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ON PROJECTIVE METHODS OF APPROXIMATE SOLUTION OF SINGULAR INTEGRAL EQUATIONS

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Abstract. The estimate for the rate of convergence of approximate projective methods with one iteration is established for one class of singular integral equations. The Bubnov–Galerkin and collocation methods are investigated.

Introduction

Let us consider an operator equation of second kind [1]

$$u - Tu = f, \quad u \in E, \quad f \in E, \tag{1}$$

where E is a Banach space and $T : E \to E$ is a linear bounded operator.

Let the sequences of closed subspaces $\{E_n\}$, $E_n \subset E$, and of the corresponding projectors $\{P_n\}$ be given so that $D(P_n) \subset E$, $E_n \subset D(P_n)$, $P_n(D(P_n)) = E_n$, $TE \subset D(P_n)$, $f \in D(P_n)$, $n = 1, 2, \dots$, where $D(P_n)$ denotes the domain of definition of P_n.

Applying the Galerkin method to equation (1), we obtain an approximate equation [1]

$$u_n - P_nTu_n = P_nf, \quad u_n \in E_n. \tag{2}$$

It is known [1] that if the operator $I - T$ is continuously invertible, and $\|P^{(n)}T\| \to 0$ for $n \to \infty$, where $P^{(n)} \equiv I - P_n$, then for sufficiently large n the approximating equation (2) has a unique solution u_n, and the estimate

$$\|u - u_n\| = O(\|P^{(n)}u\|)$$

is valid.

Assume that we have found an approximate solution u_n of equation (2).

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Take one iteration (see [2])

$$\widetilde{u}_n = Tu_n + f. \tag{3}$$

The element $\widetilde{u}_n \in E$, being the approximate solution of equation (1) by the Galerkin method, satisfies the equation

$$\widetilde{u}_n - TP_n\widetilde{u}_n = f. \tag{4}$$

From (1) and (4) we have

$$(I - TP_n)(u - \widetilde{u}_n) = TP^{(n)}u.$$

If the operator $I - T$ is continuously invertible, and $\|TP^{(n)}\| \to 0$ for $n \to \infty$, then for sufficiently large n there exists the inverse bounded operator $(I - TP_n)^{-1}$. Therefore

$$\|u - \widetilde{u}_n\| \leq \|(I - TP_n)^{-1}\| \cdot \|TP^{(n)}u\|, \quad n \geq n_0. \tag{5}$$

Since

$$\|TP^{(n)}u\| \leq \|TP^{(n)}\| \, \|P^{(n)}u\|,$$

the rate of convergence $\|u - \widetilde{u}_n\|$ compared to $\|u - u_n\|$ can be increased by means of a good estimate $\|TP^{(n)}\|$.

In the present paper we consider in the weighted space a singular integral equation of the form

$$Su + Ku = f, \tag{6}$$

where $Su \equiv \frac{1}{\pi}\int_{-1}^{1} \frac{u(t)dt}{t-x}$, $-1 < x < 1$, is a singular integral operator, and $Ku \equiv \frac{1}{\pi}\int_{-1}^{1} K(x,t)u(t)dt$ is an integral operator of the Fredholm type (see [3], [4]).

For the singular integral equation (6) we may have three index values: $\varkappa = -1, 0, 1$.

Our aim is to derive, for (6), an estimate of the convergence rate of the projective Bubnov–Galerkin and collocation methods with one iteration when Chebyshev–Jacobi polynomials are taken as a coordinate system.

Note that the results described below are also valid with required modifications for the singular integral equation of second kind

$$(a + bS + K)u = f,$$

where a and b are real numbers, $a^2 + b^2 > 0$.

§ 1. The Bubnov–Galerkin Method with One Iteration

1.1. Index $\varkappa = 1$. We take a weighted space $L_{2,\rho}[-1,1]$, where the weight $\rho = \rho_1 = (1-x^2)^{1/2}$. The scalar product $[u,v] = \int_{-1}^{1} \rho_1 uv\, dx$. For the index $\varkappa = 1$ we have the additional condition

$$\int_{-1}^{1} u(t)dt = p, \tag{7}$$

where p is a given real number.

The operator S is bounded in $L_{2,\rho}$ (see [4]). We require of the kernel $K(x,t)$ that the operator K be completely continuous in $L_{2,\rho}$. The homogeneous equation $Su = 0$ in the space $L_{2,\rho}$ has a nontrivial solution $u = (1-x^2)^{-1/2}$.

In the space $L_{2,\rho}$ the following two systems of functions are orthonormalized and complete:

(1) $\varphi_k(x) \equiv (1-x^2)^{-1/2}\widehat{T}_k(x)$, $k = 0, 1, \dots,$

$$\widehat{T}_0 = \left(\frac{1}{\pi}\right)^{1/2} T_0, \quad \widehat{T}_{k+1} = \left(\frac{2}{\pi}\right)^{1/2} T_{k+1}, \quad k = 0, 1, \dots,$$

where T_k, $k = 0, 1, \dots,$ are the Chebyshev polynomials of first kind, and

(2) $\psi_{k+1}(x) \equiv \left(\frac{2}{\pi}\right)^{1/2} U_k(x)$, $k = 0, 1, \dots,$

where U_k, $k = 0, 1, \dots,$ are the Chebyshev polynomials of second kind.

Denote $\Phi \equiv u - p\pi^{-1}(1-x^2)^{-1/2}$. Then problem (6)–(7) can be written in the form (see [5])

$$S\Phi + K\Phi = f_1, \quad \Phi \in L_{2,\rho}^{(2)}, \quad f_1 \in L_{2,\rho}, \tag{8}$$

$$\int_{-1}^{1} \Phi(t)dt = 0, \tag{9}$$

where $f_1 \equiv f - p\pi^{-1}K(1-t^2)^{-1/2}$, $L_{2,\rho} = L_{2,\rho}^{(1)} \oplus L_{2,\rho}^{(2)}$ is the orthogonal decomposition, $L_{2,\rho}^{(1)}$ is the linear span of the function $\varphi_0 = (1-x^2)^{-1/2}$, and $L_{2,\rho}^{(2)}$ is its orthogonal complement. In the sequel, under S we shall mean its restriction on $L_{2,\rho}^{(2)}$. Then $S(L_{2,\rho}^{(2)}) = L_{2,\rho}$ and $S^{-1}(L_{2,\rho}) = L_{2,\rho}^{(2)}$.

The relations

$$S\varphi_k = \psi_k, \quad k = 1, 2, \dots, \tag{10}$$

(see [6]) are valid. An approximate solution of equation (8) is sought in the form

$$\Phi_n = \sum_{k=1}^{n} a_k \varphi_k.$$

Owing to (10), the algebraic system composed of the conditions

$$[S\Phi_n + K\Phi_n - f_1, \psi_i] = 0, \quad i = 1, 2, \dots, n,$$

yields

$$a_i + \sum_{k=1}^{n} a_k [K\varphi_k, \psi_i] = [f_1, \psi_i], \quad i = 1, 2, \dots, n. \tag{11}$$

It is known [5] that if there exists the inverse operator $(I + KS^{-1})^{-1}$ mapping $L_{2,\rho}$ onto itself, then for sufficiently large n the algebraic system (11) has a unique solution $(a_1, a_2, \dots, a_n)$, and the sequence of approximate solutions

$$u_n = \Phi_n + p\pi^{-1}(1 - x^2)^{-1/2}$$

converges to the exact solution u in the metric of the space $L_{2,\rho}$. Similar results are valid for $\varkappa = -1, 0$.

With the help of the orthoprojector P_n which maps $L_{2,\rho}$ onto the linear span of the functions $\psi_1, \dots, \psi_n$ we can rewrite the algebraic system (11) as

$$w_n + P_n K S^{-1} w_n = P_n f_1, \quad w_n \equiv S\Phi_n = \sum_{k=1}^{n} a_k \psi_k. \tag{12}$$

From the initial equation (8) we have

$$w + KS^{-1} w = f_1, \quad w \in L_{2,\rho}, \quad f_1 \in L_{2,\rho}, \quad w \equiv S\Phi. \tag{13}$$

Equation (12) is the Bubnov–Galerkin approximation for (13).

As in [2], let us introduce the iteration

$$\widetilde{w}_n = -KS^{-1} w_n + f_1 = -K\Phi_n + f_1. \tag{14}$$

where $\widetilde{w}_n$ satisfies the equation

$$\widetilde{w}_n = -KS^{-1} P_n \widetilde{w}_n + f_1.$$

For $n \geq n_0$ we obtain

$$\|w - \widetilde{w}_n\| \leq C \|KS^{-1} P^{(n)}\| \cdot \|P^{(n)} w\|.$$

Let $\widetilde{\Phi}_n \equiv S^{-1}\widetilde{w}_n$. To find $\widetilde{\Phi}_n$, it is necessary to calculate the integral

$$S^{-1}\widetilde{w}_n = \frac{(1-t^2)^{-1/2}}{\pi} \int_{-1}^{1} (1-x^2)^{1/2} \frac{\widetilde{w}_n(x)dx}{t-x}.$$

We have

$$\|u-\widetilde{u}_n\| = \|\Phi-\widetilde{\Phi}_n\| = \|S^{-1}(w-\widetilde{w}_n)\| = \|w-\widetilde{w}_n\| \leq$$
$$\leq C\|KS^{-1}P^{(n)}\| \cdot \|P^{(n)}w\|,$$

where $\widetilde{u}_n = \widetilde{\Phi}_n + p\pi^{-1}(1-x^2)^{-1/2}$.

Theorem 1. *If there exists the inverse operator* $(I+KS^{-1})^{-1}$ *mapping* $L_{2,\rho}$ *onto itself, and the conditions* $w^{(n)} \in \mathrm{Lip}_M\,\alpha$, $0<\alpha\leq 1$, *and* $K^{(l)}(x,t) \in \mathrm{Lip}_M\,\alpha_1$, $0<\alpha_1\leq 1$, $\forall x\in[-1,1]$, *are fulfilled for the derivatives, then the estimate*

$$\|u-\widetilde{u}_n\| = O(n^{-(m+\alpha)-(l+\alpha_1)})$$

is valid.

Proof. We have

$$P_n w = \sum_{k=1}^{n} [w,\psi_k]\psi_k.$$

$$\|P^{(n)}w\|^2 = \int_{-1}^{1} (1-x^2)^{1/2}(w-P_n w)^2 dx =$$

$$= \int_{-1}^{1} (1-x^2)^{1/2}\Big(w-\sum_{k=1}^{n}[w,\psi_k]\psi_k\Big)^2 ds \leq$$

$$\leq \int_{-1}^{1} (1-x^2)^{1/2}(w-\mathcal{P}_{n-1})^2 dx \leq \frac{\pi}{2}\|w-\mathcal{P}_{n-1}\|_C^2,$$

where $\mathcal{P}_{n-1}$ is the polynomial of the best uniform approximation.

By Jackson's theorem [7] we have

$$\|w-\mathcal{P}_{n-1}\|_C \leq \frac{C(w)}{(n-1)^{m+\alpha}}, \quad n>1,$$

with a constant $C(w)$ depending on w and its derivatives, i.e., $\|P^{(n)}w\| = O(n^{-(m+\alpha)})$.

Furthermore,

$$\|KS^{-1}P^{(n)}v\|^2 = \|KS^{-1}\sum_{k=n+1}^{\infty}[v,\psi_k]\psi_k\|^2 = \|K\sum_{k=n+1}^{\infty}[v,\psi_k]\varphi_k\|^2 =$$

$$= \frac{1}{\pi^2}\Big\|\sum_{k=n+1}^{\infty}[v,\psi_k]\big(K(x,t),\varphi_k(t)\big)\Big\|^2 \leq$$

$$\leq \frac{1}{\pi^2}\Big\|\Big\{\sum_{k=n+1}^{\infty}[v,\psi_k]^2\Big\}^{1/2}\times\Big\{\sum_{k=n+1}^{\infty}\big(K(x,t),\varphi_k(t)\big)^2\Big\}^{1/2}\|^2 \leq$$

$$\leq \frac{\|v\|^2}{\pi^2}\int_{-1}^{1}(1-x^2)^{1/2}\Big(\sum_{k=n+1}^{\infty}\big(K(x,t),\varphi_k(t)\big)^2\Big)dx =$$

$$= \frac{\|v\|^2}{\pi^2}\int_{-1}^{1}(1-x^2)^{1/2}\Big(\sum_{k=n+1}^{\infty}\big(K(x,t),(1-t^2)^{-1/2}\widehat{T}_k(t)\big)^2\Big)dx =$$

$$= \frac{\|v\|^2}{\pi^2}\int_{-1}^{1}(1-x^2)^{1/2}\Big\|\sum_{k=n+1}^{\infty}\big[K(x,t),\widehat{T}_k(t)\big]_{L_{r,\rho^{-1}}}\widehat{T}_k(t)\Big\|^2_{L_{2,\rho^{-1}}}dx,$$

$$\Big\|\sum_{k=n+1}^{\infty}\big[K(x,t),\widehat{T}_k(t)\big]_{L_{r,\rho^{-1}}}\widehat{T}_k(t)\Big\|^2_{L_{2,\rho^{-1}}} =$$

$$= \int_{-1}^{1}(1-t^2)^{1/2}\Big(\sum_{k=n+1}^{\infty}\big[K(x,t),\widehat{T}_k(t)\big]_{L_{2,\rho^{-1}}}\widehat{T}_k(t)\Big)^2dt =$$

$$= \int_{-1}^{1}(1-t^2)^{1/2}\Big(K(x,t)-\sum_{k=0}^{n}\big[K(x,t),\widehat{T}_k(t)\big]_{L_{2,\rho^{-1}}}\widehat{T}_k(t)\Big)^2dt \leq$$

$$\leq \int_{-1}^{1}(1-t^2)^{1/2}\big(K(x,t)-\mathcal{P}_n(x,t)\big)^2dt \leq \pi\Big(E_n^t\big(K(x,t)\big)\Big)^2,$$

where x is a parameter, $\mathcal{P}_n(x,t)$ is the polynomial of the best uniform approximation with respect to t, and $E_n^t(K(x,t))$ the corresponding deviation

$$\big|K(x,t)-\mathcal{P}_n(x,t)\big| \leq E_n^t\big(K(x,t)\big), \quad -1 < x,t < 1.$$

If $K_t^{(l)}(x,t) \in \mathrm{Lip}_{M_1}\,\alpha_1$, $0 < \alpha_1 \leq 1$ $\forall x \in [-1,1]$, and is continuous with respect to x in $[-1,1]$, then (see [8, Ch.XIV, §4])

$$E_n^t\big(K(x,t)\big) = O\big(n^{-(l+\alpha_1)}\big).$$

Furthermore,

$$\big\|KS^{-1}P^{(n)}v\big\|^2 \le \frac{\|v\|^2}{\pi^2}\int_{-1}^{1}(1-x^2)^{1/2}\pi\big(E_n^t(K(x,t))\big)^2dx =$$
$$= \frac{\|v\|^2}{\pi^2}\pi\frac{\pi}{2}\big(E_n^t(K(x,t))\big)^2.$$

Thus $\big\|KS^{-1}P^{(n)}\big\| = O\big(n^{-(l+\alpha_1)}\big)$.

Finally, we get

$$\|u-\widetilde{u}_n\| \le \|KS^{-1}P^{(n)}w\| \le$$
$$\le \|KS^{-1}P^{(n)}\|\cdot\|P^{(n)}w\| = O(n^{-(m+\alpha)-(l+\alpha_1)}). \quad \square$$

For the approximate solution u_n we have

$$\|u-u_n\| \le C\|P^{(n)}w\| = O(n^{-(m+\alpha)}),$$

while for one iteration $\widetilde{u}_n$ performed over u_n when $l=m$, $\alpha_1=\alpha$, we obtain the estimate

$$\|u-\widetilde{u}_n\| = O(n^{-2(m+\alpha)}).$$

1.2. Index $\varkappa = -1$. The operator S is bounded in the weighted space $L_{2,\rho}[-1,1]$, where $\rho=\rho_2=(1-x^2)^{-1/2}$ (see [4]). We require of the kernel $K(x,t)$ that the operator K be completely continuous in $L_{2,\rho}$. The equation $Su=0$ in $L_{2,\rho}$ has the zero solution only, while the equation $S^*u=0$ has the nonzero solution $u=1$.

If in the weighted space $L_{2,\rho}$ the equation $Su+Ku=f$ has a solution u, then $[Ku-f,1]=0$. This condition will be fulfilled if $K(L_{L_{2,\rho}})\perp 1$ and $[f,1]=0$, which can be achieved by specific transform [9].

In the space $L_{2,\rho}$ the following two systems of functions are complete and orthonormal:

(1) $$\varphi_{k+1}(x) \equiv \Big(\frac{2}{\pi}\Big)^{1/2}U_k(x), \quad k=0,1,\dots,$$

where U_k, $k=0,1,\dots$ are Chebyshev polynomials of second kind, and

(2) $$\psi_{k+1}(x) \equiv -\Big(\frac{2}{\pi}\Big)T_{k+1}(x), \quad k=0,1,\dots,$$

where T_{k+1}, $k=0,1,\dots$ are Chebyshev polynomials of first kind.

Relations

$$S\varphi_k=\psi_k, \quad k=1,2,\dots$$

(see [6]) are valid.

An approximate solution is again sought in the form

$$u_n=\sum_{k=1}^{n}a_k\varphi_k.$$

The Bubnov–Galerkin method results in the algebraic system

$$a_i + \sum_{k=1}^{n} a_k [K\varphi_k, \psi_i] = [f, \psi_i], \quad i = 1, 2, \dots, n. \tag{15}$$

Denote $w_n \equiv Su_n = \sum_{k=1}^{n} a_k \psi_k$. Then using the orthoprojector P_n mapping $L_{2,\rho}$ onto the linear span of the functions $\psi_1, \psi_2, \dots, \psi_n$ the algebraic system (15) can be rewritten as

$$w_n + P_n K S^{-1} w_n = P_n f. \tag{16}$$

Let the approximate solution w_n be found.

Taking one iteration

$$\widetilde{w}_n = -KS^{-1} w_n + f = -Ku_n + f,$$

we find that $\widetilde{u}_n = S^{-1}\widetilde{w}_n = \dfrac{(1-t^2)^{1/2}}{\pi} \int_{-1}^{1} (1-x^2)^{-1/2} \dfrac{\widetilde{w}_n(x)dx}{t-x}$.

Theorem 2. *If there exists the inverse operator $(I + KS^{-1})^{-1}$ mapping $L_{2,\rho}^{(2)}$ onto itself, and the conditions $w^{(m)} \in \mathrm{Lip}_M\, \alpha$, $0 < \alpha \le 1$, $K_t^{(l)}(x,t) \in \mathrm{Lip}_M\, \alpha_1$, $0 < \alpha_1 \le 1$, $\forall x \in [-1,1]$, are fulfilled for the derivatives, then the estimate*

$$\|u - \widetilde{u}_n\| = O(n^{-(m+\alpha)-(l+\alpha_1)})$$

is valid.

This theorem as well as Theorem 3 which will be formulated in the next subsection can be proved similarly to Theorem 1.

1.3. Index $\varkappa = 0$. Here we may have two cases:

(1) $\alpha = -\frac{1}{2}$, $\beta = \frac{1}{2}$ and (2) $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$.

Let us consider the first case. The second one is considered analogously.

The operator S is bounded in the weighted space $L_{2,\rho}[-1,1]$ with the weight $\rho = \rho_3 = (1-x)^{1/2}(1+x)^{-1/2}$ [4]. We require of the kernel $K(x,t)$ that the operator K be completely continuous in $L_{2,\rho}[-1,1]$. In the space $L_{2,\rho}$ the equations $Su = 0$ and $S^*u = 0$ have only trivial solution $u = 0$, $S(L_{2,\rho}) = L_{2,\rho}$, where S is the unitary operator.

We have the equation

$$Su + Ku = f, \quad u \in L_{2,\rho}, \quad f \in L_{2,\rho}. \tag{17}$$

In $L_{2,\rho}$ we take two complete and orthonormal systems of functions (see [10]):

(1) $$\varphi_k \equiv c_k (1-x)^{1/2}(1+x)^{-1/2} P_k^{(1/2,-1/2)}, \quad k = 0, 1, \dots,$$

$$c_0 = \pi, \quad c_k = (h_k^{(-1/2,1/2)})^{-1/2}, \quad k = 1, 2, \dots,$$

$$h_k^{(-1/2,1/2)} = h_k^{(1/2,-1/2)} = \frac{2\Gamma(k+1/2)\Gamma(k+3/2)}{(2k+1)(k!)^2},$$

where $P_k^{(1/2,-1/2)}$, $k = 0, 1, \ldots$, are the Jacobi polynomials;

(2) $\psi_k \equiv -c_k P_k^{(-1/2,1/2)}$.

The relations

$$S\varphi_k = \psi_k, \quad k = 0, 1, \ldots \tag{18}$$

(see [6]) are valid.

We seek an approximate solution of equation (17) in the form

$$u_n = \sum_{k=1}^{n} a_k \varphi_k.$$

With regard to (18) the Bubnov–Galerkin method

$$[Su_n + Ku_n - f, \psi_i] = 0, \quad i = 0, 1, \ldots, n,$$

yields the algebraic system

$$a_i + \sum_{k=0}^{n} a_k [K\varphi_k, \psi_i] = [f, \psi_i] \ \ i = 0, 1, \ldots, n, \tag{19}$$

which, by means of the orthoprojector P_n mapping $L_{2,\rho}$ onto the linear span of the functions $\psi_0, \psi_1, \ldots, \psi_n$, can be written in the form

$$w_n + P_n K S^{-1} w_n = P_n f, \ \ w_n \equiv Su_n = \sum_{k=0}^{n} a_k \psi_k. \tag{20}$$

Let the approximate solution w_n be found.

Taking one iteration

$$\widetilde{w}_n = -KS^{-1} w_n + f = -Ku_n + f,$$

we find

$$\widetilde{u}_n = S^{-1}\widetilde{w}_n = \frac{(1+t)^{1/2}(1-t)^{-1/2}}{\pi} \int_{-1}^{1} (1-x)^{1/2}(1+x)^{-1/2} \frac{\widetilde{w}_n(x)dx}{t-x}.$$

Then

$$\|u - \widetilde{u}_n\| = \|S^{-1}(w - \widetilde{w}_n)\| = \|w - \widetilde{w}_n\| \le C\|KS^{-1}P^{(n)}\| \cdot \|P^{(n)}w\|.$$

Theorem 3. *If there exists the inverse operator* $(I+KS^{-1})^{-1}$ *mapping* $L_{2,\rho}$ *onto itself, and the conditions* $w^{(m)} \in \operatorname{Lip}_M \alpha$, $0<\alpha\le 1$, $K_t^{(l)}(x,t) \in \operatorname{Lip}_M \alpha_1$, $0<\alpha_1\le 1$, $\forall x \in [-1,1]$, *are fulfilled for the derivatives, then the estimate*

$$\|u-\widetilde{u}_n\| = O(n^{-(m+\alpha)-(l+\alpha_1)})$$

is valid.

§ 2. Method of Collocation with One Iteration

Using the collocation method, let us now consider the solution of equation (6). Assume that the kernel $K(x,t)$ and $f(x)$ are continuous functions.

2.1. Index $\varkappa = 1$**.** As in Subsection 1.1 we seek an approximate solution of problem (8)–(9) in the form

$$\Phi_n = \sum_{k=1}^{n} a_k \varphi_k.$$

By the collocation method the residual $S\Phi_n + K\Phi_n - f_1$ (f_1 is introduced above by (8)) at discrete points will be equated to zero,

$$[S\Phi_n + K\Phi_n - f_1]_{x_j} = 0, \quad j = 1,2,\dots,n.$$

This, owing to (10), results in the algebraic system

$$\sum_{k=1}^{n} a_k \psi_k(x_j) + \sum_{k=1}^{n} a_k (K\varphi_k)(x_j) = f_1(x_j), \;\; j=1,2,\dots,n. \qquad (21)$$

As is known [11], if there exists the operator $(I+KS^{-1})^{-1}$ mapping $L_{2,\rho}$ onto itself, and as the collocation nodes are taken the roots of the Chebyshev polynomials of the second kind U_n, then for sufficiently large n the algebraic system (21) has a unique solution, and the process converges in the space $L_{2,\rho}$. Analogous results are valid for the index $\varkappa = -1, 0$.

Let us take the space of continuous functions $C[-1,1]$.

Let Π_n be the projector defined by the Lagrange interpolation polynomial $\Pi_n v = L_n v$. With the help of this projector the algebraic system (21) can be rewritten in the form

$$w_n + \Pi_{n-1} KS^{-1} w_n = \Pi_{n-1} f_1, \quad w_n \in L_{2,\rho}^{(n)},$$

where $L_{2,\rho}^{(n)}$ is the linear span of functions $\psi_1, \psi_2, \dots, \psi_n$ (ψ_k is a polynomial of degree $k-1$).

Let the approximate solution w_n be found.

As in the Bubnov–Galerkin method we take one iteration

$$\widetilde{w}_n = -KS^{-1} w_n + f_1 = -K\Phi_n + f_1.$$

where $\widetilde{w}_n$ satisfies the equation

$$\widetilde{w}_n + KS^{-1}\Pi_{n-1}\widetilde{w}_n = f_1. \tag{22}$$

By means of (13) and (22) we obtain

$$w - \widetilde{w}_n + KS^{-1}w - KS^{-1}\Pi_{n-1}\widetilde{w}_n + KS^{-1}\widetilde{w}_n - KS^{-1}\widetilde{w}_n = 0,$$
$$(I + KS^{-1})(w - \widetilde{w}_n) = -KS^{-1}\Pi^{(n-1)}\widetilde{w}_n, \quad \Pi^{(n-1)} \equiv I - \Pi_{n-1},$$
$$w - \widetilde{w}_n = -(I + KS^{-1})^{-1}KS^{-1}\Pi^{(n-1)}(-K\phi_n + f_1),$$
$$\|w - \widetilde{w}_n\| \le C\big(\|KS^{-1}\Pi^{(n-1)}K\| \cdot \|\Phi_n\| + \|KS^{-1}\Pi^{(n-1)}f_1\|\big).$$

For sufficiently large n we have [11]

$$w_n = (I + \Pi_{n-1}KS^{-1})^{-1}\Pi_{n-1}f_1,$$
$$\|\Phi_n\| = \|S^{-1}w_n\| \le C\|\Pi_{n-1}f_1\|,$$
$$\Pi_{n-1}f_1 \to f_1, \quad \text{for} \quad n \to \infty, \quad \forall f_1 \in C[-1,1],$$

i.e., $\|\Phi_n\|$ are uniformly bounded owing to the Erdös–Turan [10] and Banach–Steinhaus [8] theorems. Therefore

$$\|w - \widetilde{w}_n\| \le C\big(\|KS^{-1}\Pi^{(n-1)}K\| + \|KS^{-1}\Pi^{(n-1)}f_1\|\big).$$

We find that

$$\widetilde{\Phi}_n \equiv S^{-1}\widetilde{w}_n = \frac{1}{\pi}(1-t^2)^{-1/2}\int_{-1}^{1}\frac{(1-x^2)^{1/2}\widetilde{w}_n(x)dx}{t-x}.$$

Then

$$\|u - \widetilde{u}_n\| = \|\Phi - \widetilde{\Phi}_n\| = \|S^{-1}(w - \widetilde{w}_n)\| = \|w - \widetilde{w}_n\| \le$$
$$\le C\big(\|KS^{-1}\Pi^{(n-1)}K\| + \|KS^{-1}\Pi^{(n-1)}f_1\|\big), \tag{23}$$

where $\widetilde{u}_n \equiv \widetilde{\Phi}_n + p\pi^{-1}(1-x^2)^{-1/2}$.

It is known [12] that

$$\Pi^{(n-1)}v(t) = \omega(t)\delta^{(n)}v(t), \quad \forall v \in C[-1,1],$$

where $\omega(t) \equiv \prod_{i=1}^{n}(t - t_i)$ and $\delta^{(n)}v(t)$ is the divided difference of the continuous function $v(t)$. If the roots of the Chebyshev polynomial of second kind U_n are taken as interpolation nodes, then (see [13])

$$\Pi^{(n-1)}v(t) = \Big(\frac{\pi}{2}\Big)^{1/2}\frac{\widehat{U}_n(t)}{2^n}\delta^{(n)}v(t) \quad \Big(\widehat{U}_n(t) = \Big(\frac{2}{\pi}\Big)^{1/2}U_n(t)\Big).$$

Denote $K_1(x,t) \equiv (1-t^2)^{-1/2}K(x,t)$.

Theorem 4. *If there exists the inverse operator $(I+KS^{-1})^{-1}$ mapping $L_{2,\rho}$, $\rho=\rho_1$, onto itself, the roots of the second kind Chebyshev polynomial U_n are taken as collocation nodes, $(SK_1(x,t))^m \in L_{P_M}\alpha$, $0<\alpha\le 1$, $\forall x\in[-1,1]$, and the divided differences of all orders of the functions $f_1(x)$ and $K_1(x,t)$ with respect to x are uniformly bounded $\forall t\in[-1,1]$, then the estimate*

$$\|u-\widetilde{u}_n\| = O\Big(\frac{1}{2^n}\cdot\frac{1}{(n-1)^{m+\alpha}}\Big)$$

is valid.

Proof. Let us estimate the norms on the right-hand side of inequality (23). We have

$$\|KS^{-1}\Pi^{(n-1)}Kv\|_{L_{2,\rho}} = \Big\|\frac{1}{\pi}\big(K(x,t),$$
$$S^{-1}\Pi^{(n-1)}(K(x,\tau),v(\tau))\big)\Big\| = \frac{1}{\pi^2}\Big\|\big((1-t^2)^{1/2}(1-t^2)^{-1/2}K(x,t),$$
$$S^{-1}\Pi^{(n-1)}(1-\tau^2)^{1/2}(1-\tau^2)^{-1/2}(K(x,\tau),v(\tau))\big)\Big\| =$$
$$= \frac{1}{\pi^2}\Big\|\big[SK_1(x,t),\ \Pi^{(n-1)}[K_1(x,\tau),v(\tau)]\big]\Big\| =$$
$$= \frac{1}{\pi^2}\Big\|\big[SK_1(x,t),\ [\Pi^{(n-1)}K_1(x,\tau),v(\tau)]\big]\Big\| =$$
$$= \frac{1}{\pi^2}\Big\|\big[[SK_1(x,t),\ \Pi^{(n-1)}K_1(x,\tau)],v(\tau)\big]\Big\| =$$
$$= \frac{1}{\pi^2}\Big(\frac{\pi}{2}\Big)^{1/2}\Big\|\big[[SK_1(x,t),\ \frac{\widehat{U}_n(t)}{2^n}\delta^{(n)}K_1(x,\tau),v(\tau)\big]\Big\| =$$
$$= \frac{1}{\pi^2}\Big(\frac{\pi}{2}\Big)^{1/2}\frac{1}{2^n}\Big\|\big[\delta^{(n)}K_1(x,\tau)SK_1(x,t),\ P^{(n-1)}\widehat{U}_n(t)]v(\tau)\big]\Big\| =$$
$$= \frac{1}{\pi^2}\Big(\frac{\pi}{2}\Big)^{1/2}\frac{1}{2^n}\Big\|\big[[P^{(n-1)}\delta^{(n)}K_1(t,\tau)SK_1(x,t),\widehat{U}_n(t)]v(\tau)\big]\Big\| \le$$
$$\le \frac{1}{\pi^2}\Big(\frac{\pi}{2}\Big)^{1/2}\frac{1}{2^n}\Big\|\ \big\|\big[P^{(n-1)}\delta^{(n)}K_1(t,\tau)SK_1(x,t),\widehat{U}_n(t)]\big\|\times\|v(\tau)\|\ \Big\| \le$$
$$\le \frac{\|v\|}{\pi^2}\Big(\frac{\pi}{2}\Big)^{1/2}\frac{1}{2^n}\Big\|\ \Big\|\ \Big\|P^{(n-1)}\delta^{(n)}K_1(t,\tau)SK_1(x,t)\Big\|\times\|\widehat{U}_n(t)\|\ \Big\|\ \Big\| \le$$
$$\le \frac{\|v\|}{\pi^2}\Big(\frac{\pi}{2}\Big)^{1/2}\frac{1}{2^n}\Big\|\ \Big\|\ \big\|P^{(n-1)}\delta^{(n)}K_1(t,\tau)SK_1(x,t)\big\|\Big\|\ \Big\|.$$

Under the conditions of the theorem

$$(\delta^{(n)}K_1(t,\tau)SK_1(x,t))^{(m)} \in \mathrm{Lip}_M\,\alpha,\ \ 0<\alpha\le 1,\ \ \forall x,\tau\in[-1,1].$$

By Jackson's theorem (see [7], [8])

$$\Big\|P^{(n-1)}\delta^{(n)}K_1(t,\tau)SK_1(x,t)\Big\| \le \frac{c'_m 2^{m+\alpha}M}{(n-1)^{m+\alpha}},$$

where $c'_m \equiv 12\frac{6^m m^m}{m!}\left(\frac{m+1}{2}\right)^\alpha$.

Therefore we obtain

$$\|KS^{-1}\Pi^{(n-1)}K\| = O\Big(\frac{1}{2^n}\cdot\frac{1}{(n-1)^{m+\alpha}}\Big). \tag{24}$$

Furthermore,

$$\begin{aligned}\|KS^{-1}\Pi^{(n-1)}f_1\|_{L_{2,\rho}} &= \frac{1}{\pi}\|(K(x,t),S^{-1}\Pi^{(n-1)}f_1)\| = \\ &= \frac{1}{\pi}\Big(\frac{\pi}{2}\Big)^{1/2}\Big\|[SK_1(x,t),\frac{\widehat{U}_n(t)}{2^n}\delta^{(n)}f_1\Big\| = \\ &= \frac{1}{\pi}\Big(\frac{\pi}{2}\Big)^{1/2}\frac{1}{2^n}\Big\|[\delta^{(n)}f_1SK_1(x,t),P^{(n-1)}\widehat{U}_n(t)]\Big\| = \\ &= \frac{1}{\pi}\Big(\frac{\pi}{2}\Big)^{1/2}\frac{1}{2^n}\Big\|[P^{(n-1)}\delta^{(n)}f_1SK_1(x,t),\widehat{U}_n(t)]\Big\| \le \\ &\le \frac{1}{\pi}\Big(\frac{\pi}{2}\Big)^{1/2}\frac{1}{2^n}\Big\|\ \Big\|P^{(n-1)}\delta^{(n)}f_1SK_1(x,t)\Big\|\times\|\widehat{U}_n(t)\|\ \Big\| \le \\ &\le \frac{1}{\pi}\Big(\frac{\pi}{2}\Big)^{1/2}\frac{1}{2^n}\Big\|\ \Big\|P^{(n-1)}\delta^{(n)}f_1SK_1(x,t)\Big\|\ \Big\|.\end{aligned}$$

Under the conditions of the theorem

$$\Big(\delta^{(n)}f_1SK_1(x,t)\Big)^{(m)} \in \mathrm{Lip}_M\,\alpha,\quad 0<\alpha\le 1,\quad \forall x\in[-1,1].$$

Therefore we have $\|P^{(n-1)}\delta^{(n)}f_1SK_1(x,t)\| \le \frac{C'_m 2^{m+\alpha}M}{(n-1)^{m+\alpha}}$,

$$\Big\|KS^{-1}\Pi^{(n-1)}f_1\Big\| = O\Big(\frac{1}{2^n}\cdot\frac{1}{(n-1)^{m+\alpha}}\Big). \tag{25}$$

With the help of the obtained estimates (24) and (25), from (23) we finally get

$$\|u-\widetilde{u}_n\| = O\Big(\frac{1}{2^n}\cdot\frac{1}{(n-1)^{m+\alpha}}\Big). \qquad \square$$

Remark 1. Under the conditions of the theorem we obtain for the approximate solution u_n that

$$\|u-u_n\| \le C\|\Pi^{(n-1)}w\| = C\Big(\frac{\pi}{2}\Big)^{1/2}\Big\|\frac{\widehat{U}_n(t)}{2^n}\delta^{(n)}w\Big\| \le$$

$$\le \frac{C_1}{2^n}\|\widehat{U}_n(t)\delta^{(n)}w(t)\| = \frac{C_1}{2^n}\Big\{\int_{-1}^{1}(1-t^2)^{1/2}\widehat{U}_n^2(t)(\delta^{(n)}w)^2dt\Big\}^{1/2} \le$$

$$\leq \frac{C_1}{2^n}\|\delta^{(n)}w\|_C\Big\{\int\limits_{-1}^{1}(1-t^2)^{1/2}\widehat{U}_n^2(t)dt\Big\}^{1/2}=$$

$$=\frac{C_1}{2^n}\|\delta^{(n)}w\|_C\times\|\widehat{U}_n(t)\|=\frac{C_1}{2^n}\|\delta^{(n)}w\|_C=\frac{C_1}{2^n}\|\delta^{(n)}(f_1-K\Phi)\|_C=$$

$$=\frac{C_1}{2^n}\Big\|\delta^{(n)}(f_1-\frac{1}{\pi}\big[K_1(x,t),\Phi(t)\big]\Big\|_C\leq$$

$$\leq\frac{C_1}{2^n}\Big(\|\delta^{(n)}f_1\|_C+\frac{1}{\pi}\Big\|\big[\delta^{(n)}K_1(x,t),\Phi(t)\big]\Big\|_C\leq$$

$$\leq\frac{C_1}{2^n}\Big(\|\delta^{(n)}f_1\|_C+\frac{1}{\pi}\Big\|\ \|\delta^{(n)}K_1(x,t)\|\times\|\Phi(t)\|\ \Big\|_C\leq$$

$$\leq\frac{C_1}{2^n}\Big(\|\delta^{(n)}f_1\|_C+\frac{1}{\pi}\Big\|\ \|\delta^{(n)}K_1(x,t)\|_C\Big)\leq\frac{C_2}{2^n}.$$

2.2. Index $\varkappa=-1$. Introduce the subspace $C_0[-1,1]\subset C[-1,1]$; $v\in C_0[-1,1]$ if $[v,1]=0$. $C^{(n)}[-1,1]\subset C[-1,1]$ is a linear span of polynomials $\psi_0,\psi_1,\dots,\psi_n$. The projector can be determined as follows [11]:

$$\Pi_n v=L_n v-a_0^{(n)}\psi_0,$$

where $L_n v\in C^{(n)}[-1,1]$ is the Lagrange polynomial and $a_0^{(n)}$ is the coefficient of the Fourier series expansion $a_0^{(n)}\equiv[L_n v,\psi_0]$.

Again, as in Subsection 1.2, we seek an approximate solution in the form

$$u_n=\sum_{k=1}^{n}a_k\varphi_k.$$

We compose the algebraic system by the condition

$$\Pi_n(Su_n+Ku_n-f)=0,$$

which results in

$$a_0\psi_0+\sum_{k=1}^{n}a_k\psi_k(x_j)+\sum_{k=1}^{n}a_k(K\varphi_k)(x_j)=f(x_j),\ \ j=0,1,\dots,n.$$

Using the projector, we can rewrite this algebraic system as

$$w_n+\Pi_nKS^{-1}w_n=\Pi_nf,\ \ w_n\in L_{2,\rho}^{(n)},$$

where $L_{2,\rho}^{(n)}$ is the linear span of the system of functions $\psi_0,\psi_1,\dots,\psi_n$.

Let the approximate solution w_n be found. Taking one iteration

$$\widetilde{w}_n=-KS^{-1}w_n+f=-Ku_n+f,$$

we find

$$\widetilde{u}_n = S^{-1}\widetilde{w}_n = \frac{(1-t^2)^{1/2}}{\pi}\int_{-1}^{1}(1-x^2)^{-1/2}\frac{\widetilde{w}_n(x)dx}{t-x}.$$

Denote $K_1(x,t) \equiv (1-t^2)^{1/2}K(x,t)$.

Theorem 5. *If there exists the inverse operator $(I+KS^{-1})^{-1}$ mapping $L_{2,\rho}^{(r)}$, $\rho=\rho_2$, onto itself, the roots of the Chebyshev polynomial of the first kind T_{n+1} are taken as collocation nodes, $(SK_1(x,t))^m \in \mathrm{Lip}_M\,\alpha$, $0<\alpha\le 1$, $\forall x \in [-1,1]$, and the divided differences of all orders of the functions $f(x)$ and $K_1(x,t)$ with respect to x are uniformly bounded $\forall t \in [-1,1]$, then the estimate*

$$\|u-\widetilde{u}_n\| = O\Big(\frac{1}{2^n}\cdot\frac{1}{n^{m+\alpha}}\Big)$$

is valid.

This theorem as well as the next one can be proved similarly to Theorem 4.

Remark 2. As in Subsection 2.1, under the conditions of the theorem we obtain

$$\|u-u_n\| = O\Big(\frac{1}{2^n}\Big).$$

2.3. Index $\varkappa = 0$. As in Subsection 1.3, an approximate solution is again sought in the form

$$u_n = \sum_{k=1}^{n} a_k\varphi_k.$$

Equating the residuals to zero at the points $x_1,\dots,x$, we obtain

$$\big[Su_n + Ku_n - f\big]_{x_j} = 0 \quad j=0,1,\dots,n,$$

which yields the algebraic system

$$\sum_{k=0}^{n} a_k\psi_k(x_j) + \sum_{k=0}^{n} a_k(K\varphi_k)(x_j) = f(x_j), \quad j=0,1,\dots,n.$$

Just as for the index $\varkappa = 1$ we can rewrite this system as

$$w_n + \Pi_n KS^{-1}w_n = \Pi_n f, \quad w_n \in L_{2,\rho}^{(n)},$$

where $L_{2,\rho}^{(n)}$ is the linear span of functions $\psi_0,\dots,\psi_n$.

Let the approximate solution w_n be found.

Taking one iteration

$$\widetilde{w}_n = -KS^{-1}w_n + f = -Ku_n + f$$

we find

$$\widetilde{u}_n = S^{-1}\widetilde{w}_n = \frac{(1+t)^{1/2}(1-t)^{-1/2}}{\pi}\int_{-1}^{1}(1-x)^{1/2}(1+x)^{-1/2}\frac{\widetilde{w}_n(x)dx}{t-x}.$$

Denote $K_1(x,t) \equiv (1+t)^{1/2}(1-t)^{-1/2}K(x,t)$.

Theorem 6. *If there exists the inverse operator $(I+KS^{-1})^{-1}$ mapping $L_{2,\rho}$, $\rho=\rho_3$, onto itself, the roots of the Jacobi polynomial $P_{n+1}^{(\frac{1}{2},-\frac{1}{2})}$ are taken as collocation nodes, $(SK_1(x,t))^{(m)} \in L:p_M\alpha$, $0<\alpha\le 1$, $\forall x\in[-1,1]$, and the divided differences of all orders of the functions $f(x)$ and $K_1(x,t)$ with respect to x are uniformly bounded $\forall t\in[-1,1]$, then the estimate*

$$\|u-\widetilde{u}_n\| = O\Big(\frac{1}{2^n}\cdot\frac{1}{n^{m+\alpha}}\Big)$$

is valid.

Remark 3. Under the conditions of the theorem

$$\|u-u_n\| = O\Big(\frac{1}{2^n}\Big).$$

Remark 4. If we require only that $w^{(m)}\in \operatorname{Lip}_M\alpha$, $0<\alpha\le 1$, then for the collocation method for all values of the index $\varkappa = 1,0,-1$ we obtain the same order of convergence

$$\|u-u_n\| = O\Big(\frac{\ln\, n}{n^{m+\alpha}}\Big)$$

for an approximate solution u_n in the respective weighted spaces.

Indeed, for any $v\in C[-1,1]$ we have $\Pi^{(n)}v = \Pi^{(n)}P^{(n)}v$, where $\Pi^{(n)}\equiv I-\Pi_n$, $P^{(n)}\equiv I-P_n$, where Π_n is the Lagrange interpolation operator, and P_n is the orthoprojector with respect to polynomials $\widehat{T}_0,\widehat{T}_1,\dots,\widehat{T}_n$. If the nodes in the interpolation Lagrange polynomial are taken with respect to the weight, then $L_n: C\to L_{2,\rho}$ are bounded by the Erdös–Turan theorem. Therefore (see [13])

$$\|u-u_n\|_{L_{2,\rho}} \le C\|\Pi^{(n)}u\|_{L_{2,\rho}} \le C_1\|P^{(n)}u\|_C = \\ = O\Big(\frac{\ln\, n}{n^{m+\alpha}}\Big)\quad (\Pi_n = L_n).$$

As an example of the application of the above methods in the case of the index $\varkappa=-1$, let us consider the equation

$$\frac{1}{\pi}\int_{-1}^{1}\frac{u(t)dt}{t-x} + \frac{1}{\pi}\int_{-1}^{1}(x^8t^8+x^7t^7)u(t)dt =$$

$$= \left(\frac{2}{\pi}\right)^{1/2}\left(\frac{3}{128}x^7 - 32x^6 + 48x^4 - 18x^2 + 1\right)$$

with the exact solution

$$u(x) = \varphi_6(x) = \left(\frac{2}{\pi}\right)^{1/2}(32x^5 - 32x^3 + 6x)(1-x^2)^{1/2},$$

where $\{\varphi_k(x)\}$, $k = 0, 1, \dots$, is the orthonormal system of functions in L_{2,ρ_2}.

We find the fifth approximation

$$u_5(x) = \sum_{k=1}^{5} a_k\varphi_k(x), \quad \varphi_k(x) = \left(\frac{2}{\pi}\right)^{1/2}(1-x^2)^{1/2}U_{k-1}(x),$$

where $U_{k-1}(x)$, $k = 1, 2, \dots$, are the Chebyshev polynomials of the second kind.

Computations are carried out to within 10^{-7}. $u_5(x)$ and $\widetilde{u}_5(x)$ are calculated.

In the case of the Bubnov–Galerkin method we have

$$\|\Delta u_5\| = 1,0001151, \quad \|\Delta\widetilde{u}_5\| = 0,0151855,$$
$$\frac{\|\Delta u_5\|}{\|u\|} \approx 100,01\%, \quad \frac{\|\Delta\widetilde{u}_5\|}{\|u\|} \approx 1,52\%$$

for an absolute and a relative error, respectively, while in the case of the collocation method we obtain

$$\|\Delta u_5\| = 1,0002931, \quad \|\Delta\widetilde{u}_5\| = 0,0159781,$$
$$\frac{\|\Delta u_5\|}{\|u\|} \approx 100,03\%, \quad \frac{\|\Delta\widetilde{u}_5\|}{\|u\|} \approx 1,60\%.$$

The result is the expected one for $u_5(x)$, since in our example the function $\varphi_6(x)$ is the exact solution.

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