

ON THE TWO QUESTIONS OF LOHWATER AND PIRANIAN

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ABSTRACT. The problem we are dealing with consists in the following: find the necessary and sufficient conditions for the zero measure subset of the circumference at which points the bounded analytic function has no radial limits.

1. For a function $f : D \rightarrow \mathbb{C}$ analytic and bounded in the unit disk $D = \{z : z \in \mathbb{C}, |z| < 1\}$ and any point $e^{i\theta} \in C = \{e^{i\tau} : 0 \leq \tau < \pi\}$ let $f(e^{i\theta}) = \lim_{r \rightarrow 1} f(re^{i\theta})$ denote the radial limit of f . Fatou proved in 1906 that for such a function there exist radial limits except maybe for a set of points $e^{i\theta}$ of linear measure 0. Conversely, as Lusin showed in [1], for any set E of measure zero on C there exists a function analytic and bounded in D having no radial limits at the points of E .

Lohwater and Piranian noticed in [2] that “the set of nonexistence of radial limit is of second category for some bounded regular functions; it can even be a residual set on C ; but we do not know of any case where a set E of second category, prescribed without reference to function theory, has been established as the precise set where the radial limit of some bounded regular function fails to exist.”

This is the first question we have to answer in this note. The second one is connected with the following statement (Theorem 8 in [2]).

Theorem. “Let the set E on C be of types F_σ and G_δ and of measure zero. Then there exists a function $f(z)$, regular and bounded in D , which has the following properties: for each point $e^{i\theta}$ in E ,

$$\liminf_{r \rightarrow 1} |f(re^{i\theta})| = 0, \quad \underline{\lim}_{r \rightarrow 1} |f(re^{i\theta})| = 1;$$

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for each point $e^{i\theta}$ in $C \setminus E$ the radial limit $f(e^{i\theta})$ exists and has modulus 1, except for a denumerable set of points where $f(e^{i\theta}) = 0$."

In 1956 Lohwater and Piranian did not know "whether the converse of the theorem is true."

Theorems 1 and 2 proved below give answers to these two questions.

2. To prove the theorems we need several lemmas. Some of them are of independent interest. In what follows

$$C(a, r) = \{z : |z - a| \leq r\}, \quad D(a, r) = \{z : |z - a| < r\};$$

the length of an interval I will be denoted by the same letter I .

Lemma 1. Let $z_0 = e^{it_0}$ and $0 < \alpha < 1$.

If $z \in D(0, 1) \setminus D(\alpha z_0, 1 - \alpha)$, then $\operatorname{Re} \frac{z_0 + z}{z_0 - z} \leq \frac{\alpha}{1 - \alpha}$.

If $z \in C(\alpha z_0, 1 - \alpha)$, then $\operatorname{Re} \frac{z_0 + z}{z_0 - z} = \frac{\alpha}{1 - \alpha}$.

If $z \in D(\alpha z_0, 1 - \alpha)$, then $\operatorname{Re} \frac{z_0 + z}{z_0 - z} > \frac{\alpha}{1 - \alpha}$.

If $0 < t < \pi/2$, then

$$\sup_{0 \leq r \leq 1} \left| \frac{1 + re^{it}}{1 - re^{it}} \right| < \left| \cot \frac{t}{2} \right|.$$

For $0 < r < 1$

$$\begin{aligned} & \left| \frac{z_0 + re^{it}}{z_0 - re^{it}} - i \cot \frac{t - t_0}{2} \right| = \\ & = \frac{1 - r^2}{(1 - r)^2 + 4r \sin^2 \frac{t - t_0}{2}} \left| 1 - i \frac{1 - r}{1 + r} \cot \frac{t - t_0}{2} \right|. \end{aligned}$$

Proof. The linear-fractional mapping $\frac{z_0 + z}{z_0 - z}$ maps the circle $C(\alpha z_0, 1 - \alpha)$ on the straight line $x = \alpha/(1 - \alpha)$ and the disk $D(\alpha z_0, 1 - \alpha)$ on the halfplane $\{z : \operatorname{Re} z > \alpha/(1 - \alpha)\}$ which imply the first three assertions of the lemma. To prove the next inequality note that the straight line containing the radius $\{re^{it} : 0 \leq r \leq 1\}$ is mapped by the mapping $\frac{1+z}{1-z}$ on the circle having the center on the imaginary axis and intersecting this axis at the points $-i \tan \frac{t}{2}$ and $a = i \cot \frac{t}{2}$. The radius itself is mapped onto the smaller arc of this circle (lying in the first quadrant) with the endpoints 1 and $b = \frac{1+e^{it}}{1-e^{it}}$. Simple analysis of the triangle with vertices 0, a , and b shows that $|a| > |b|$.

The last assertion is proved as follows ($\tau = t - t_0$):

$$\frac{z_0 + re^{it}}{z_0 - re^{it}} - i \cot \frac{\tau}{2} = \frac{1 + re^{i\tau}}{1 - re^{i\tau}} - i \cot \frac{\tau}{2} =$$

$$\begin{aligned}
 &= \frac{1-r^2}{(1-r)^2 + 4r \sin^2 \frac{\tau}{2}} + i \frac{2r \sin \tau}{(1-r)^2 + 4r \sin^2 \frac{\tau}{2}} - i \cot \frac{\tau}{2} = \\
 &= \frac{1-r^2}{(1-r)^2 + 4r \sin^2 \frac{\tau}{2}} \left(1 - i \frac{1-r}{1+r} \cot \frac{\tau}{2} \right). \quad \square
 \end{aligned}$$

Lemma 2. $\rho(e^{it}, C(\alpha, 1-\alpha)) \geq 2\alpha \sin^2 \frac{t}{2}$, where $\rho(E, F)$ is a distance between the sets E and F , and $\alpha \in (0, \frac{1}{2})$.

Proof. It is evident that $\rho(e^{it}, C(\alpha, 1-\alpha)) = |e^{it} - \alpha| - (1-\alpha) = \sqrt{(1-\alpha)^2 + 4\alpha \sin^2 \frac{t}{2}} - (1-\alpha) \geq 2\alpha \sin^2 \frac{t}{2}$ (equality holds if $t = k\pi$, $k \in \mathbb{Z}$). \square

Lemma 3. Suppose $\theta_n = \arg z_n$ and $\sum_{n=1}^{\infty} I_n |\cot \frac{\theta - \theta_n}{2}| < \infty$. Then the function $g(z) = \sum_{n=1}^{\infty} I_n \frac{z_n + z}{z_n - z}$ has a radial limit of modulus one at the point $e^{i\theta}$.

Proof. By Lemma 1, $|\sum_{n=1}^{\infty} I_n \frac{z_n + z}{z_n - z}| \leq \sum_{n=1}^{\infty} I_n |\cot \frac{\theta - \theta_n}{2}|$. Therefore g is continuous on the radius $[0, e^{i\theta}]$. \square

Let $d(E, F)$ denote an arcdistance between the subsets E and F of C and ∂G be a boundary of G .

Lemma 4. Let $G \subset G_1 \subset G_0$ be open subsets of the circle. Suppose $G = \bigcup_{n=1}^{\infty} I_n$, $G_1 = \bigcup_{n=1}^{\infty} J_n$ and $z_n \in \overline{I_n}$. If $e^{i\theta} \notin G_0$ and $d(J_k, \partial G_0) > 2^k J_k$, then there exists $\lim_{r \rightarrow 1} \sum_{n=1}^{\infty} I_n \frac{z_n + re^{i\theta}}{z_n - ze^{i\theta}}$.

Proof. If $I_n \subset J_k$, then

$$\sup_{0 \leq r \leq 1} \left| \frac{z_n + re^{i\theta}}{z_n - re^{i\theta}} \right| \leq \left| \cot \frac{\theta - \theta_n}{2} \right| \leq \cot \frac{d(e^{i\theta}, J_k)}{2},$$

and therefore

$$\begin{aligned}
 \sum_{k=1}^{\infty} \sum_{I_n \subset J_k} \left| I_n \frac{z_n + re^{i\theta}}{z_n - re^{i\theta}} \right| &\leq \sum_{k=1}^{\infty} \cot \frac{d(e^{i\theta}, J_k)}{2} \sum_{I_n \subset J_k} I_n \leq \\
 &\leq \sum_{k=1}^{\infty} J_k \left| \cot \frac{d(e^{i\theta}, J_k)}{2} \right| < \infty.
 \end{aligned}$$

Using Lemma 3, we can prove Lemma 4. \square

The following lemma is the basic one.

Lemma 5. Suppose $G = \bigcup_{n=1}^{\infty} I_n$ to be an open set. Let $z_n \in I_n$, $d(z_n, \partial I_n) > I_n/4$ and let

$$\sum_{n \notin \{n_k\}} \frac{I_n}{d(e^{i\theta}, I_n)} < \infty, \lim_{k \rightarrow \infty} \frac{I_{n_{k+1}}}{I_{n_k}} = 0.$$

Then the function $g(z) = \sum_{n=1}^{\infty} I_n \frac{z_n+z}{z_n-z}$ has radial limit at the point $e^{i\theta}$ if and only if the series

$$\sum_{n=1}^{\infty} I_n \cot \frac{\theta - \arg z_n}{2} \tag{1}$$

is convergent.

Proof. By Lemma 3 since the function $\sum_{n \notin \{n_k\}} I_n \frac{z_n+z}{z_n-z}$ has radial limit at the point $e^{i\theta}$, then without loss of generality we may assume that

$$\lim_{n \rightarrow \infty} \frac{I_{n+1}}{I_n} = 0, \quad \frac{I_{n+1}}{I_n} \leq \frac{1}{2}.$$

Denote $\rho_n = 1 - I_{n+1}$ and suppose that $\rho_{N-1} < |z| \leq \rho_N$. We have

$$\begin{aligned} g(z) &= \sum_{n=1}^{N-1} I_n \left(\frac{z_n+z}{z_n-z} - i \cot \frac{\theta - \arg z_n}{2} \right) + I_N \frac{z_N+z}{z_N-z} + \\ &+ I_{N+1} \frac{z_{N+1}+z}{z_{N+1}-z} + \sum_{N+2}^{\infty} I_n \frac{z_n+z}{z_n-z} + i \sum_{n=1}^{N-1} I_n \cot \frac{\theta - \arg z_n}{2}. \end{aligned} \tag{2}$$

If $n \geq N + 2$, then

$$I_n \left| \frac{z_n+z}{z_n-z} \right| \leq 2 \frac{I_n}{1-|z|} \leq 2 \frac{I_n}{I_{N+1}} \leq 2^{N-n} \frac{I_{N+2}}{I_{N+1}},$$

whence

$$\left| \sum_{n=N+2}^{\infty} I_n \frac{z_n+z}{z_n-z} \right| \leq \frac{I_{N+2}}{I_{N+1}}. \tag{3}$$

By Lemma 1

$$I_n \left| \frac{z_n+z}{z_n-z} - i \cot \frac{\theta - \arg z_n}{2} \right| \leq 4\pi^3 I_n \frac{1-r}{(\theta - \arg z_n)^2} \leq 16\pi^3 \frac{I_N}{I_n}.$$

Therefore

$$\left| \sum_{n=1}^{N-1} I_n \left(\frac{z_n+z}{z_n-z} - i \cot \frac{\theta - \arg z_n}{2} \right) \right| \leq 16\pi^3 \frac{I_N}{I_{N-1}}. \tag{4}$$

If $\lim_{n \rightarrow \infty} I_n \cot \frac{\theta - \arg z_n}{2} = 0$, then by Lemma 1, $\lim_{n \rightarrow \infty} I_n \frac{z_n + z}{z_n - z} = 0$. Thus, by (3) and (4) the right-hand side of (2) has radial limit if and only if (1) is convergent.

Finally, if $\overline{\lim}_{n \rightarrow \infty} I_n |\cot \frac{\theta - \arg z_n}{2}| \geq \delta > 0$, we have

$$\begin{aligned} \operatorname{Re} I_{N+1} \frac{z_{N+1} + \rho_N e^{i\theta}}{z_{N+1} - \rho_N e^{i\theta}} &\geq I_{N+1} \frac{1 - \rho_N^2}{(1 - \rho_N)^2 + 4\rho_N \sin^2 \frac{\theta - \theta_N}{2}} \geq \\ &\geq I_{N+1} \frac{1 - \rho_N}{(1 - \rho_N)^2 + 4 \frac{I_{N+1}^2}{\delta^2}} = \frac{I_{N+1}^2}{I_{N+1}^2 + \frac{4}{\delta^2} I_{M+1}^2}, \end{aligned}$$

since $\sin^2 \frac{\theta - \theta_{N+1}}{2} \leq \frac{I_{N+1}^2}{\delta^2}$, where $\theta_N = \arg z_{N+1}$;

$$\begin{aligned} \operatorname{Re} I_N \frac{z_N + \rho_N e^{i\theta}}{z_N - \rho_N e^{i\theta}} &\leq I_N \frac{2(1 - \rho_N)}{(1 - \rho_N)^2 + 4\rho_N \sin^2 \frac{\theta - \theta_N}{2}} \leq \\ &\leq \frac{2I_N I_{N+1}}{4\rho_N (\frac{2}{\pi} \frac{\theta - \theta_N}{2})^2} \leq \frac{2I_N I_{N+1}}{4 \frac{1}{2} \frac{1}{\pi^2} (\frac{1}{4} I_N)^2} = 16\pi^2 \frac{I_{N+1}}{I_N}, \end{aligned}$$

since $|\theta - \theta_N| \geq \frac{1}{4} I_N$, $e^{i\theta_N} \notin I_N$, $\rho_N \geq \frac{1}{2}$.

By the notation $\rho'_N = 1 - \sqrt{I_N I_{N+1}}$ we get

$$\begin{aligned} \operatorname{Re} I_{N+1} \frac{z_{N+1} + \rho'_N e^{i\theta}}{z_{N+1} - \rho'_N e^{i\theta}} &\leq I_{N+1} \frac{2(1 - \rho'_N)}{(1 - \rho'_N)^2} \leq \\ &\leq 2 \frac{I_{N+1}}{1 - \rho'_N} = 2 \frac{I_{N+1}}{\sqrt{I_N I_{N+1}}} = 2\sqrt{\frac{I_{N+1}}{I_N}} \end{aligned}$$

and

$$\begin{aligned} \operatorname{Re} I_N \frac{z_N + \rho'_N e^{i\theta}}{z_N - \rho'_N e^{i\theta}} &\leq I_N \frac{2(1 - \rho'_N)}{(I_N^2)/(16\pi^2)} = \\ &= 32\pi^2 \frac{I_N \sqrt{I_N I_{N+1}}}{I_N^2} = 32\pi^2 \sqrt{\frac{I_{N+1}}{I_N}}. \end{aligned}$$

Taking into account these inequalities we can conclude that the right-hand side of (2) has no radial limit. \square

Definition 1. We say that E is an *arrangeable* set if there exist a countable set $(z_{mn}) = (e^{i\theta_{mn}})$, a sequence of intervals (arcs) (I_{mn}) , and a sequence of positive numbers (α_k) satisfying the following conditions:

1. $z_{mn} \in I_{mn}$, $4d(z_{mn}, \partial I_{mn}) > I_{mn}$;
2. $\lim_{n \rightarrow \infty} \frac{I_{m(n+1)}}{I_{mn}} = 0$, $m = 1, 2, \dots$;
3. $I_{(m+1)n} \subset \bigcup_{i=1}^{\infty} (I_{mi} \setminus \{z_{mi}\})$, $m = 1, 2, \dots$;

4. $\sqrt{I_{mn}} < 2^{-n}\alpha_m\alpha_{m+1}d(I_{mn}, \partial G_{m-1})$, $m > 1$, where $G_m = \bigcup_{n=1}^{\infty} (I_{mn} \setminus \{z_{mn}\})$, $a_k \searrow 0$, $\sum \alpha_k < \infty$;

5. $E = (\bigcap_{m=1}^{\infty} G_m) \cup (\bigcup_{m=1}^{\infty} Q_m)$, where

$$Q_m = \left\{ e^{i\theta} : e^{i\theta} \notin G_m, \sum_{n=1}^{\infty} I_{mn} \cot \frac{\theta - \theta_{mn}}{2} \text{ is divergent} \right\}.$$

Lemma 6. *Let E be an arrangeable set. Then the function $f(z) = \exp\{-g(z)\}$, with $g(z) = \sum_{m=1}^{\infty} g_m(z)$, $g_m(z) = \frac{1}{\alpha_m} \sum_{n=1}^{\infty} I_{mn} \frac{z_{mn}+z}{z_{mn}-z}$, has no radial limit on the set E only.*

Proof. Since

$$\sum_{m=1}^{\infty} \frac{1}{\alpha_m} \sum_{n=1}^{\infty} I_{mn} \leq \sum_{m=1}^{\infty} \frac{1}{\alpha_m} \sum_{n=1}^{\infty} 2^{-2n} \alpha_m^2 \alpha_{m+1}^2 < \infty,$$

we conclude that f is analytic in the unit disk.

Suppose $e^{i\theta} \notin G_1$. We have

$$\begin{aligned} \sum_{m=2}^{\infty} \frac{1}{\alpha_m} \sum_{n=1}^{\infty} I_{mn} \left| \cot \frac{\theta - \theta_{mn}}{2} \right| &< \sum_{m=2}^{\infty} \frac{1}{\alpha_m} \sum_{n=1}^{\infty} I_{mn} \frac{1}{d(I_{mn}, \partial G_{m-1})} < \\ &< \sum_{m=2}^{\infty} \frac{1}{\alpha_m} \sum_{n=1}^{\infty} 2^{-4n} \alpha_m^2 \alpha_{m+1}^2 d(I_{mn}, \partial G_{m-1}) < \sum_{m=2}^{\infty} \alpha_m \alpha_{m+1}^2 < \infty. \end{aligned}$$

Hence by Lemma 3, $g - g_1$ has radial limit at $e^{i\theta}$, and thus by our basic lemma we can conclude that g has no radial limit only at Q_1 .

Now if $e^{i\theta} \in G_m \setminus G_{m+1}$, $m = 1, 2, \dots$, then

$$\sum_{k=1}^m \frac{1}{\alpha_k} \sum_{n=1}^{\infty} I_{kn} \left| \cot \frac{\theta - \theta_{kn}}{2} \right| < \frac{1}{d(e^{i\theta}, \partial G_m)} \sum_{k=1}^m \frac{1}{\alpha_k} \sum_{n=1}^{\infty} I_{kn} < \infty,$$

and

$$\begin{aligned} \sum_{k=m+2}^{\infty} \frac{1}{\alpha_k} \sum_{n=1}^{\infty} I_{kn} \left| \cot \frac{\theta - \theta_{kn}}{2} \right| &< \\ &< \sum_{k=m+2}^{\infty} \frac{1}{\alpha_k} \sum_{n=1}^{\infty} I_{kn} \frac{1}{d(I_{kn}, \partial G_{k-1})} < \sum_{k=m+2}^{\infty} \alpha_k \alpha_{k+1}^2 < \infty. \end{aligned}$$

Therefore arguing as above we come to the conclusion that g has no radial limit at Q_{m+1} . Suppose now that $e^{i\theta} \in \bigcap_{m=1}^{\infty} G_m$, and denote $O_m = \bigcup_{n=1}^{\infty} D(\alpha_m^2 z_{mn}, 1 - \alpha_m^2)$, $C_m = \partial O_m$.

Let $z \in C_m$ and $k \leq m$; then since $z \notin D(\alpha_m^2 z_{mn}, 1 - \alpha_m^2)$, we have

$$\operatorname{Re} g_k(z) \leq \frac{1}{\alpha_k} \sum_{n=1}^{\infty} I_{kn} \frac{\alpha_m^2}{1 - \alpha_m^2} \leq \frac{2\alpha_m^2}{\alpha_k} \mu G_k \leq \alpha_m.$$

Further, if $z \in C_m$ and $k > m$, then according to Lemma 2,

$$\operatorname{Re} \frac{z_{kn} + z}{z_{kn} - z} \leq \frac{2}{\rho(z_{kn}, C_m)} < \frac{1}{\alpha_m^2 \sin^2 \frac{\theta_{kn} - \theta_{mi}}{2}} < \frac{\pi^2}{\alpha_m^2 d^2(I_{kn}, \partial G_{k-1})}.$$

Hence

$$\begin{aligned} \operatorname{Re} g_k(z) &\leq \frac{1}{\alpha_k} \sum_{n=1}^{\infty} 2^{-2n} \alpha_k^2 \alpha_{k+1}^2 d^2(I_{kn}, \partial G_{k-1}) \frac{\pi^2}{\alpha_m^2 d^2(I_{kn}, \partial G_{k-1})} \leq \\ &\leq \frac{\alpha_k \alpha_{k+1}^2}{\alpha_m^2} \leq \alpha_{k+1}. \end{aligned}$$

Thus, if $z \in C_m$, then

$$\operatorname{Re} g(z) \leq \sum_{k=1}^m \alpha_m + \sum_{k=m+1}^{\infty} \alpha_k = m\alpha_m + \sum_{k=m+1}^{\infty} \alpha_k,$$

whence

$$\varliminf_{r \rightarrow 1} \operatorname{Re} g(re^{i\theta}) = 0. \tag{5}$$

Since $e^{i\theta} \in G_m$, there exists an integer j such that $e^{i\theta} \in I_{mj}$. It is clear that the radius $[0, e^{i\theta}]$ intersects the circle $C((1 - I_{mj})z_{mj}, I_{mj})$ in which

$$\operatorname{Re} g(re^{i\theta}) \geq \operatorname{Re} g_m(re^{i\theta}) \geq \operatorname{Re} \frac{1}{\alpha_m} I_{mj} \frac{z_{mj} + z}{z_{mj} - z} \geq \frac{1}{2\alpha_m}.$$

Hence, $\overline{\lim}_{r \rightarrow 1} \operatorname{Re} g(re^{i\theta}) = \infty$, which together with (5) gives

$$\varliminf_{r \rightarrow 1} |f(re^{i\theta})| = 0, \quad \overline{\lim}_{r \rightarrow 1} |f(re^{i\theta})| = 1. \quad \square$$

3. Let us now formulate and prove our theorems.

Theorem 1. *There exists an arrangeable set of second category.*

Proof. Let $Q = \{x_n\}$ be a countable subset of a unit circle and G_0 be an open subset covering Q and $\mu G_0 < 2\pi$ (μ denotes the Lebesgue measure).

Cover x_1 by the interval J_1 such that $\partial J_1 \cap Q = \emptyset$ and $\sqrt{J_1} < 2^{-1} \alpha_1 \alpha_2 d(J_1, \partial G_0)$. Let x_{k_2} be the first element of the sequence Q not belonging to J_1 . Cover x_{k_2} by the interval J_2 such that $\partial J_2 \cap Q = \emptyset$, $J_1 \cap J_2 = \emptyset$, $\sqrt{J_2} < 2^{-2} \alpha_1 \alpha_2 d(J_2, \partial G_0)$, $\frac{J_2}{J_1} < \frac{1}{2}$.

Given J_1, J_2, \dots, J_{n-1} , let x_{k_n} be the first element of Q not belonging to $\bigcup_{k=1}^{n-1} J_k$. Cover x_{k_n} by the interval J_n such that $\partial J_n \cap Q = \emptyset$, $J_n \cap (\bigcup_{k=1}^{n-1} G_k) = \emptyset$, $\sqrt{J_n} < 2^{-n} \alpha_1 \alpha_2 d(J_n, \partial G_0)$, $\frac{J_n}{J_{n-1}} < \frac{1}{n}$. Select z_{1n} such that $z_{1n} \in J_n$, $4d(z_{1n}, \partial J_n) > J_n$ and $z_{1n} \notin Q$. Denote $I_{1n} = J_n \setminus \{z_{1n}\}$ and $G_1 = \bigcup_{n=1}^{\infty} I_{1n}$.

Taking G_1 instead of G_0 and arguing analogously, we will obtain new sequences of the intervals J_n and points z_{2n} with the following properties:

- (a) $z_{2n} \in J_n, z_{2n} \notin Q$;
- (b) $J_n \cap J_k = \emptyset, n \neq k$;
- (c) $4d(z_{2n}, J_n) > J_n$;
- (d) $\sqrt{J_n} < 2^{-n} \alpha_2 \alpha_3 d(J_n, \partial G_1)$;
- (e) $\lim_{n \rightarrow \infty} \frac{J_{n+1}}{J_n} = 0$.

Denote $I_{2n} = J_n \setminus \{z_{2n}\}$ and $G_2 = \bigcup_{n=1}^{\infty} I_{2n}$.

Repeating the above process for G_2, G_3 , etc., we shall get the sequences z_{mn} and I_{mn} which define the arrangeable set E . If Q is a countable everywhere dense subset, then the obtained set E will be of second category. \square

Theorem 2. *There exist a set E and a bounded analytic function f with the following conditions:*

- (a) E is of G_δ type;
- (b) E is not of F_σ type;
- (c) if $e^{i\theta} \in E$, then $\lim_{r \rightarrow 1} |f(re^{i\theta})| = 0$, $\overline{\lim}_{r \rightarrow 1} |f(re^{i\theta})| = 1$;
- (d) the function f on $C \setminus E$ has radial limit of modulus one except a countable set where the radial limit is zero.

Proof. Consider the Cantor set E without ends on its adjacent intervals. It is clear that E is of type G_δ and not of type F_σ .

Suppose $E = \bigcap_{k=1}^{\infty} G_k$ and $\sum_{k=1}^{\infty} \alpha_k < 1$. Without loss of generality we may assume that the ends of component intervals I_{1n} of G_1 belong to adjacent intervals of Cantor's set. Cover every $I_{1n} \cap \overline{E}$ by an open set H_{1n} such that $\sqrt{\mu H_{1n}} < 2^{-n} \alpha_1 \alpha_2 d(H_{1n}, \partial I_{1n})$, where H_{1n} is a finite union of intervals. Put $R_1 = \bigcup_{n=1}^{\infty} H_{1n} = \bigcup_{k=1}^{\infty} J_{1k}$. Suppose z_{1k} is the right end of the interval J_{1k} .

If $e^{i\theta} \notin G_1$, we have

$$\sum_{k=1}^{\infty} J_{1k} \left| \cot \frac{\theta - \theta_{1k}}{2} \right| \leq \sum_{n=1}^{\infty} \sum_{J_{1k} \subset I_{1n}} J_{1k} \frac{\pi}{d(H_{1n}, \partial I_{1n})} \leq$$

$$\leq \pi \sum_{n=1}^{\infty} \frac{\mu H_{1n}}{d(H_{1n}, \partial I_{1n})} \leq \pi \alpha_1 \alpha_2$$

and if $e^{i\theta} \in I_{1n}$, then

$$\sum_{J_{1k} \cap I_{1n} = \emptyset} J_{1k} \left| \cot \frac{\theta - \theta_{1k}}{2} \right| \leq \frac{\pi}{d(e^{i\theta}, \partial I_{1n})} \sum_{n=1}^{\infty} \mu H_{1n} = \frac{\pi \mu R_1}{d(e^{i\theta}, \partial I_{1n})}.$$

Hence, according to Lemma 3 and the fact that intervals contained in I_{1n} are finite, we conclude that $f_1(z) = \exp\{-g_1(z)\}$, and

$$g_1(z) = \frac{1}{\alpha_1} \sum_{k=1}^{\infty} J_{1k} \frac{z_{1k} + z}{z_{1k} - z}$$

has everywhere a radial limit. Put $R_1 \cap G_2 = \bigcup_{n=1}^{\infty} I_{2n}$. We assume again that the ends of I_{2n} belong to the adjacent intervals of Cantor's set. Cover every $I_{2n} \cap \bar{E}$ by open sets H_{2n} such that $\sqrt{\mu H_{2n}} < 2^{-n} \alpha_2 \alpha_3 d(H_{2n}, \partial I_{2n})$, where H_{2n} is a finite union of intervals. Put $R_2 = \bigcup_{n=1}^{\infty} H_{2n} = \bigcup_{k=1}^{\infty} J_{2k}$. Suppose z_{2k} is the right end of J_{2k} .

If $e^{i\theta} \notin R_1 \cap G_2$, we have

$$\sum_{k=1}^{\infty} J_{2k} \left| \cot \frac{\theta - \theta_{2k}}{2} \right| \leq \pi \alpha_2 \alpha_3$$

and if $e^{i\theta} \in I_{2n}$, then

$$\sum_{J_{2k} \cap I_{2n} = \emptyset} J_{2k} \left| \cot \frac{\theta - \theta_{2k}}{2} \right| \leq \frac{\pi \mu R_2}{d(e^{i\theta}, \partial I_{2n})}.$$

Hence, according to Lemma 3 and the fact that intervals contained in I_{2n} are finite, we conclude that $f_2(z) = \exp\{-g_2(z)\}$, and

$$g_2(z) = \frac{1}{\alpha_2} \sum_{k=1}^{\infty} J_{2k} \frac{z_{2k} + z}{z_{2k} - z}$$

has everywhere a radial limit.

Given R_1, R_2, \dots, R_{m-1} , put $R_{m-1} \cap G_m = \bigcup_{n=1}^{\infty} I_{mn}$. Assume that the ends of I_{mn} belong to adjacent intervals of Cantor's set. Cover every $I_{mn} \cap \bar{E}$ by open sets H_{mn} such that $\sqrt{\mu H_{mn}} < 2^{-n} \alpha_m \alpha_{m+1} d(H_{mn}, \partial I_{mn})$, where H_{mn} is a finite union of intervals. Put $R_m = \bigcup_{n=1}^{\infty} H_{mn} = \bigcup_{k=1}^{\infty} J_{mk}$. Suppose z_{mk} is the right end of J_{mk} .

If $e^{i\theta} \notin R_{m-1} \cap G_m$, we have

$$\sum_{k=1}^{\infty} J_{mk} \left| \cot \frac{\theta - \theta_{mk}}{2} \right| \leq \pi \alpha_m \alpha_{m+1} \quad (6)$$

and if $e^{i\theta} \in I_{mn}$, then

$$\sum_{J_{mk} \cap I_{mn} = \emptyset} J_{mk} \left| \cot \frac{\theta - \theta_{mk}}{2} \right| \leq \frac{\pi \mu R_m}{d(e^{i\theta}, \partial I_{mn})}.$$

The function $f_m(z) = \exp\{-g_m(z)\}$ with

$$g_m(z) = \frac{1}{\alpha_m} \sum_{k=1}^{\infty} J_{mk} \frac{z_{mk} + z}{z_{mk} - z}$$

has everywhere a radial limit.

Put

$$f(z) = \prod_{m=1}^{\infty} f_m(z) = \exp \left\{ - \sum_{m=1}^{\infty} g_m(z) \right\}.$$

It is evident that if $e^{i\theta} = z_{mk}$, then $\lim_{r \rightarrow 1} f(re^{i\theta}) = 0$. Since $R_m \subset G_m$ and $R_m \supset E$, then $\bigcap_{m=1}^{\infty} R_m = E$.

If $e^{i\theta} \notin E$, then there exists an integer n such that $e^{i\theta} \notin R_n$. By (6) we have

$$\sum_{m=n}^{\infty} \frac{1}{\alpha_m} \sum_{k=1}^{\infty} J_{mk} \left| \cot \frac{\theta - \theta_{mk}}{2} \right| \leq \pi \sum_{m=n}^{\infty} \alpha_{m+1} < \infty.$$

Therefore, by Lemma 3, $\operatorname{Re} \sum_{m=n}^{\infty} g_m(re^{i\theta})$ has zero radial limit.

Suppose finally $e^{i\theta} \in E$. Then using the definitions and arguing as in Lemma 6, we may conclude that

$$\underline{\lim}_{r \rightarrow 1} |f(re^{i\theta})| = 0, \quad \overline{\lim}_{r \rightarrow 1} |f(re^{i\theta})| = 1. \quad \square$$

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