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ON THE TWO QUESTIONS OF LOHWATER AND PIRANIAN

M. GVARADZE

Abstract. The problem we are dealing with consists in the following: find the necessary and sufficient conditions for the zero measure subset of the circumference at which points the bounded analytic function has no radial limits.

1. For a function $f : D \to \mathbb{C}$ analytic and bounded in the unit disk $D = \{z : z \in \mathbb{C}, |z| < 1\}$ and any point $e^{i\theta} \in C = \{e^{i\tau} : 0 \le \tau < \pi\}$ let $f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ denote the radial limit of f. Fatou proved in 1906 that for such a function there exist radial limits except maybe for a set of points $e^{i\theta}$ of linear measure 0. Conversely, as Lusin showed in [1], for any set E of measure zero on C there exists a function analytic and bounded in D having no radial limits at the points of E.

Lohwater and Piranian noticed in [2] that "the set of nonexistence of radial limit is of second category for some bounded regular functions; it can even be a residual set on C; *but we do not know of any case where a set E of second category, prescribed without reference to function theory, has been established as the precise set where the radial limit of some bounded regular function fails to exist.*"

This is the first question we have to answer in this note. The second one is connected with the following statement (Theorem 8 in [2]).

Theorem. *"Let the set E on C be of types F_σ and G_δ and of measure zero. Then there exists a function $f(z)$, regular and bounded in D, which has the following properties: for each point $e^{i\theta}$ in E,*

$$\liminf_{r\to 1} \left|f(re^{i\theta})\right| = 0, \quad \varlimsup_{r\to 1} \left|f(re^{i\theta})\right| = 1;$$

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for each point $e^{i\theta}$ in $C\backslash E$ the radial limit $f(e^{i\theta})$ exists and has modulus 1, except for a denumerable set of points where $f(e^{i\theta})=0$."

In 1956 Lohwater and Piranian did not know *"whether the converse of the theorem is true."*

Theorems 1 and 2 proved below give answers to these two questions.

2. To prove the theorems we need several lemmas. Some of them are of independent interest. In what follows

$$C(a,r)=\{z:\ |z-a|\le r\},\quad D(a,r)=\{z:\ |z-a|<r\};$$

the length of an interval I will be denoted by the same letter I.

Lemma 1. *Let $z_0=e^{it_0}$ and $0<\alpha<1$.*

If $z\in D(0,1)\backslash D(\alpha z_0,1-\alpha)$, then $\operatorname{Re}\dfrac{z_0+z}{z_0-z}\le\dfrac{\alpha}{1-\alpha}$.

If $z\in C(\alpha z_0,1-\alpha)$, then $\operatorname{Re}\dfrac{z_0+z}{z_0-z}=\dfrac{\alpha}{1-\alpha}$.

If $z\in D(\alpha z_0,1-\alpha)$, then $\operatorname{Re}\dfrac{z_0+z}{z_0-z}>\dfrac{\alpha}{1-\alpha}$.

If $0<t<\pi/2$, then

$$\sup_{0\le r\le 1}\left|\frac{1+re^{it}}{1-re^{it}}\right|<\left|\cot\frac{t}{2}\right|.$$

For $0<r<1$

$$\left|\frac{z_0+re^{it}}{z_0-re^{it}}-i\cot\frac{t-t_0}{2}\right|=$$
$$=\frac{1-r^2}{(1-r)^2+4r\sin^2\frac{t-t_0}{2}}\left|1-i\frac{1-r}{1+r}\cot\frac{t-t_0}{2}\right|.$$

Proof. The linear-fractional mapping $\frac{z_0+z}{z_0-z}$ maps the circle $C(\alpha z_0,1-\alpha)$ on the straight line $x=\alpha/(1-\alpha)$ and the disk $D(\alpha z_0,1-\alpha)$ on the halfplane $\{z:\ \operatorname{Re}z>\alpha/(1-\alpha)\}$ which imply the first three assertions of the lemma. To prove the next inequality note that the straight line containing the radius $\{re^{it}:\ 0\le r\le 1\}$ is mapped by the mapping $\frac{1+z}{1-z}$ on the circle having the center on the imaginary axis and intersecting this axis at the points $-i\tan\frac{t}{2}$ and $a=i\cot\frac{t}{2}$. The radius itself is mapped onto the smaller arc of this circle (lying in the first quadrant) with the endpoints 1 and $b=\frac{1+e^{it}}{1-e^{it}}$. Simple analysis of the triangle with vertices 0, a, and b shows that $|a|>|b|$.

The last assertion is proved as follows ($\tau=t-t_0$):

$$\frac{z_0+re^{it}}{z_0-re^{it}}-i\cot\frac{\tau}{2}=\frac{1+re^{i\tau}}{1-re^{i\tau}}-i\cot\frac{\tau}{2}=$$

$$= \frac{1-r^2}{(1-r)^2+4r\sin^2\frac{\tau}{2}} + i\frac{2r\sin\tau}{(1-r)^2+4r\sin^2\frac{\tau}{2}} - i\cot\frac{\tau}{2} =$$
$$= \frac{1-r^2}{(1-r)^2+4r\sin^2\frac{\tau}{2}}\Big(1 - i\frac{1-r}{1+r}\cot\frac{\tau}{2}\Big). \quad \square$$

Lemma 2. *$\rho(e^{it}, C(\alpha, 1-\alpha)) \geq 2\alpha\sin^2\frac{t}{2}$, where $\rho(E,F)$ is a distance between the sets E and F, and $\alpha \in (0, \frac{1}{2})$.*

Proof. It is evident that $\rho(e^{it}, C(\alpha, 1-\alpha)) = |e^{it}-\alpha| - (1-\alpha) =$ $= \sqrt{(1-\alpha)^2 + 4\alpha\sin^2\frac{t}{2}} - (1-\alpha) \geq 2\alpha\sin^2\frac{t}{2}$ (equality holds if $t = k\pi$, $k \in \mathbb{Z}$). $\square$

Lemma 3. *Suppose $\theta_n = \arg z_n$ and $\sum\limits_{n=1}^{\infty} I_n|\cot\frac{\theta-\theta_n}{2}| < \infty$. Then the function $g(z) = \sum\limits_{n=1}^{\infty} I_n\frac{z_n+z}{z_n-z}$ has a radial limit of modulus one at the point $e^{i\theta}$.*

Proof. By Lemma 1, $\big|\sum_{n=1}^{\infty} I_n\frac{z_n+z}{z_n-z}\big| \leq \sum_{n=1}^{\infty} I_n\big|\cot\frac{\theta-\theta_n}{2}\big|$. Therefore g is continuous on the radius $[0, e^{i\theta}]$. $\square$

Let $d(E,F)$ denote an arcdistance between the subsets E and F of C and ∂G be a boundary of G.

Lemma 4. *Let $G \subset G_1 \subset G_0$ be open subsets of the circle. Suppose $G = \bigcup\limits_{n=1}^{\infty} I_n$, $G_1 = \bigcup\limits_{n=1}^{\infty} J_n$ and $z_n \in \overline{I_n}$. If $e^{i\theta} \notin G_0$ and $d(J_k, \partial G_0) > 2^k J_k$, then there exists $\lim\limits_{r\to 1}\sum\limits_{n=1}^{\infty} I_n\frac{z_n+re^{i\theta}}{z_n-ze^{i\theta}}$.*

Proof. If $I_n \subset J_k$, then

$$\sup_{0\leq r\leq 1}\Big|\frac{z_n+re^{i\theta}}{z_n-re^{i\theta}}\Big| \leq \Big|\cot\frac{\theta-\theta_n}{2}\Big| \leq \cot\frac{d(e^{i\theta}, J_k)}{2},$$

and therefore

$$\sum_{k=1}^{\infty}\sum_{I_n\subset J_k}\Big|I_n\frac{z_n+re^{i\theta}}{z_n-re^{i\theta}}\Big| \leq \sum_{k=1}^{\infty}\cot\frac{d(e^{i\theta}, J_k)}{2}\sum_{I_n\subset J_k} I_n \leq$$
$$\leq \sum_{k=1}^{\infty} J_k\Big|\cot\frac{d(e^{i\theta}, J_k)}{2}\Big| < \infty.$$

Using Lemma 3, we can prove Lemma 4. $\square$

The following lemma is the basic one.

Lemma 5. *Suppose* $G=\bigcup_{n=1}^{\infty} I_n$ *to be an open set. Let* $z_n \in I_n$, $d(z_n, \partial I_n) > I_n/4$ *and let*

$$\sum_{n\notin\{n_k\}} \frac{I_n}{d(e^{i\theta}, I_n)} < \infty, \lim_{k\to\infty} \frac{I_{n_{k+1}}}{I_{n_k}} = 0.$$

Then the function $g(z) = \sum_{n=1}^{\infty} I_n \frac{z_n+z}{z_n-z}$ *has radial limit at the point* $e^{i\theta}$ *if and only if the series*

$$\sum_{n=1}^{\infty} I_n \cot \frac{\theta - \arg z_n}{2} \tag{1}$$

is convergent.

Proof. By Lemma 3 since the function $\sum_{n\notin\{n_k\}} I_n \frac{z_n+z}{z_n-z}$ has radial limit at the point $e^{i\theta}$, then without loss of generality we may assume that

$$\lim_{n\to\infty} \frac{I_{n+1}}{I_n} = 0, \quad \frac{I_{n+1}}{I_n} \le \frac{1}{2}.$$

Denote $\rho_n = 1 - I_{n+1}$ and suppose that $\rho_{N-1} < |z| \le \rho_N$. We have

$$g(z) = \sum_{n=1}^{N-1} I_n \Big(\frac{z_n+z}{z_n-z} - i\cot\frac{\theta-\arg z_n}{2}\Big) + I_N \frac{z_N+z}{z_N-z} +$$
$$+ I_{N+1}\frac{z_{N+1}+z}{z_{N+1}-z} + \sum_{N+2}^{\infty} I_n \frac{z_n+z}{z_n-z} + i\sum_{n=1}^{N-1} I_n \cot\frac{\theta-\arg z_n}{2}. \tag{2}$$

If $n \ge N+2$, then

$$I_n \Big|\frac{z_n+z}{z_n-z}\Big| \le 2\frac{I_n}{1-|z|} \le 2\frac{I_n}{I_{N+1}} \le 2^{N-n}\frac{I_{N+2}}{I_{N+1}},$$

whence

$$\Big|\sum_{n=N+2}^{\infty} I_n \frac{z_n+z}{z_n-z}\Big| \le \frac{I_{N+2}}{I_{N+1}}. \tag{3}$$

By Lemma 1

$$I_n\Big|\frac{z_n+z}{z_n-z} - i\cot\frac{\theta-\arg z_n}{2}\Big| \le 4\pi^3 I_n \frac{1-r}{(\theta-\arg z_n)^2} \le 16\pi^3 \frac{I_N}{I_n}.$$

Therefore

$$\Big|\sum_{n=1}^{N-1} I_n \Big(\frac{z_n+z}{z_n-z} - i\cot\frac{\theta-\arg z_n}{2}\Big)\Big| \le 16\pi^3 \frac{I_N}{I_{N-1}}. \tag{4}$$

If $\lim\limits_{n\to\infty} I_n \cot\frac{\theta-\arg z_n}{2}=0$, then by Lemma 1, $\lim\limits_{n\to\infty} I_n\frac{z_n+z}{z_n-z}=0$. Thus, by (3) and (4) the right-hand side of (2) has radial limit if and only if (1) is convergent.

Finally, if $\overline{\lim\limits_{n\to\infty}}\, I_n|\cot\frac{\theta-\arg z_n}{2}|\geq\delta>0$, we have

$$\operatorname{Re} I_{N+1}\frac{z_{N+1}+\rho_N e^{i\theta}}{z_{N+1}-\rho_N e^{i\theta}}\geq I_{N+1}\frac{1-\rho_N^2}{(1-\rho_N)^2+4\rho_N\sin^2\frac{\theta-\theta_N}{2}}\geq$$
$$\geq I_{N+1}\frac{1-\rho_N}{(1-\rho_N)^2+4\frac{I_{N+1}^2}{\delta^2}}=\frac{I_{N+1}^2}{I_{N+1}^2+\frac{4}{\delta^2}I_{M+1}^2},$$

since $\sin^2\frac{\theta-\theta_{N+1}}{2}\leq\frac{I_{N+1}^2}{\delta^2}$, where $\theta_N=\arg z_{N+1}$;

$$\operatorname{Re} I_N\frac{z_N+\rho_N e^{i\theta}}{z_N-\rho_N e^{i\theta}}\leq I_N\frac{2(1-\rho_N)}{(1-\rho_N)^2+4\rho_N\sin^2\frac{\theta-\theta_N}{2}}\leq$$
$$\leq\frac{2I_NI_{N+1}}{4\rho_N(\frac{2}{\pi}\frac{\theta-\theta_N}{2})^2}\leq\frac{2I_NI_{N+1}}{4\frac{1}{2}\frac{1}{\pi^2}(\frac{1}{4}I_N)^2}=16\pi^2\frac{I_{N+1}}{I_N},$$

since $|\theta-\theta_N|\geq\frac{1}{4}I_N$, $e^{i\theta_N}\notin I_N$, $\rho_N\geq\frac{1}{2}$.

By the notation $\rho_N'=1-\sqrt{I_NI_{N+1}}$ we get

$$\operatorname{Re} I_{N+1}\frac{z_{N+1}+\rho_N' e^{i\theta}}{z_{N+1}-\rho_N' e^{i\theta}}\leq I_{N+1}\frac{2(1-\rho_N')}{(1-\rho_N')^2}\leq$$
$$\leq 2\frac{I_{N+1}}{1-\rho_N'}=2\frac{I_{N+1}}{\sqrt{I_NI_{N+1}}}=2\sqrt{\frac{I_{N+1}}{I_N}}$$

and

$$\operatorname{Re} I_N\frac{z_N+\rho_N' e^{i\theta}}{z_N-\rho_N' e^{i\theta}}\leq I_N\frac{2(1-\rho_N')}{(I_N^2)/(16\pi^2)}=$$
$$=32\pi^2\frac{I_N\sqrt{I_NI_{N+1}}}{I_N^2}=32\pi^2\sqrt{\frac{I_{N+1}}{I_N}}.$$

Taking into acount these inequalities we can conclude that the right-hand side of (2) has no radial limit. □

Definition 1. We say that E is an *arrangeable* set if there exist a countable set $(z_{mn})=(e^{i\theta_{mn}})$, a sequence of intervals (arcs) (I_{mn}), and a sequence of positive numbers (α_k) satisfying the following conditions:

1. $z_{mn}\in I_{mn}$, $4d(z_{mn},\partial I_{mn})>I_{mn}$;
2. $\lim\limits_{n\to\infty}\frac{I_{m(n+1)}}{I_{mn}}=0$, $m=1,2,\dots$;
3. $I_{(m+1)n}\subset\bigcup\limits_{i=1}^{\infty}(I_{mi}\backslash\{z_{mi}\})$, $m=1,2,\dots$;

4. $\sqrt{I_{mn}} < 2^{-n}\alpha_m\alpha_{m+1}d(I_{mn},\partial G_{m-1})$, $m>1$, where $G_m=\bigcup\limits_{n=1}^{\infty}(I_{mn}\setminus\{z_{mn}\})$, $a_k\searrow 0$, $\sum\alpha_k<\infty$;

5. $E=(\bigcap\limits_{m=1}^{\infty}G_m)\cup(\bigcup\limits_{m=1}^{\infty}Q_m)$, where

$$Q_m=\Big\{e^{i\theta}:\ e^{i\theta}\notin G_m,\ \sum_{n=1}^{\infty}I_{mn}\cot\frac{\theta-\theta_{mn}}{2}\ \text{ is divergent}\Big\}.$$

Lemma 6. *Let E be an arrangeable set. Then the function $f(z)=\exp\{-g(z)\}$, with $g(z)=\sum\limits_{m=1}^{\infty}g_m(z)$, $g_m(z)=\frac{1}{\alpha_m}\sum\limits_{n=1}^{\infty}I_{mn}\frac{z_{mn}+z}{z_{mn}-z}$, has no radial limit on the set E only.*

Proof. Since

$$\sum_{m=1}^{\infty}\frac{1}{\alpha_m}\sum_{n=1}^{\infty}I_{mn}\le\sum_{m=1}^{\infty}\frac{1}{\alpha_m}\sum_{n=1}^{\infty}2^{-2n}\alpha_m^2\alpha_{m+1}^2<\infty,$$

we conclude that f is analytic in the unit disk.

Suppose $e^{i\theta}\notin G_1$. We have

$$\sum_{m=2}^{\infty}\frac{1}{\alpha_m}\sum_{n=1}^{\infty}I_{mn}\Big|\cot\frac{\theta-\theta_{mn}}{2}\Big|<\sum_{m=2}^{\infty}\frac{1}{\alpha_m}\sum_{n=1}^{\infty}I_{mn}\frac{1}{d(I_{mn},\partial G_{m-1})}<$$
$$<\sum_{m=2}^{\infty}\frac{1}{\alpha_m}\sum_{n=1}^{\infty}2^{-4n}\alpha_m^2\alpha_{m+1}^2d(I_{mn},\partial G_{m-1})<\sum_{m=2}^{\infty}\alpha_m\alpha_{m+1}^2<\infty.$$

Hence by Lemma 3, $g-g_1$ has radial limit at $e^{i\theta}$, and thus by our basic lemma we can conclude that g has no radial limit only at Q_1.

Now if $e^{i\theta}\in G_m\setminus G_{m+1}$, $m=1,2,\dots$, then

$$\sum_{k=1}^{m}\frac{1}{\alpha_k}\sum_{n=1}^{\infty}I_{kn}|\cot\frac{\theta-\theta_{kn}}{2}|<\frac{1}{d(e^{i\theta},\partial G_m)}\sum_{k=1}^{m}\frac{1}{\alpha_k}\sum_{n=1}^{\infty}I_{kn}<\infty,$$

and

$$\sum_{k=m+2}^{\infty}\frac{1}{\alpha_k}\sum_{n=1}^{\infty}I_{kn}|\cot\frac{\theta-\theta_{kn}}{2}|<$$
$$<\sum_{k=m+2}^{\infty}\frac{1}{\alpha_k}\sum_{n=1}^{\infty}I_{kn}\frac{1}{d(I_{kn},\partial G_{k-1})}<\sum_{k=m+2}^{\infty}\alpha_k\alpha_{k+1}^2<\infty.$$

Therefore arguing as above we come to the conclusion that g has no radial limit at Q_{m+1}. Suppose now that $e^{i\theta}\in\bigcap\limits_{m=1}^{\infty}G_m$, and denote $O_m=\bigcup\limits_{n=1}^{\infty}D(\alpha_m^2z_{mn},1-\alpha_m^2)$, $C_m=\partial O_m$.

Let $z \in C_m$ and $k \leq m$; then since $z \notin D(\alpha_m^2 z_{mn}, 1-\alpha_m^2)$, we have

$$\operatorname{Re} g_k(z) \leq \frac{1}{\alpha_k} \sum_{n=1}^{\infty} I_{kn} \frac{\alpha_m^2}{1-\alpha_m^2} \leq \frac{2\alpha_m^2}{\alpha_k} \mu G_k \leq \alpha_m.$$

Further, if $z \in C_m$ and $k > m$, then according to Lemma 2,

$$\operatorname{Re} \frac{z_{kn}+z}{z_{kn}-z} \leq \frac{2}{\rho(z_{kn}, C_m)} < \frac{1}{\alpha_m^2 \sin^2 \frac{\theta_{kn}-\theta_{mi}}{2}} < \frac{\pi^2}{\alpha_m^2 d^2(I_{kn}, \partial G_{k-1})}\ .$$

Hence

$$\operatorname{Re} g_k(z) \leq \frac{1}{\alpha_k} \sum_{n=1}^{\infty} 2^{-2n} \alpha_k^2 \alpha_{k+1}^2 d^2(I_{kn}, \partial G_{k-1}) \frac{\pi^2}{\alpha_m^2 d^2(I_{kn}, \partial G_{k-1})} \leq$$

$$\leq \frac{\alpha_k \alpha_{k+1}^2}{\alpha_m^2} \leq \alpha_{k+1}.$$

Thus, if $z \in C_m$, then

$$\operatorname{Re} g(z) \leq \sum_{k=1}^{m} \alpha_m + \sum_{k=m+1}^{\infty} \alpha_k = m\alpha_m + \sum_{k=m+1}^{\infty} \alpha_k,$$

whence

$$\varliminf_{r\to 1} \operatorname{Re} g(re^{i\theta}) = 0. \tag{5}$$

Since $e^{i\theta} \in G_m$, there exists an integer j such that $e^{i\theta} \in I_{mj}$. It is clear that the radius $[0, e^{i\theta}]$ intersects the circle $C((1-I_{mj})z_{mj}, I_{mj})$ in which

$$\operatorname{Re} g(re^{i\theta}) \geq \operatorname{Re} g_m(re^{i\theta}) \geq \operatorname{Re} \frac{1}{\alpha_m} I_{mj} \frac{z_{mj}+z}{z_{mj}-z} \geq \frac{1}{2\alpha_m}.$$

Hence, $\varlimsup_{r\to 1} \operatorname{Re} g(re^{i\theta}) = \infty$, which together with (5) gives

$$\varliminf_{r\to 1} |f(re^{i\theta})| = 0, \quad \varlimsup_{r\to 1} |f(re^{i\theta})| = 1. \qquad \square$$

3. Let us now formulate and prove our theorems.

Theorem 1. *There exists an arrangeable set of second category.*

Proof. Let $Q = \{x_n\}$ be a countable subset of a unit circle and G_0 be an open subset covering Q and $\mu G_0 < 2\pi$ (μ denotes the Lebesgue measure).

Cover x_1 by the interval J_1 such that $\partial J_1 \cap Q = 0$ and $\sqrt{J_1} < 2^{-1}\alpha_1\alpha_2$ $d(J_1, \partial G_0)$. Let x_{k_2} be the first element of the sequence Q not belonging to J_1. Cover x_{k_2} by the interval J_2 such that $\partial J_2 \cap Q = 0$, $J_1 \cap J_2 = \varnothing$, $\sqrt{J_2} < 2^{-2}\alpha_1\alpha_2 d(J_2, \partial G_0)$, $\frac{J_2}{J_1} < \frac{1}{2}$.

Given $J_1, J_2, \dots, J_{n-1}$, let x_{k_n} be the first element of Q not belonging to $\bigcup\limits_{k=1}^{n-1} J_k$. Cover x_{k_n} by the interval J_n such that $\partial J_n \cap Q = \varnothing$, $J_n \cap (\bigcup\limits_{k=1}^{n-1} G_k) = \varnothing$, $\sqrt{J_n} < 2^{-n}\alpha_1\alpha_2 d(J_n, \partial G_0)$, $\frac{J_n}{J_{n-1}} < \frac{1}{n}$. Select z_{1n} such that $z_{1n} \in J_n, 4d(z_{1n}, \partial J_n) > J_n$ and $z_{1n} \notin Q$. Denote $I_{1n} = J_n \backslash \{z_{1n}\}$ and $G_1 = \bigcup\limits_{n=1}^{\infty} I_{1n}$.

Taking G_1 instead of G_0 and arguing analogously, we will obtain new sequences of the intervals J_n and points z_{2n} with the following properties:

(a) $z_{2n} \in J_n, z_{2n} \notin Q$;

(b) $J_n \cap J_k = \varnothing$, $n \neq k$;

(c) $4d(z_{2n}, J_n) > J_n$;

(d) $\sqrt{J_n} < 2^{-n}\alpha_2\alpha_3 d(J_n, \partial G_1)$;

(e) $\lim\limits_{n\to\infty} \dfrac{J_{n+1}}{J_n} = 0$.

Denote $I_{2n} = J_n \backslash \{z_{2n}\}$ and $G_2 = \bigcup\limits_{n=1}^{\infty} I_{2n}$.

Repeating the above process for G_2, G_3, etc., we shall get the sequences z_{mn} and I_{mn} which define the arrangeable set E. If Q is a countable everywhere dense subset, then the obtained set E will be of second category. $\square$

Theorem 2. *There exist a set E and a bounded analytic function f with the following conditions:*

(a) *E is of G_δ type;*

(b) *E is not of F_σ type;*

(c) *if $e^{i\theta} \in E$, then $\varliminf\limits_{r\to 1} |f(re^{i\theta})| = 0$, $\varlimsup\limits_{r\to 1} |f(re^{i\theta})| = 1$;*

(d) *the function f on $C\backslash E$ has radial limit of modulus one except a countable set where the radial limit is zero.*

Proof. Consider the Cantor set E without ends on its adjacent intervals. It is clear that E is of type G_δ and not of type F_σ.

Suppose $E = \bigcap\limits_{k=1}^{\infty} G_k$ and $\sum\limits_{k=1}^{\infty} \alpha_k < 1$. Without loss of generality we may assume that the ends of component intervals I_{1n} of G_1 belong to adjacent intervals of Cantor's set. Cover every $I_{1n} \cap \overline{E}$ by an open set H_{1n} such that $\sqrt{\mu H_{1n}} < 2^{-n}\alpha_1\alpha_2 d(H_{1n}, \partial I_{1n})$, where H_{1n} is a finite union of intervals. Put $R_1 = \bigcup\limits_{n=1}^{\infty} H_{1n} = \bigcup\limits_{k=1}^{\infty} J_{1k}$. Suppose z_{1k} is the right end of the interval J_{1k}.

If $e^{i\theta} \notin G_1$, we have

$$\sum_{k=1}^{\infty} J_{1k} \Big|\cot\frac{\theta - \theta_{1k}}{2}\Big| \le \sum_{n=1}^{\infty} \sum_{J_{1k}\subset I_{1n}} J_{1k} \frac{\pi}{d(H_{1n}, \partial I_{1n})} \le$$

$$\leq \pi \sum_{n=1}^{\infty} \frac{\mu H_{1n}}{d(H_{1n}, \partial I_{1n})} \leq \pi\alpha_1\alpha_2$$

and if $e^{i\theta} \in I_{1n}$, then

$$\sum_{J_{1k}\cap I_{1n}=\varnothing} J_{1k}|\cot\frac{\theta-\theta_{1k}}{2}| \leq \frac{\pi}{d(e^{i\theta}, \partial I_{1n})}\sum_{n=1}^{\infty}\mu H_{1n} = \frac{\pi\mu R_1}{d(e^{i\theta}, \partial I_{1n})}.$$

Hence, according to Lemma 3 and the fact that intervals contained in I_{1n} are finite, we conclude that $f_1(z) = \exp\{-g_1(z)\}$, and

$$g_1(z) = \frac{1}{\alpha_1}\sum_{k=1}^{\infty} J_{1k}\frac{z_{1k}+z}{z_{1k}-z}$$

has everywhere a radial limit. Put $R_1 \cap G_2 = \bigcup_{n=1}^{\infty} I_{2n}$. We assume again that the ends of I_{2n} belong to the adjacent intervals of Cantor's set. Cover every $I_{2n}\cap\overline{E}$ by open sets H_{2n} such that $\sqrt{\mu H_{2n}} < 2^{-n}\alpha_2\alpha_3 d(H_{2n}, \partial I_{2n})$, where H_{2n} is a finite union of intervals. Put $R_2 = \bigcup_{n=1}^{\infty} H_{2n} = \bigcup_{k=1}^{\infty} J_{2k}$. Suppose z_{2k} is the right end of J_{2k}.

If $e^{i\theta} \notin R_1 \cap G_2$, we have

$$\sum_{k=1}^{\infty} J_{2k}\left|\cot\frac{\theta-\theta_{2k}}{2}\right| \leq \pi\alpha_2\alpha_3$$

and if $e^{i\theta} \in I_{2n}$, then

$$\sum_{J_{2k}\cap I_{2n}=\varnothing} J_{2k}\left|\cot\frac{\theta-\theta_{2k}}{2}\right| \leq \frac{\pi\mu R_2}{d(e^{i\theta}, \partial I_{2n})}\ .$$

Hence, according to Lemma 3 and the fact that intervals contained in I_{2n} are finite, we conclude that $f_2(z) = \exp\{-g_2(z)\}$, and

$$g_2(z) = \frac{1}{\alpha_2}\sum_{k=1}^{\infty} J_{2k}\frac{z_{2k}+z}{z_{2k}-z}$$

has everywhere a radial limit.

Given $R_1, R_2, \dots, R_{m-1}$, put $R_{m-1} \cap G_m = \bigcup_{n=1}^{\infty} I_{mn}$. Assume that the ends of I_{mn} belong to adjacent intervals of Cantor's set. Cover every $I_{mn}\cap\overline{E}$ by open sets H_{mn} such that $\sqrt{\mu H_{mn}} < 2^{-n}\alpha_m\alpha_{m+1} d(H_{mn}, \partial I_{mn})$, where H_{mn} is a finite union of intervals. Put $R_m = \bigcup_{n=1}^{\infty} H_{mn} = \bigcup_{k=1}^{\infty} J_{mk}$. Suppose z_{mk} is the right end of J_{mk}.

If $e^{i\theta} \notin R_{m-1} \cap G_m$, we have

$$\sum_{k=1}^{\infty} J_{mk} \Big| \cot \frac{\theta - \theta_{mk}}{2} \Big| \leq \pi \alpha_m \alpha_{m+1} \tag{6}$$

and if $e^{i\theta} \in I_{mn}$, then

$$\sum_{J_{mk} \cap I_{mn} = \varnothing} J_{mk} | \cot \frac{\theta - \theta_{mk}}{2} | \leq \frac{\pi \mu R_m}{d(e^{i\theta}, \partial I_{mn})} .$$

The function $f_m(z) = \exp\{-g_m(z)\}$ with

$$g_m(z) = \frac{1}{\alpha_m} \sum_{k=1}^{\infty} J_{mk} \frac{z_{mk} + z}{z_{mk} - z}$$

has everywhere a radial limit.

Put

$$f(z) = \prod_{m=1}^{\infty} f_m(z) = \exp\Big\{ - \sum_{m=1}^{\infty} g_m(z) \Big\}.$$

It is evident that if $e^{i\theta} = z_{mk}$, then $\lim\limits_{r \to 1} f(re^{i\theta}) = 0$. Since $R_m \subset G_m$ and $R_m \supset E$, then $\bigcap\limits_{m=1}^{\infty} R_m = E$.

If $e^{i\theta} \notin E$, then there exists an integer n such that $e^{i\theta} \notin R_n$. By (6) we have

$$\sum_{m=n}^{\infty} \frac{1}{\alpha_m} \sum_{k=1}^{\infty} J_{mk} \Big| \cot \frac{\theta - \theta_{mk}}{2} \Big| \leq \pi \sum_{m=n}^{\infty} \alpha_{m+1} < \infty.$$

Therefore, by Lemma 3, Re $\sum\limits_{m=n}^{\infty} g_m(re^{i\theta})$ has zero radial limit.

Suppose finally $e^{i\theta} \in E$. Then using the definitions and arguing as in Lemma 6, we may conclude that

$$\varliminf_{r \to 1} \big| f(re^{i\theta}) \big| = 0, \quad \varlimsup_{r \to 1} \big| f(re^{i\theta}) \big| = 1. \qquad \square$$

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Author's address:
A. Razmadze Mathematical Institute of Georgian Academy of Sciences
1, M. Alexidze St., Tbilisi 380093, Republic of Georgia