\times n}$ whose components are summable on the segment $[0, a]$ for every $a \in]0, +\infty[$.

Along with problem (1), (2) let us consider the perturbed problem

$$\frac{dy}{dt} = \widetilde{\mathcal{P}}(t)x + \widetilde{q}(t), \tag{3}$$

$$y(\widetilde{t}_0) = \widetilde{c}_0 \tag{4}$$

and introduce the following

Definition 1. Problem (1), (2) is *correct* if for every arbitrarily small $\varepsilon > 0$ and arbitrarily large $\rho > 0$ there exists $\delta > 0$ such that for any $\widetilde{t}_0 \in R_+$, $\widetilde{c}_0 \in R^n$, $\widetilde{\mathcal{P}} \in L_{loc}(R_+; R^{n\times n})$, and $\widetilde{q} \in L_{loc}(R_+; R^n)$ satisfying the conditions

$$|t_0 - \widetilde{t}_0| < \delta, \qquad \|c_0 - \widetilde{c}_0\| < \delta, \tag{5}$$

$$\Big\| \int_{t_0}^t [\widetilde{\mathcal{P}}(s) - \mathcal{P}(s)]ds \Big\| < \delta, \quad \Big\| \int_{t_0}^t [\widetilde{q}(s) - q(s)]ds \Big\| < \delta \quad \text{for } t \in R_+ \tag{6}$$

and

$$\int_0^{+\infty} \|\widetilde{\mathcal{P}}(s) - \mathcal{P}(s)\|ds < \rho, \tag{7}$$

the inequality

$$\|x(t) - y(t)\| < \varepsilon \quad \text{for} \quad t \in R_+ \tag{8}$$

holds, where x and y are the solutions of problems (1), (2) and (3), (4), respectively.

Definition 2. Problem (1), (2) is *weakly correct* if for arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that for any $\widetilde{t}_0 \in R_+$, $\widetilde{c}_0 \in R^n$, $\widetilde{\mathcal{P}} \in L_{loc}(R_+; R^{n\times n})$, $\widetilde{q} \in L_{loc}(R_+; R^n)$ satisfying the conditions (5) and

$$\int_0^{+\infty} \big\|\widetilde{\mathcal{P}}(s) - \mathcal{P}(s)\big\| ds < \delta, \qquad \int_0^{+\infty} \big\|\widetilde{q}(s) - q(s)\big\| ds < \delta, \tag{6'}$$

inequality (8) holds, where x and y are the solutions of problems (1), (2) and (3), (4), respectively.

Theorem 1. *If*

$$\int_0^{+\infty} \|\mathcal{P}(t)\|\, dt < +\infty, \qquad \int_0^{+\infty} \|q(t)\|\, dt < +\infty, \tag{9}$$

then problem (1)*,* (2) *is correct.*

Proof. According to the conditions (9),

$$\|x(t)\| \le \rho_* \quad \text{for} \quad t \in R_+$$

and

$$\|x(t)\| + \int_0^{+\infty} \|x'(\tau)\|\, d\tau \le \rho_0 \quad \text{for} \quad t \in R_+, \tag{10}$$

where

$$\rho_* = \bigg(\|c_0\| + \int_0^{+\infty} \|q(\tau)\|\, d\tau\bigg) \exp(\rho_1), \quad \rho_0 = \int_0^{+\infty} \big(\|\mathcal{P}(\tau)\|\rho_* + \|q(\tau)\|\big)\, d\tau + \rho_*$$

and

$$\rho_1 = \int_0^{+\infty} \|\mathcal{P}(\tau)\|\, d\tau. \tag{11}$$

Therefore

$$\omega(\eta) = \sup\bigg\{ \int_s^t \|x'(\tau)\|\, d\tau : 0 \le s \le t < +\infty,\ \ t - s \le \eta \bigg\} \to \\ \to 0 \quad \text{for} \quad \eta \to 0. \tag{12}$$

Because of (12) for arbitrarily given $\varepsilon > 0$ and $\rho > 0$ there exists $\delta > 0$ such that

$$\big[2(1+\rho_0)\delta + \omega(\delta)\big]\exp(\rho+\rho_1) < \varepsilon. \tag{13}$$

Consider problem (3), (4), where $\widetilde{t}_0 \in R_+$, $\widetilde{c}_0 \in R^n$. $\widetilde{\mathcal{P}} \in L_{loc}(R_+; R^{n\times n})$ and $\widetilde{q} \in L_{loc}(R_+; R^n)$ satisfy the conditions (5)–(7).

Let y be the solution of problem (3), (4), and let

$$z(t) = y(t) - x(t).$$

Then

$$z'(t) = \widetilde{\mathcal{P}}(t)z(t) + [\widetilde{\mathcal{P}}(t) - \mathcal{P}(t)]x(t) + \widetilde{q}(t) - q(t). \tag{14}$$

On the other hand, because of (5),

$$\begin{aligned} \|z(\widetilde{t}_0)\| = \|\widetilde{c}_0 - x(\widetilde{t}_0)\| = \|\widetilde{c}_0 - c_0 + x(t_0) - x(\widetilde{t}_0)\| \le \\ \le \|\widetilde{c}_0 - c_0\| + \|x(\widetilde{t}_0) - x(t_0)\| \le \delta + \omega(\delta). \end{aligned} \tag{15}$$

If we integrate equality (14) from $\widetilde{t}_0$ to t, then taking into account (6) and (15), we find

$$\begin{aligned} \|z(t)\| \le 2\delta + \omega(\delta) + \bigg|\int_{\widetilde{t}_0}^{t} \|\widetilde{\mathcal{P}}(s)\|\,\|z(s)\|\,ds\bigg| + \\ + \bigg\|\int_{\widetilde{t}_0}^{t} \big[\widetilde{\mathcal{P}}(s) - \mathcal{P}(s)\big]x(s)\,ds\bigg\| \quad \text{for} \quad t \in R_+. \end{aligned} \tag{16}$$

According to the formula of integration by parts,

$$\begin{aligned} \int_{\widetilde{t}_0}^{t} \big[\widetilde{\mathcal{P}}(s) - \mathcal{P}(s)\big]x(s)\,ds = \bigg(\int_{\widetilde{t}_0}^{t} \big[\widetilde{\mathcal{P}}(\tau) - \mathcal{P}(\tau)\big]\,d\tau\bigg)x(t) + \\ + \int_{\widetilde{t}_0}^{t} \bigg(\int_{t_0}^{s} \big[\widetilde{\mathcal{P}}(\tau) - \mathcal{P}(\tau)\big]\,d\tau\bigg)x'(s)\,ds. \end{aligned}$$

However, because of (6),

$$\Big\| \int_{\widetilde{t}_0}^{t} \big[\widetilde{\mathcal{P}}(s) - \mathcal{P}(s)\big]\, ds \Big\| \le \Big\| \int_{t_0}^{\widetilde{t}_0} \big[\widetilde{\mathcal{P}}(\tau) - \mathcal{P}(\tau)\big]\, d\tau \Big\| + \\ + \Big\| \int_{t_0}^{t} \big[\widetilde{\mathcal{P}}(\tau) - \mathcal{P}(\tau)\big]\, d\tau \Big\| \le 2\delta \quad \text{for} \quad t \in R_+.$$

Therefore

$$\Big\| \int_{t_0}^{t} \big[\widetilde{\mathcal{P}}(s) - \mathcal{P}(s)\big] x(s)\, ds \Big\| \le \\ \le 2\delta \Big(\|x(t)\| + \int_0^{+\infty} \|x'(s)\|\, ds \Big) \quad \text{for} \quad t \in R_+.$$

Taking the above-said and inequality (10) into consideration, we find from (16) that

$$\|z(t)\| \le 2(1+\rho_0)\delta + \omega(\delta) + \Big| \int_{\widetilde{t}_0}^{t} \|\widetilde{\mathcal{P}}(s)\|\, \|z(s)\|\, ds \Big| \quad \text{for} \quad t \in R_+.$$

This, owing to Gronwall's lemma, implies

$$\|z(t)\| \le \big[2(1+\rho_0)\delta + \omega(\delta)\big] \exp\Big(\Big| \int_{\widetilde{t}_0}^{t} \|\widetilde{\mathcal{P}}(s)\|\, ds \Big| \Big) \quad \text{for} \quad t \in R_+.$$

But because of (7) and (11),

$$\Big| \int_{t_0}^{t} \|\widetilde{\mathcal{P}}(s)\|\, ds \Big| \le \int_0^{+\infty} \|\widetilde{\mathcal{P}}(s) - \mathcal{P}(s)\|\, ds + \int_0^{+\infty} \|\mathcal{P}(s)\|\, ds \le \rho + \rho_1.$$

Therefore

$$\|z(t)\| \le \big[2(1+\rho_0)\delta + \omega(\delta)\big] \exp(\rho + \rho_1) \quad \text{for} \quad t \in I,$$

whence, by virtue of (13), there follows estimate (8). □

It is not difficult to see that the problems

$$\frac{dx}{dt} = x; \quad x(0) = 0 \tag{17}$$

and

$$\frac{dx}{dt} = 1; \quad x(0) = 0 \tag{18}$$

are incorrect, although all the conditions of Theorem 1, except for summability of $\mathcal{P}$ (of q) on R_+, are fulfilled for problem (17) (for problem (18)).

These examples show that if one of the conditions (9) is violated, then for problem (1), (2) to be correct on R_+, one should impose an additional restriction on the matrix function $\mathcal{P}$. However, as it easily follows from Theorem 1, problem (1), (2) is correct on every finite interval irrespective of the fact whether the conditions (9) are fulfilled or not. More precisely, the following corollary is valid.

Corollary 1. *For any* $t^* \in]t_0 + 1, +\infty[$*,* $\varepsilon > 0$*, and* $\rho > 0$ *there exists* $\delta > 0$ *such that if* $\widetilde{t}_0 \in [0, t^*]$*,* $\widetilde{c}_0 \in R^n$*, and* $\widetilde{\mathcal{P}} \in L_{loc}(R_+; R^{n\times n})$ *satisfy the conditions* (5)–(7)*, then*

$$\|x(t) - y(t)\| < \varepsilon \quad \textit{for} \quad 0 \le t \le t^*,$$

where x and y are the solutions of problems (1)*,* (2) *and* (3)*,* (4)*, respectively.*

In the case where one of the conditions (9) is violated, to investigate the correctness of problem (1), (2) on R_+ we shall need some definitions from the stability theory (see [3]).

Definition 3. The zero solution of the differential system

$$\frac{dx}{dt} = \mathcal{P}(t)x \tag{19}$$

is *uniformly stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that an arbitrary solution x of that system, satisfying for some $t_0 \in R_+$ the inequality

$$\|x(t_0)\| < \delta,$$

admits the estimate

$$\|x(t)\| < \varepsilon \quad \text{for} \quad t \ge t_0.$$

Definition 4. The zero solution of (19) is *exponentially asymptotically stable*, if there exists a $\eta > 0$ and, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that an arbitrary solution of that system, satisfying for some $t_0 \in R_+$ the inequality

$$\|x(t_0)\| < \delta,$$

admits the estimate

$$\|x(t)\| < \varepsilon \exp\big(-\eta(t - t_0)\big) \quad \text{for} \quad t \ge t_0.$$

Definition 5. The matrix function $\mathcal{P}$ is *uniformly stable* (*exponentially asymptotically stable*) if the zero solution of the system (9) is uniformly stable (exponentially asymptotically stable).

Theorem 2. *If*

$$\limsup_{t\to+\infty}\int_t^{t+1}\|\mathcal{P}(\tau)\|\,d\tau<+\infty,\qquad \lim_{t\to+\infty}\int_t^{t+1}\|q(\tau)\|\,d\tau=0, \tag{20}$$

then the exponentially asymptotic stability of $\mathcal{P}$ *ensures the correctness of problem* (1), (2).

Proof. Let $t_0\in R_+$ and $c_0\in R^n$ be arbitrarily fixed, and let x be the solution of problem (1), (2). Owing to the exponentially asymptotic stability of $\mathcal{P}$ and because of the conditions (21):

(i) the solution x satisfies the condition

$$\lim_{t\to+\infty}\|x(t)\|=0; \tag{21}$$

(ii) there exist positive numbers ρ_0 and η such that the Cauchy matrix C of the system (19) admits the estimate

$$\|C(t,\tau)\|\le\rho_0\exp\big(-\eta(t-\tau)\big)\quad\text{for}\quad t\ge\tau\ge0; \tag{22}$$

(iii) there exists $\rho_1>0$ such that

$$\int_0^t\exp\big(-\eta(t-\tau)\big)\|\mathcal{P}(\tau)\|\,d\tau\le\rho_1\quad\text{for}\quad t\in R_+. \tag{23}$$

Owing to (21), for arbitrarily given $\varepsilon\in\,[0,1[$ and $\rho>0$ there exists $t^*=t^*(\varepsilon,\rho)\in\,]t_0+1,+\infty[$ such that

$$\|x(t)\|<(4\rho\rho_0)^{-1}\exp(-\rho_0\rho)\varepsilon\quad\text{for}\quad t\ge t^*. \tag{24}$$

By Corollary 1, there exists

$$\delta\in\Big]0,\frac{\varepsilon}{4(1+\rho_0\rho)}\,\exp(-\rho_0\rho)\Big[\tag{25}$$

such that for any $\widetilde{t}_0\in R_+$, $\widetilde{c}_0\in R^n$, $\widetilde{\mathcal{P}}\in L_{loc}(R_+;R^{n\times n})$, and $\widetilde{q}\in L_{loc}(R_+;R^n)$ satisfying the conditions (5)–(7), the inequality

$$\|x(t)-y(t)\|<\frac{\varepsilon}{4\rho_0}\exp(-\rho_0\rho)\quad\text{for}\quad 0\le t\le t^* \tag{26}$$

holds, where y is the solution of problem (3), (4).

To prove the theorem, it remains to show that

$$\|x(t)-y(t)\|<\varepsilon\quad\text{for}\quad t>t^*. \tag{27}$$

Assume

$$z(t) = x(t) - y(t).$$

Then

$$z'(t) = \mathcal{P}(t)z(t) + \big[\widetilde{\mathcal{P}}(t) - \mathcal{P}(t)\big]z(t) + \big[\mathcal{P}(t) - \widetilde{\mathcal{P}}(t)\big]x(t) + q(t) - \widetilde{q}(t)$$

and

$$z(t) = C(t,t^*)z(t^*) + \int\limits_{t^*}^{t} C(t,\tau)\big[\widetilde{\mathcal{P}}(\tau) - \mathcal{P}(\tau)\big]z(\tau)\,d\tau + \\ + \int\limits_{t^*}^{t} C(t,\tau)\big[\mathcal{P}(\tau) - \widetilde{\mathcal{P}}(\tau)\big]x(\tau)\,d\tau + \\ + \int\limits_{t^*}^{t} C(t,\tau)\big[q(\tau) - \widetilde{q}(\tau)\big]\,d\tau. \tag{28}$$

However, by the inequalities (7), (22), (24), and (26),

$$\big\|C(t,t^*)z(t^*)\big\| \leq \rho_0\|z(t^*)\| < \frac{\varepsilon}{4}\exp(-\rho_0\rho) \quad \text{for} \quad t \geq t^*, \tag{29}$$

$$\bigg\|\int\limits_{t^*}^{t} C(t,\tau)\big[\widetilde{\mathcal{P}}(\tau) - \mathcal{P}(\tau)\big]z(\tau)\,d\tau\bigg\| \leq \\ \leq \rho_0 \int\limits_{t^*}^{t} \big\|\widetilde{\mathcal{P}}(\tau) - \mathcal{P}(\tau)\|\,\|z(\tau)\|\,d\tau \quad \text{for} \quad t \geq t^* \tag{30}$$

and

$$\bigg\|\int\limits_{t^*}^{t} C(t,\tau)\big[\mathcal{P}(\tau) - \widetilde{\mathcal{P}}(\tau)\big]x(\tau)\,d\tau\bigg\| \leq \\ \leq \rho_0(4\rho\rho_0)^{-1}\exp(-\rho_0\rho)\varepsilon \int\limits_{t^*}^{t} \big\|\widetilde{\mathcal{P}}(\tau) - \mathcal{P}(\tau)\big\|\,d\tau \leq \\ \leq \frac{\varepsilon}{4}\exp(-\rho_0\rho) \quad \text{for} \quad t \geq t^*. \tag{31}$$

According to the formula of integration by parts and because of the equality

$$\frac{\partial C(t,\tau)}{\partial \tau} = -C(t,\tau)\mathcal{P}(\tau)$$

we have

$$\int_{t^*}^{t} C(t,\tau)\big[q(\tau)-\widetilde{q}(\tau)\big]\,d\tau = \int_{t^*}^{t} \big[q(\tau)-\widetilde{q}(\tau)\big]\,d\tau + \\ + \int_{t^*}^{t} C(t,\tau)\mathcal{P}(\tau)\int_{t^*}^{\tau} \big[q(s)-\widetilde{q}(s)\big]\,ds,$$

whence, by virtue of the conditions (6), (22), (23), and (25), we obtain

$$\left\|\int_{t^*}^{t} C(t,\tau)\big[q(\tau)-\widetilde{q}(\tau)\big]\,d\tau\right\| \le 2\delta + 2\delta\rho_0\int_{t^*}^{t}\exp\big(-\eta(t-\tau)\big)\|\mathcal{P}(\tau)\|\,d\tau \le \\ \le 2(1+\rho_0\rho_1)\delta < \frac{\varepsilon}{2}\exp(-\rho\rho_0) \quad \text{for} \quad t \ge t^*. \tag{32}$$

Taking into account the inequalities (29)–(32), we find from (28) that

$$\|z(t)\| < \varepsilon\exp(-\rho\rho_0) + \rho_0\int_{t^*}^{t}\|\widetilde{\mathcal{P}}(\tau)-\mathcal{P}(\tau)\|\,\|z(\tau)\|\,d\tau \quad \text{for} \quad t \ge t^*$$

which, owing to Gronwall's lemma and inequality (7), implies

$$\|z(t)\| < \varepsilon\exp(-\rho\rho_0)\exp\left(\rho_0\int_{t^*}^{t}\big\|\widetilde{\mathcal{P}}(\tau)-\mathcal{P}(\tau)\big\|\,d\tau\right) < \varepsilon \quad \text{for} \quad t \ge t^*.$$

Consequently, estimate (27) is valid. □

Theorem 3. *If*

$$\int_{0}^{+\infty}\|q(t)\|\,dt < +\infty, \tag{33}$$

then the uniform stability of $\mathcal{P}$ *ensures the weak correctness of problem* (1), (2).

Proof. Let $t_0 \in R_+$ and $c_0 \in R^n$ be arbitrarily fixed, and let x be the solution of problem (1), (2). Due to the uniform stability of $\mathcal{P}$ and because of (33), there exists a positive number ρ_0 such that

$$\|x(t)\| \le \rho_0 \quad \text{for} \quad t \ge 0, \qquad \|C(t,\tau)\| \le \rho_0 \quad \text{for} \quad t \ge \tau \ge 0, \tag{34}$$

where C is the Cauchy matrix of the system (19).

Let $t^* = t_0 + 1$. By Corollary 1, for arbitrarily given $\varepsilon \in]0,1[$ there exists

$$\delta \in \Big]0, \frac{\varepsilon}{4\rho_0^2} \exp(-\rho_0)\Big[\tag{35}$$

such that for any $\widetilde{t}_0 \in R_+$, $\widetilde{c}_0 \in R^n$, $\widetilde{\mathcal{P}} \in L_{loc}(R_+; R^{n\times n})$ and $\widetilde{q} \in L_{loc}(R_+; R^n)$, satisfying the conditions (5)–(7), the inequality

$$\|x(t) - y(t)\| < \frac{\varepsilon}{2\rho_0} \exp(-\rho_0) \quad \text{for} \quad 0 \le t \le t^* \tag{36}$$

holds, where y is the solution of problem (3), (4). Our aim is to establish estimate (8). With this end in view it suffices to prove estimate (27).

Assume $z(t) = x(t) - y(t)$. Then representation (28) is valid. From the latter, owing to (5)–(6) and (34)–(36), we find

$$\|z(t)\| \le \rho_0 \|z(t^*)\| + \rho_0 \int_{t^*}^{t} \big\|\widetilde{\mathcal{P}}(\tau) - \mathcal{P}(\tau)\big\| \, \|z(\tau)\| \, d\tau + \rho_0^2 \delta + \rho_0 \delta <$$

$$< \varepsilon \exp(-\rho_0) + \rho_0 \int_{t^*}^{t} \big\|\widetilde{\mathcal{P}}(\tau) - \mathcal{P}(\tau)\big\| \, \|z(\tau)\| \, d\tau \quad \text{for} \quad t \ge t^*,$$

whence by virtue of Gronwall's lemma we find

$$\|z(t)\| \le \varepsilon \exp\bigg(-\rho_0 + \rho_0 \int_{t^*}^{t} \big\|\widetilde{\mathcal{P}}(\tau) - \mathcal{P}(\tau)\big\| \, d\tau \bigg) <$$

$$< \varepsilon \exp(-\rho_0 + \rho_0 \delta) < \varepsilon \quad \text{for} \quad t \ge t^*.$$

Consequently, estimate (27) is valid. □

References

1. Z. Opial, Continuous parameter dependence in linear systems of differential equations. *J. Diff. Eqs.* **3**(1967), No. 4, 580–594.

2. I. Kiguradze, Boundary value problems for systems of ordinary differential equations. (Russian) *Current Problems in Mathematics. Newest results, vol.* **30**, 3–103, *Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vses. Inst. Nauchn. i Tekh. Inform., Moscow,* 1987.

3. T. Yoshizawa, Stability theory and the existence of periodic solutions and almost periodic solutions. *Springer-Verlag, New York, Heidelberg, Berlin,* 1975.

(Received 15.02.1995)

Author's address:
A. Razmadze Mathematical Institute

Georgian Academy of Sciences
1, M. Alexidze St., Tbilisi 380093
Republic of Georgia