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COMPUTING GAUSS–MANIN SYSTEMS FOR COMPLETE INTERSECTION SINGULARITIES S_μ

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Abstract. The Gauss–Manin systems with coefficients having logarithmic poles along the discriminant sets of the principal deformations of complete intersection quasihomogeneous singularities S_μ are calculated. Their solutions in the form of generalized hypergeometric functions are presented.

Introduction

It is well known that the notion of hypergeometric series appeared in the work of L. Euler in 1769 (see [1]) where he studied expansion in series of a special type integral. This series satisfies a certain differential equation which is called the hypergeometric equation. Its particular case is known as the classical Legendre equation. In 1813 C. F. Gauss [2] investigated the properties of the hypergeometric series and its generalization called the hypergeometric function. After that many investigations were devoted to the study of various generalizations of the hypergeometric functions (HGF) as well as of the Legendre equation. The latter may be regarded as a special case of more general type equations called Fuchsian equations. In turn, Fuchsian equations belong to the class of equations with regular singularities. During the last two decades systems with regular singularities have been systematically and extensively studied by many authors. We shall mention here only the names of P. Deligne, B. Malgrange, C. Sabbah, M. Kashiwara and T. Kawai.

The theory of hypergeometric functions developed in a rather complicated and intriguing way. Since the beginning of the 19th century there has appeared a great many papers containing the description of various ap-

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proaches to this subject. Among them the investigations carried out by L. Pochhammer have significant meaning in our studies (see Theorem 2.6).

The theory of singularities enables us to give a very fruitful and clear interpretation of the previously obtained results from a sufficiently general point of view. Thus, the classical Legendre equation can be considered as a coordinate representation of the Gauss–Manin connection associated with the minimal versal deformation of simple hypersurface singularity A_2.

E. Brieskorn was probably one of the first who developed these ideas. In his famous work [3] he proved that the connection associated with 1-parameter principal deformations of isolated hypersurface singularities could be represented by systems of ordinary differential equations with regular singularities. Furthermore, it follows from his results that after matrix transformations with meromorphic entries such equations reduce to the ones having poles of the first order. The next step was taken by K. Saito. Having calculated the connection associated with the total 3-parameter miniversal deformation of A_3-singularity he came to the conclusion [4] that a very convenient representation of the corresponding system may be obtained if one considers the coefficients of this system as logarithmic differential forms. We will establish a similar result for the Gauss–Manin connections associated with a series of complete intersection singularities S_μ (see Theorem 3.1).

Another approach was developed by S. Ishiura and M. Noumi [5] who described the Gauss–Manin systems in the A_μ-case by means of K. Saito's Hamiltonian representation. M. Noumi also treated some particular cases of linear deformations of the Pham singularities [6]. More exactly, he gave a concrete representation of solutions to the Gauss–Manin system in terms of the known generalized hypergeometric functions.

It should be remarked that integrals of the type

$$J(t) = \int F_0^{\lambda_0}(z) F_1^{\lambda_1}(z,t) \dots F_m^{\lambda_m}(z,t) dz$$

have been studied by K. Aomoto [7], [8], I. M. Gelfand [9], and their followers in the case where $F_j(z,t),\ 1 \le j \le m$, are linear functions with respect to the variables $z = (z_1, \dots, z_n)$ and $F_0(z,t)$ is a linear function or a quadric. In this work, we consider situations where both $F_0(z,t)$ and $F_1(z,t)$ are quadrics or higher-order polynomials ($m = 1$). This point is an essential difference from the earlier investigations which dealt with various representations of Gauss–Manin connections for nonisolated singularities given by special arrangements of hyperplanes treated (see Remark 6).

So far we do not know of any publications concerning concrete calculations of the connections associated with deformations of isolated complete intersection singularities. Herein we give some computational results in this direction. It should be noted that similar calculations were carried out by

S. Guzev [10] several years ago. We are grateful to V. P. Palamodov who gave us a chance to become acquainted with this work.

1. Notion of Gauss–Manin Connection for Complete Intersections

Following B. Malgrange's approach, E. Brieskorn [3] calculated Gauss–Manin connections associated with isolated hypersurface singularities. The main idea was used in the case of isolated complete intersection singularities by G.-M. Greuel (see [11]).

Let us investigate a smooth mapping with isolated complete intersection singularities $f : X \to S$. We consider a flat deformation with smooth fibres outside a certain hypersurface D in S called the discriminant set or, equivalently, the set of critical values of the mapping f. The coherent sheaf $\mathcal{H}^p = H^p(f_*\Omega^\bullet_{X/S}, d)$ is defined as the pth cohomology group of a relative de Rham complex. The restriction of this sheaf on $S \setminus D$ is isomorphic to the pth cohomology group $H^p(X_t, \mathbf{C})$, $t \in S \setminus D$. The transference of cohomology classes H^p along the tangent directions on the complement $S \setminus D$ induces a connection

$$\nabla_{X/S} : \mathcal{H}^p(f_*\Omega^\bullet_{X/S}) \longrightarrow \mathcal{H}^p(f_*\Omega^\bullet_{X/S}) \otimes \Omega^1_S(D) \tag{1}$$

given by the rule

$$\nabla_{X/S}[\omega] = \sum_{i=1}^{k} h[\alpha_i] \otimes dt_i/h,$$

where h is the defining function of the discriminant set D and $\Omega^1_S(D)$ denotes the sheaf of meromorphic 1-forms with poles of the first order along D. Here the symbol $[\alpha]$ means the corresponding relative de Rham cohomology class of α in $f_*\Omega^p_{X/S}/d(f_*\Omega^{p-1}_{X/S})$. Thus we have the decomposition

$$d\omega = \sum_{i=1}^{k} df_i \wedge \alpha_i, \qquad \alpha_i \in f_*\Omega^p_X.$$

From the formula (1) it follows immediately that the Gauss–Manin connection has poles of the first order in the case where h has no multiple factors. In fact, Saito's considerations [4], [12] imply that the connection can also be rewritten as

$$\nabla_{X/S} : \mathcal{H}^p(f_*\Omega^\bullet_{X/S}) \to \mathcal{H}^p(f_*\Omega^\bullet_{X/S}) \otimes \Omega^1_S(\log D) \tag{1'}$$

if we define the connection on a slightly larger module. Here we will pay attention to the difference between (1) and (1′) that consists in the presence of the factor $\Omega^1_S(\log D)$ in (1'). As usual, $\Omega^1_S(\log D)$ denotes the $\mathcal{O}_S$-module of logarithmic differential forms which is a submodule of $\Omega^1_S(D)$ possessing various interesting properties (see [13], [14]).

2. Period Integrals Associated with Deformations of S_μ-Singularities

The isolated complete intersection singularity of type $S_\mu,\ \mu \geq 5$, from the list of M. Giusti [15] is determined by the following pair of equations defined on $\mathbf{C}^3$:

$$\begin{cases} f_1(x,y,z) &= x^2+y^2+z^\nu &= 0 \\ f_2(x,y,z) &= yz &= 0, \end{cases}$$

where $\nu = \mu - 3$. Let

$$\begin{cases} F_1(x,y,z,s) &= f_1(x,y,z)+s_\nu z^{\nu-1}+\cdots+s_2 z+s_1 y-t_1 &= 0 \\ F_2(x,y,z,s) &= f_2+s_{\nu+1}x-t_2 &= 0 \end{cases}$$

be the minimal versal deformation of the germ X_0 given by the system $f_1 = 0,\ f_2 = 0$. Denote by $X_{(s,t)}$ the fiber of the miniversal deformation over the point (s,t) in the μ-dimensional base space S. Here $s = (s_1, s_2, \ldots, s_{\nu+1}),\ t = (t_1, t_2)$.

We will consider integrals of the type

$$I_j(s,t) = \int_{\gamma(s,t)} q_j(x,y,z)dx \wedge dy \wedge dz/dF_1 \wedge dF_2,$$

where the integration is taken along some regularization of a real vanishing cycle $\gamma(s,t) \in H_1(X_{(s,t)}, \mathbf{C})$ (see [9]) and $q_j(x,y,z)$ are polynomials. It is well known that there is an isomorphism

$$H^1(X_{(s,t)}, \mathbf{C}) \cong \Omega^3_{\mathbf{C}^3}/(f_1\Omega^3_{\mathbf{C}^3} + f_2\Omega^3_{\mathbf{C}^3} + df_1 \wedge df_2 \wedge \Omega^1_{\mathbf{C}^3}).$$

This can be realized by multiplying the 2-form $df_1 \wedge df_2$ by the elements of the cohomology group on the left side. It is not difficult to see that in the case of $S_{\nu+3}$-singularity the above quotient space is generated by the following monomial forms:

$$\{1, z, z^2, \ldots, z^\nu, y, x\} dx \wedge dy \wedge dz.$$

Our aim is to compute the Gauss–Manin system (1) associated with the so-called principal deformation of the singularity X_0, that is, the deformation over the (t_1, t_2)-parameter subspace T in the base space S of the miniversal deformation:

$$\nabla_{X/T} : \mathcal{H}^1(f_*\Omega^\bullet_{X/T}) \longrightarrow \mathcal{H}^1(f_*\Omega^\bullet_{X/T}) \otimes \Omega^1_T(D).$$

In effect, Leray's residue theorem yields the identity

$$\frac{d}{dt_i}\int_{\gamma(s,t)}\omega = \int_{\gamma(s,t)}\frac{d\omega}{df_i}, \quad i=1,2,$$

which has been proved in [3]. This fact implies that we can calculate the Gauss–Manin system as the relation between the integrals instead of that between cohomology classes.

Thus, the corresponding system of differential equations describes non-trivial relations between the integrals

$$I_j(s,t) = \int_{\gamma(s,t)} q_j(x,y,z)dx\wedge dy\wedge dz/df_1\wedge df_2$$

and their derivatives. Here the functions $q_j(x,y,z)$, $j=1,\dots,\mu$, are monomials $1, z, z^2, \dots, z^\nu, y, x$ (cf. [11], 2.3), that is,

$$\begin{aligned}\frac{z^j\,dx\wedge dy\wedge dz}{df_1\wedge df_2} &= \frac{z^{j-1}dz}{2x}, \qquad 0\le j\le \nu,\\ \frac{ydx\wedge dy\wedge dz}{df_1\wedge df_2} &= \frac{t_2dz}{z^2x},\\ \frac{xdx\wedge dy\wedge dz}{df_1\wedge df_2} &= \frac{-dy}{2y}.\end{aligned}$$

Evidently, the last form can be easily integrated. So *nontrivial* integrals for which the differential equation will be calculated are given by the following set of differential forms consisting of $\mu-1$ elements:

$$\{z^{-2}dz/x, z^{-1}dz/x, dz/x, \dots, z^{\nu-1}dz/x\}.$$

Suppose that $s_{\nu+1}=0$. Then the integrals $I_j(s,t)$ can be expressed in the following manner:

$$\begin{aligned}I_j(s,t) &= \int_{\gamma(s,t)}\frac{z^{j-1}dz}{x} =\\ &= \int\frac{z^{j-1}dz}{((t_2/z)^2+z^\nu+s_\nu z^{\nu-1}+\dots+t_1+s_1(t_2/z))^{1/2}} =\\ &= \int\frac{z^j\,dz}{(z^{\nu+2}+s_\nu z^{\nu+1}+\dots+t_1z^2+s_1t_2z+{t_2}^2)^{1/2}},\end{aligned}$$

where $(x,y,z)\in\gamma(s,t)$, $-1\le j\le\nu$. Denote

$$J_{j+2}(t) = -2\frac{\partial}{\partial t_1}I_j(0,t) = \int\frac{z^{j+2}dz}{(z^{\nu+2}+t_1z^2+{t_2}^2)^{3/2}}.$$

Now the system of equations satisfying the integrals $J_1(t),\dots,J_{\nu+2}(t)$ will be investigated. Let us consider the period integrals for the miniversal deformation of the singularity $A_{\nu+1}$, given by the equation $z^{\nu+2}+s_\nu z^\nu+\cdots+s_1z+s_0=0$:

$$K_i^\lambda(s)=\int z^i(z^{\nu+2}+s_\nu z^\nu+\cdots+s_1z+s_0)^\lambda dz,\quad i=0,\ldots,\nu+2.$$

It is evident that the following relations between $J_i(t)$ and $K_i(t)$ hold:

$$-1/2K_i^{-3/2}(t_2^2,0,t_1,0,\ldots,0)=J_i(t),\ \ i=1,\ldots,\nu+2. \tag{2}$$

Proposition 2.1 ([16]). *The period integrals $K_0^\lambda(s),\ldots,K_{\nu+2}^\lambda(s)$ satisfy the following overdetermined system of differential equations:*

$$\sum_{\ell=0}^{\nu}s_\ell\frac{\partial}{\partial s_0}K_{\ell+i}^\lambda+\frac{\partial}{\partial s_0}K_{\nu+2+i}^\lambda=\lambda K_i^\lambda,\quad 0\le i\le\nu, \qquad (3)_i$$

$$\sum_{\ell=1}^{\nu}\ell s_\ell\frac{\partial}{\partial s_0}K_{\ell+j}^\lambda+(\nu+2)\frac{\partial}{\partial s_0}K_{\nu+2+j}^\lambda=-(j+1)K_j^\lambda,\ \ -1\le j\le\nu. \qquad (4)_j$$

As remarked above for $S_{\nu+3}$-singularities, we have $\nu+2$ nontrivial period integrals $J_1(t),\ldots,J_{\nu+2}(t)$ which correspond to the integrals $K_1(s_0,s_2,0,\ldots,0),\ldots,\ K_{\nu+2}(s_0,s_2,0,\ldots,0)$ in view of the relation (2). In order to simplify the system that appeared in Proposition 2.1, we consider a set of μ period integrals

$$K_0(s',0,\ldots,0),\ \ K_1(s',0,\ldots,0),\ \ldots,\ K_{\nu+2}(s',0,\ldots,0)$$

(the notation $s'=(s_0,s_2)$ will be used in the sequel). The superscript λ can be omitted when no specification in needed. As a matter of fact, these μ period integrals are not independent elements of a certain $\mathcal{D}$-module over $\mathbf{C}[s][\frac{\partial}{\partial s_0}]$, that is, there are relations between the integrals. The first one is as follows (see $(4)_0$):

$$s_0\frac{\partial}{\partial s_0}K_2(s',0)+(\nu+2)\frac{\partial}{\partial s_0}K_{\nu+2}(s',0)=-K_0(s',0).$$

This means that $K_{\nu+2}$ is uniquely determined by K_0 and K_2 if we take into account the homogeneity of $K_{\nu+2}$. The latter follows easily from the definition of the integral

$$\Big(s_0\frac{\partial}{\partial s_0}+\frac{\nu}{\nu+2}s_2\frac{\partial}{\partial s_2}\Big)K_{\nu+2}^\lambda(s',0)=\Big(\lambda+\frac{\nu+3}{\nu+2}\Big)K_{\nu+2}^\lambda(s',0).$$

Notice that $K_{\nu+1}$ has no relation with K_j's except that with K_1. It gives us the second relation:

$$(2s_2,\nu+2)\frac{\partial}{\partial s_0}(K_1,K_{\nu+1})^t=0.$$

Using the homogeneity of the relation one can rewrite it as

$$K_{\nu+1}(s',0) = \frac{-2s_2}{(\nu+2)}\, K_1(s',0).$$

Hence it is enough to calculate the system of equations for $\nu+1$ integrals $K_0,\dots,K_\nu$ in order to get the corresponding system for $K_1,\dots,K_{\nu+2}$.

Proposition 2.2. *The integrals $K_0(s',0),\dots,K_\nu(s',0)$ satisfy the following system of differential equations:*

$$(s_0\,\mathrm{id}_{\nu+1} + C(s_2))\frac{\partial}{\partial s_0}\mathbf{K} = (L + V(s_2))\mathbf{K},$$

where $\mathbf{K}$ denotes the vector column $(K_0,\dots,K_\nu)$,

$$C(s_2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & \frac{\nu}{\nu+2}s_2 \\ 0 & 0 & 0 & \frac{\nu}{\nu+2}s_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \frac{\nu}{\nu+2}s_2 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & \ddots & \frac{\nu}{\nu+2}s_2 \\ 0 & \frac{-2\nu}{(\nu+2)^2}s_2^2 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \frac{-2\nu}{(\nu+2)^2}s_2^2 & 0 & \cdots & \cdots & 0 \end{bmatrix},$$

$$L = \mathrm{diag}(\lambda + \frac{1}{\nu+2},\dots,\lambda+\frac{\nu+1}{\nu+2}),$$

$$V(s_2) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \frac{-2\nu}{(\nu+2)^2}s_2 & 0 & \cdots & 0 \end{bmatrix}.$$

Corollary 2.3. *The period integrals $J_0(t),\dots,J_{\nu+2}(t)$ for the complete intersection singularity $S_{\nu+3}$ satisfy the following system of differential equations:*

$$(t_2^2\,\mathrm{id}_{\nu+1} + C(t_1))\frac{\partial}{\partial t_2}\mathbf{J}(t) = 2t_2(L + V(t_1))\mathbf{J}(t),$$

$$J_{\nu+1}(t) = \frac{-2t_1}{(\nu+2)}J_1(t),$$

$$\frac{\partial}{\partial t_2}J_{\nu+2}(t) = \frac{-2}{(\nu+2)}(t_1\frac{\partial}{\partial t_2}J_2(t) + t_2J_0(t)),$$

where

$$\mathbf{J}(t) = (J_0(t),\dots,J_\nu(t))^t.$$

The matrix L is the same as in Proposition 2.2 and the matrices $C(t_1)$ and $V(t_1)$ can be obtained from the corresponding matrices by substituting the variable t_1 for s_2.

Taking into consideration the system obtained in Proposition 2.2, let us try to solve the system of equations to get an explicit form of the integrals $K_0(s,0),\dots,K_{\nu+2}(s,0)$:

$$s_0\frac{\partial}{\partial s_0}K_j^\lambda + s_2\frac{\partial}{\partial s_0}K_{j+2}^\lambda + \frac{\partial}{\partial s_0}K_{\nu+2+j} = \lambda K_j^\lambda, \qquad 0\le j\le \nu, \qquad (5)_j$$

$$2s_2\frac{\partial}{\partial s_0}K_{j+2}^\lambda + (\nu+2)\frac{\partial}{\partial s_0}K_{\nu+2+j}^\lambda = -(j+1)K_j^\lambda, \qquad -1\le j\le \nu. \quad (6)_j$$

Subtract relation $(6)_j$ multiplied by $1/(\nu+2)$ from $(5)_j$. We obtain recursive relations between the period integrals $K_0(s,0),\dots,K_{\nu+2}(s,0)$:

$$\Big(s_0\frac{\partial}{\partial s_0} - (\lambda+(j+1)/(\nu+2))\Big)K_j^\lambda = -\frac{\nu s_2}{(\nu+2)}\frac{\partial}{\partial s_0}K_{j+2}^\lambda,\ \ 0\le j\le \nu-2,$$

$$\Big(s_0\frac{\partial}{\partial s_0} - (\lambda+\nu/(\nu+2))\Big)K_{\nu-1}^\lambda = \frac{-\nu s_2}{(\nu+2)}\frac{\partial}{\partial s_0}K_{\nu+1}^\lambda = \frac{\nu s_2^2}{(\nu+2)^2}\frac{\partial}{\partial s_0}K_1^\lambda,$$

$$\Big(s_0\frac{\partial}{\partial s_0} - (\lambda+(\nu+1)/(\nu+2))\Big)K_\nu^\lambda = \frac{-\nu s_2}{(\nu+2)}\frac{\partial}{\partial s_0}K_{\nu+2}^\lambda.$$

By virtue of the commutation relation

$$\Big[s_0\frac{\partial}{\partial s_0}, \frac{\partial}{\partial s_0}\Big] = -\frac{\partial}{\partial s_0},$$

that is to say,

$$\Big(s_0\frac{\partial}{\partial s_0} - \alpha\Big)\frac{\partial}{\partial s_0} = \frac{\partial}{\partial s_0}\Big(s_0\frac{\partial}{\partial s_0} - \alpha - 1\Big),$$

we can deduce differential equations satisfied by $K_j(s,0)$ from the above recursive relations. Thus we obtain

Proposition 2.4. (1) *Assume $\nu = 2m$. Then the following differential equations of order $(m+1)$ are satisfied by the integrals K_{2j}^λ, $0\le j\le m$:*

$$S_{0,j-1}(\vartheta+m-j,\lambda)S_{j,m}(\vartheta-j,\lambda)K_{2j} = \\ = \big(-2s_2/(\nu+2)\big)\big(\vartheta-(\lambda+1/2)+m-j+1\big)\psi^m K_{2j}. \qquad (7)_{2j,e}$$

For K_{2j+1}, $0\le j\le m$ we have similar equations of order m:

$$T_{0,j-1}(\vartheta+m-j,\lambda)T_{j,m-1}(\vartheta-j,\lambda)K_{2j+1} = \\ = \big(-2s_2/(\nu+2)\big)\psi^m K_{2j+1}. \qquad (7)_{2j+1,e}$$

(2) *Assume* $\nu = 2m{+}1$. *The differential operators annihilating the period integrals* K^{λ}_j, $0 \le j \le \nu$, *are of order* $(\nu+1)$ *for both even and odd cases:*

$$\begin{aligned} &S_{0,j-1}(\vartheta+\nu-j,\lambda)T_{0,m-1}(\vartheta+m-j+1,\lambda)\ \times \\ &\times S_{j,m}(\vartheta-j,\lambda)\big(\vartheta-\big(\lambda+(\nu+1)/(\nu+2)\big)+\nu-j\big)K_{2j}(s) = \\ &= \big(2s_2/(\nu+2)\big)^2\psi^{2m+1}\big(\vartheta-(\lambda+1/2+j)\big)K_{2j}(s), \qquad (7)_{2j,o} \\ &T_{0,j-1}(\vartheta+\nu-j,\lambda)S_{0,m-1}(\vartheta+m-j+1,\lambda)\ \times \\ &\times T_{j,m}(\vartheta-m,\lambda)\big(\vartheta-\big(\lambda+\nu/(\nu+2)\big)+\nu-j-1\big)K_{2j+1} = \\ &= (2s_2/(\nu+2))\psi^{2m+1}(\vartheta-(\lambda+j+m))K_{2j+1}. \qquad (7)_{2j+1,o} \end{aligned}$$

Herein the following notation have been used both for $\nu = 2m$ *and for* $\nu = 2m+1$*:*

$$\begin{aligned} S_{\alpha,\beta}(X,\lambda) &= \prod_{\ell=\alpha}^{\beta}\big(X-\big(\lambda+(-\nu\ell+1)/(\nu+2)\big)\big), \\ T_{\alpha,\beta}(X,\lambda) &= \prod_{\ell=\alpha}^{\beta}\big(X-\big(\lambda+(-\nu\ell+2)/(\nu+2)\big)\big), \\ \psi &= \big(-\nu s_2/(\nu+2)\big)\frac{\partial}{\partial s_0}, \quad \vartheta = s_0\frac{\partial}{\partial s_0}. \end{aligned}$$

Remark 1. The differential operators annihilating integrals K^{λ}_j, $0 \le j \le \nu$, contain only the derivatives with respect to the variable s_0. Therefore the variable s_2 can be regarded as a parameter in their expressions. In other words, the differential operators calculated above are essentially ordinary differential operators.

Corollary 2.5. (1) *Assume* $\nu = 2m$. *Then we get the following differential equations of order* $(m+1)$ *satisfied by the period integrals* $J_{2j}(t_1,t_2)$, $J_{2j+1}(t_1,t_2)$, $0 \le j \le m$, *depending on parameters of the principal deformation for* $S_{\nu+3}$:

$$\begin{aligned} &S_{0,j-1}(\theta+m-j,-3/2)S_{j,m}(\theta-j,-3/2)J_{2j}(t) = \\ &= \big(-2t_1/(\nu+2)\big)(\theta+m-j+2)\phi^m J_{2j}(t), \qquad (8)_{2j,e} \\ &T_{0,j-1}(\theta+m-j,-3/2)T_{j,m-1}(\theta-j,-3/2)J_{2j+1} = \\ &= -2t_1/(\nu+2)\phi^m J_{2j+1}. \qquad (8)_{2j+1,e} \end{aligned}$$

(2) *Assume* $\nu = 2m{+}1$. *Then we get the following differential equations*

of order $(2m+2)$ *for the period integrals:*

$$S_{0,j-1}(\theta+\nu-j,-3/2)T_{0,m-1}(\theta+m-j+1,-3/2)\ \times$$
$$\times\, S_{j,m}(\theta-j,-3/2)\big(\theta-(\nu+3)/2(\nu+2)+\nu-j\big)J_{2j}(t)=$$
$$=\big(2t_1/(\nu+2)\big)^2\phi^{2m+1}\big(\theta-(\lambda+1/2+j)\big)J_{2j}(t), \qquad (8)_{2j,o}$$
$$T_{0,j-1}(\theta+\nu-j,-3/2)S_{0,m-1}(\theta+m-j+1,-3/2)\ \times$$
$$\times\, S_{j,m}(\theta\ -m,\ -3/2)\big(\theta\ -(\nu+6)/2(\nu+2)\big)+\nu-j-1)J_{2j}(t)=$$
$$=\big(2t_1/(\nu+2)\big)\phi^{2m-1}(\theta-j-m)J_{2j+1}(t),, \qquad (8)_{2j+1,o}$$

where the differential operators $S_{\alpha,\beta}$, $T_{\alpha,\beta}$ *are those defined in Proposition* 2.4, *while the other notations are as follows:*

$$\phi=\frac{-\nu t_1}{2(\nu+2)t_2}\frac{\partial}{\partial t_2},\qquad \theta=\frac{t_2}{2}\frac{\partial}{\partial t_2}.$$

Remark 2. All of the equations $(8)_{2j,e}$ have the singular locus $D=\{t_2(t_2^{2m}-2\nu^m(-t_1/(\nu+2))^{m+1})=0\}$ included in the discriminant set of the principal deformation. The equations $(8)_{2j+1,e}$, however, have the singular locus $D_0=\{(t_2^{2m}-2\nu^m(-t_1/(\nu+2))^{m+1})=0\}=D\setminus\{t_2=0\}$. Here we observe a phenomenon which can be interpreted as the splitting of a system of differential equations into two subsystems corresponding to different singular loci.

In the case of S_5-singularity (i.e. $\nu=2$) one can check that the set D defined above coincides with the discriminant set. It is obtained by computing the determinant of the matrix defined by the coefficients of vector fields tangent to the discriminant set. Explicit expressions of such vector fields for S_5 and S_6 are presented in [14].

Let us try to write solutions of the differential equations obtained in Corollary 2.5. As they have quite similar forms, we restrict ourselves to writing solutions for equations $(8)_{0,e}$ and $(8)_{0,o}$ only.

Theorem 2.6. (1) *The case* $\nu=2m$. *Equation* $(8)_{0,e}$ *has* $(m+1)$ *solutions* $\mathcal{U}_k(t)$, $0\le k\le m$, *that can be expressed by the series*

$$\mathcal{U}_k(t)=t_1^{-(3\nu+4)/2\nu}\tau^{\rho_k}U_k(\tau^m,-3/2),$$

where

$$U_k(x,\lambda)=\sum_{\ell\ge 0}a_{m\ell+k}(\lambda)x^\ell,$$

$$a_{m\ell+k}(\lambda)=\prod_{j=0}^{m-1}\frac{((k-\lambda-1)/m-j/(m+1);\ell)}{((j+k)/m-1;\ell)}\frac{(-\lambda+1/(\nu+2);\ell)}{(m-(\lambda+1/2);\ell)},$$

$$\tau=(1/\nu)^{1/m}\big(-\nu t_1/(\nu+2)\big)^{\frac{-m-1}{m}}t_2^2,$$

$(\alpha;\ell) = \Gamma(\alpha+\ell)/\Gamma(\alpha)$, *and* $0 \le k \le m$. *For* $k = 0,\dots,m-1$ *the characteristic exponents* $\rho_k = k$, *while* $\rho_m = \lambda + 1/2 = -1$.

(2) *The case* $\nu = 2m{+}1$. *Equation* $(8)_{0,o}$ *has* $(2m{+}2)$ *solutions* $\mathcal{V}_k(t)$, $0 \le k \le 2m+1$, *that can be expressed by the series*

$$\mathcal{V}_k(t) = t_1^{-(3\nu+4)/2\nu}\sigma^{\rho_k} V_k(\sigma^{2m+1}, -3/2),$$

where

$$V_k(x,\lambda) = \sum_{\ell \ge 0} a_{(2m+1)\ell+k}(\lambda) x^\ell,$$

$$a_{(2m+1)\ell+k}(\lambda) =$$

$$= \frac{\prod_{j=1}^{m+1}(\frac{(k-\lambda-1+(\nu j+1)/(\nu+2))}{\nu} - 1;\ell)(j/(\nu+2) + (k+m-\lambda)/\nu - 1;\ell)}{\prod_{j=1}^{2m+1}((k+j)/\nu - 1;\ell)((k-\lambda-1/2)/\nu;\ell)},$$

$$\sigma = \Big(-4\Big(\frac{t_1}{\nu+2}\Big)^{\nu+2}\Big)^{-1/\nu} t_2^2, \quad 0 \le k \le 2m+1.$$

For $k = 0,\dots,2m$ *the characteristic exponent* $\rho_k = k$, *while* $\rho_{2m+1} = \lambda + 1/2 = -1$.

Proof. In view of the substitution $t_1 = s_2$, $t_2^2 = s_0$, we solve equation $(7)_{0,e}$ (respectively $(7)_{0,o}$) to get a solution of equation $(8)_{0,e}$ (respectively $(8)_{0,o}$). To obtain the recursive relation between $a_{m\ell+k}$ and $a_{m(\ell+1)+k}$ (respectively, between $a_{(2m+1)\ell+k}$ and $a_{(2m+1)(\ell+1)+k}$) it is enough to take into account the following trivial equality:

$$\Big(\tau\frac{\partial}{\partial\tau} - \alpha\Big)\tau^r = (r-\alpha)\tau^r.$$

Further calculations are performed in an elementary manner. □

Remark 3. The functions $U_k(x,\lambda)$ ($V_k(x,\lambda)$) introduced in the above theorem can be regarded as generalized hypergeometric functions

$$_{m+1}F_m(\alpha_1^{(k)},\dots,\alpha_{m+1}^{(k)};\beta_1^{(k)},\dots,\beta_m^{(k)}|x)$$
$$\big(\,_{2m+2}F_{2m+1}(\gamma_1^{(k)},\dots,\gamma_{2m+2}^{(k)};\delta_1^{(k)},\dots,\delta_{2m+1}^{(k)}|x)\,\big)$$

in Pochhammer's notation (see [16]). This can be checked easily, as we find the factor $(0;\ell) = \ell!$ in every denominator of the expansion coefficients. The indices $\alpha_1^{(k)}, \beta_1^{(k)}, \dots.$ are obtained from the expansion coefficients.

Remark 4. The expressions obtained in Theorem 2.6 permit one to describe the monodromy of the period integral $J_0(t)$ around the origin. It is enough to see what happens by translation along a loop $\gamma : (t_1, t_2) \to (e^{2\pi i} t_1, e^{2\pi i} t_2)$. One observes that

$$\gamma_* \mathcal{U}_k(t,\lambda) = \exp\Big(2\pi i (\frac{m+1}{m}\Big(\lambda - \rho_k) + \frac{1}{2m}\Big)\Big)\,\mathcal{U}_k(t,\lambda), \quad \nu = 2m,$$
$$\gamma_* \mathcal{V}_k(t,\lambda) = \exp\Big(2\pi i \Big(\lambda - \frac{2+(\nu+2)\rho_k}{\nu}\Big)\Big)\,\mathcal{V}_k(t,\lambda), \quad \nu = 2m+1.$$

Namely, monodromy is described by the matrix

$$M = \begin{cases} \text{diag}\left[e^{-2\pi i(\frac{3m+1}{2m})}, e^{-2\pi i(\frac{3m+1+m+1}{2m})}, \dots, e^{-2\pi i(\frac{3m+1+m(m+1)}{2m})}\right], \ \nu = 2m, \\ \text{diag}\left[e^{-2\pi i(\frac{3\nu+2}{2\nu})}, e^{-2\pi i(\frac{3\nu+2+\nu+2}{2\nu})}, \dots, e^{-2\pi i(\frac{3\nu+2+\nu(\nu+2)}{2\nu})}\right], \ \nu = 2m+1. \end{cases}$$

This result is compatible with the well-known fact (see [17], Prop. 3.4.1) that the Coxeter number for S_{2m+3} (resp. S_{2m+4}) singularity is equal to $2m$ (resp. to $2(2m+1)$). We also remark here that the monodromy for quasihomogeneous hypersurface singularity has been calculated by means of the Gauss–Manin system in [3].

3. Connection with the Logarithmic Differential Forms

As for the relationship with the logarithmic differential forms studied by K. Saito, we obtain

Theorem 3.1. *Let $D = \{t \in \mathbf{C}^2; t_2\varphi(t_1,t_2) = 0\}$ be the discriminant set of the principal deformation for $S_{\nu+3}$-singularity. Then the Gauss–Manin system for the period integrals $\mathbf{J} = (J_0, \dots, J_\nu)^t$ associated with $S_{\nu+3}$-singularity permits a representation as a Picard–Fuchs system (total differential system) with coefficients from $\Omega^1_S(\log D)$ as follows:*

$$d\mathbf{J} = L\Big(A\frac{d(t_2\varphi)}{t_2\varphi} + \big(H_1(t_1) + H_2(t_1,t_2)\big)\frac{\iota_\eta(\omega)}{t_2\varphi}\Big)\mathbf{J}, \tag{9}$$

where L is the diagonal matrix corresponding to the weights of the basis of the cohomology group which appeared in Proposition 2.2,

$$A = 2\ \text{diag}(1/(2m+1), \dots, 1/(2m+1)), \quad \nu = 2m,$$
$$A = 4\ \text{diag}((2m+1)/(4m+3), \dots, (2m+1)/(4m+3)), \quad \nu = 2m+1,$$

$$H_1(t_1) = -m^{m-1}\frac{t_1^m}{(2m+1)(-m-1)^{m+1}}\begin{bmatrix} 2m & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$(\nu = 2m),$$

$$H_1(t_1) = \frac{8(2m+1)^{2m+1}t_1^{2m+2}}{(4m+3)(2m+3)^{2m+3}}\begin{bmatrix} 2(2m+1) & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{bmatrix}$$

$$(\nu = 2m+1),$$

$H_2(t)$ is a matrix with polynomial entries with the zero diagonal part. All the matrices given above have the size $(\nu+1)\times(\nu+1)$. The functions that appear in the definition of the divisor D have the forms

$$\varphi(t_1,t_2) = t_2^{2m} - 2(2m)^m\big(-t_1/(2m+2)\big)^{(m+1)}, \quad \nu = 2m,$$

$$\varphi(t_1,t_2) = t_2^{4m+2} + 4(2m+1)^{2m+1}\big(t_1/(2m+3)\big)^{(2m+3)}, \quad \nu = 2m+1.$$

The holomorphic 1-form $\iota_\eta(\omega)$ is defined by the interior product between the Euler vector field $\eta = \nu t_1 \frac{\partial}{\partial t_1} + \frac{(\nu+2)t_2}{2}\frac{\partial}{\partial t_2}$ and the volume form $\omega = dt_1 \wedge dt_2$.

Remark 5. As we have mentioned in the introduction, the logarithmic differential forms $\Omega^1_S(\log D)$ constitute a strictly narrower class than the differential forms $\Omega^1_S(D)$ with poles along D. In fact, the two meromorphic forms $d(t_2\varphi)/t_2\varphi,\ \iota_\eta(\omega)/t_2\varphi$ form a free basis of $\Omega^1_S(\log D)$ in this situation where the divisor D is a generalized cusp.

Remark 6. Let us consider

$$J(t) = \int F_0^{\lambda_0}(z)F_1^{\lambda_1}(z,t)\ldots F_m^{\lambda_m}(z,t)dz,$$

where $F_j(z,t)$ $(1 \le j \le m)$ are the linear functions with respect to the variables $z = (z_1,\ldots,z_n)$, and $F_0(z)$ is a quadric (or a linear function). As in [8], the Gauss–Manin systems defined for integrals $J(t)$ admit as their coefficients logarithmic differential forms of type $d\psi(t)/\psi(t)$ only. Our Theorem 3.1 states, however, that if $\deg F_1(z,t) \geq 2$ there may appear coefficients corresponding to the torsion element of Ω^1_D with $D = \{t \in \mathbf{C}^n : \psi(t) = 0\}$, which cannot be expressed in terms of logarithmic differential forms of type $d\psi(t)/\psi(t)$. Thus one can see an essential difference between the *hyperplane* arrangement case and cases associated with configurations of *hypersurfaces*.

Proof of Theorem 3.1. Before going into an analysis of period integrals $J_0(t)$, $\dots$, $J_\nu(t)$, let us consider the integrals $K_0(s), \dots, K_\nu(s)$. The quasihomogeneity of these integrals implies

$$\Big(w_0 s_0 \frac{\partial}{\partial s_0} + w_2 s_2 \frac{\partial}{\partial s_2}\Big)\mathbf{K} = L\mathbf{K},$$

where $w_0 = 1$, $w_2 = \nu/(\nu+2)$. The equation obtained in Proposition 2.2 and the quasihomogeneity yield

$$\begin{aligned} \frac{\partial}{\partial s_0}\mathbf{K} &= \frac{1}{s_0}\big(\operatorname{id}_{\nu+1} + C(s_2)/s_0\big)^{-1}(L+V)\mathbf{K},\\ \frac{\partial}{\partial s_2}\mathbf{K} &= \frac{w_0}{w_2 s_2}\big(\operatorname{id}_{\nu+1} + C(s_2)/s_0\big)^{-1}(C(s_2)L/s_0 - V)\mathbf{K}. \end{aligned}$$

In summary,

$$\begin{gathered} d\,\mathbf{K} = \frac{\partial}{\partial s_0}\mathbf{K}ds_0 + \frac{\partial}{\partial s_2}\mathbf{K}ds_2 = \\ = \big(s_0 \operatorname{id}_{\nu+1} + C(s_2)\big)^{-1}\Big(\big(L + V(s_2)\big)ds_0 + \frac{w_0}{w_2}\big(C(s_2)L - s_0 V(s_2)\big)\frac{ds_2}{s_2}\Big)\mathbf{K}. \end{gathered}$$

Consequently, for $\mathbf{J}$ we have a Picard–Fuchs system of the form

$$\begin{aligned} d\,\mathbf{J} &= \frac{\partial}{\partial t_1}\mathbf{J}dt_1 + \frac{\partial}{\partial t_2}\mathbf{J}dt_2 = \\ &= \big(t_2^2 \operatorname{id}_{\nu+1} + C(t_1)\big)^{-1}\Big(\big(2t_2(L + V(t_1)\big)dt_2 + \\ &\quad + \frac{w_0}{w_2}\big(C(t_1)L - t_2^2 V(t_1)\big)\frac{dt_1}{t_1}\Big)\mathbf{J}. \end{aligned} \tag{10}$$

Thus to show the statement, it is enough to calculate the expressions

$$\big(t_2^2 \operatorname{id}_{\nu+1} + C(t_1)\big)^{-1}, \quad \big(t_2^2 \operatorname{id}_{\nu+1} + C(t_1)\big)^{-1}C(t_1). \tag{11}$$

As for the remaining part of (10), it is easy to see that the expression

$$\big(t_2^2 \operatorname{id}_{\nu+1} + C(t_1)\big)^{-1}\Big(2t_2 V(t_1)dt_2 - \frac{w_0}{w_2}t_2^2 V(t_1)\frac{dt_1}{t_1}\Big)$$

belongs to $\mathrm{End}(\mathbf{C}^{\nu+1}) \otimes \mathbf{C}[t_1, t_2] \cdot \iota_\eta(\omega)$. This is evident from the equality

$$2w_2 t_1 dt_2 - w_0 t_2 dt_1 = 2\iota_\eta(\omega)/(\nu+2)$$

and the explicit form of the matrix V obtained in Proposition 2.2.

Let us show how to calculate the two expressions in (11). In view of the substitution $s_0 = t_2^2, s_2 = t_1$ the calculation is reduced to that of the differential form-valued matrix

$$\big(s_0 \,\mathrm{id}_{\nu+1} + C(s_2)\big)^{-1} ds_0 + \frac{w_0}{w_2}\big(s_0 \,\mathrm{id}_{\nu+1} + C(s_2)\big)^{-1} C(s_2) \frac{ds_2}{s_2}. \qquad (12)$$

We divide the expression into two parts to be calculated below:

$$\big(s_0 \,\mathrm{id}_{\nu+1} + C(s_2)\big)^{-1}, \qquad (13)$$

$$\big(s_0 \,\mathrm{id}_{\nu+1} + C(s_2)\big)^{-1} C(s_2). \qquad (14)$$

First of all we remark that

$$(s_0 \,\mathrm{id}_{\nu+1} + C(s_2))^{-1} =$$

$$= \frac{1}{s_0 \psi(s)} \times \begin{bmatrix} \psi(s) & * & * & \cdots & * \\ 0 & & & & \\ \vdots & & s_0(s_0 \,\mathrm{id}_\nu + \tilde{C}(s_2))^{-1} & & \\ 0 & & & & \end{bmatrix},$$

where $\widetilde{C}(s_2)$ is the $(\nu \times \nu)$-matrix defined as follows:

$$\widetilde{C}(s_2) = \begin{bmatrix} 0 & 0 & \frac{\nu}{\nu+2}s_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \frac{\nu}{\nu+2}s_2 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & \frac{\nu}{\nu+2}s_2 \\ \frac{-2\nu}{(\nu+2)^2}s_2^2 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \frac{-2\nu}{(\nu+2)^2}s_2^2 & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

The function $\psi(s_0, s_2)$ denotes a polynomial defined by

$$\psi = \det\big(s_0 \,\mathrm{id}_\nu + \widetilde{C}(s_2)\big)$$

and the first row of matrix (12) is uniquely determined by $(s_0 \,\mathrm{id}_\nu + \tilde{C}(s_2))^{-1}$. For the sake of simplicity, further we will use the notation

$$u_1 = -2\nu s_2^2/(\nu+2)^2, \quad u_2 = \nu s_2/(\nu+2).$$

By induction with respect to the size of matrix, one can show

$$\psi(s_0, s_2) = s_0^{2m+1} + u_1^2 u_2^{2m-1}, \quad when \;\; \nu = 2m+1;$$
$$\psi(s_0, s_2) = \big(s_0^m + u_1(-u_2)^{m-1}\big)^2, \quad when \;\; \nu = 2m.$$

Below we will make use of the notation of function $\phi(s_0, s_2)$ defined as follows:

$$\phi = \psi, \quad when \;\; \nu = 2m+1;$$
$$\phi = s_0^m + u_1(-u_2)^{m-1}, \; when \;\; \nu = 2m.$$

We divide the calculation procedure into several steps that are formulated in the form of lemmas.

Lemma 3.2. *Let us define a $(\nu \times \nu)$-matrix R as follows:*

$$R = \psi(s_0 \operatorname{id}_\nu + \tilde{C}(s_2))^{-1}.$$

The entries of R are given by the relations shown below.
The case $\nu = 2m$:

$$\begin{aligned}
R_{i,i} &= s_0^{m-1}, && 1 \le i \le 2m,\\
R_{2i,2j+1} = R_{2i+1,2j} &= 0, && 1 \le i, j \le m,\\
R_{i+1,j+1} &= R_{i,j}, && 1 \le i, j \le 2m,\\
R_{1,2j+1} &= (-s_0)^{j-1} u_1 u_2^{m-j-1}, && 1 \le j \le m,\\
R_{2j+1,1} &= s_0^{m-j-1}(-u_2)^j, && 1 \le j \le 2m,\\
R_{i,j} \cdot R_{j,i} &= -s_0^{m-2} u_1 u_2^{m-1}, && 1 \le i, j \le 2m.
\end{aligned}$$

The case $\nu = 2m+1$:

$$\begin{aligned}
R_{i,i} &= s_0^{2m}, && 1 \le i \le 2m+1,\\
R_{i+1,j+1} &= R_{i,j}, && 1 \le i, j \le 2m,\\
R_{1,2j} &= (-s_0)^{m+j-1} u_1 u_2^{m-j}, && 1 \le j \le m,\\
R_{1,2j+1} &= (-s_0)^{j-1} u_1^2 u_2^{2m-j-1}, && 1 \le j \le m,\\
R_{i,j} \cdot R_{j,i} &= (-s_0)^{2m-1} u_1^2 u_2^{2m-1}, && 1 \le i, j \le 2m+1.
\end{aligned}$$

Using the matrix R defined above we get a concrete expression of matrix (13).

Lemma 3.3. *If we set*

$$S = s_0 \phi\big(s_0 \operatorname{id}_{\nu+1} + C(s_2)\big)^{-1},$$

then the $(\nu+1)\times(\nu+1)$*-matrix* S *admits the following expression:*

$$S = \begin{bmatrix} \phi & s_0R_{2,1} & s_0R_{3,1} & \cdots & s_0R_{\nu,1} & S_{\nu,1} \\ 0 & & & & & \\ \vdots & & & s_0R & & \\ 0 & & & & & \end{bmatrix},$$

where

$$\begin{aligned} S_{i+1,j+1} &= s_0R_{i,j}, & 1 \le i,j \le \nu-1, \\ S_{j,1} &= s_0R_{j,1}, & 2 \le j \le \nu, \\ S_{2m,1} &= (-u_2)^m, & \nu = 2m, \\ S_{2m+1,1} &= -u_1u_2^{2m}, & \nu = 2m+1. \end{aligned}$$

Calculating matrix (14), one gets the following

Lemma 3.4. *Let us set*

$$T = s_0\phi\big(s_0 \operatorname{id}_{\nu+1} + C(s_2)\big)^{-1}C(s_2).$$

Then the following equality holds for the off-diagonal entries of the matrices S *and* T*:*

$$T_{i,j} = -s_0S_{i,j}, \quad \textit{when } i \ne j.$$

Proof. It is easy to see that the $(i+2)$th column of the matrix product RC coincides with the ith column of the matrix R multiplied by u_2. The latter in turn is equal to the $(i+2)$th column of the matrix R multiplied by $(-s_0)$. These equalities immediately follow from Lemma 3.2. □

On rewriting (12) in terms of the matrices S and T defined above, we have

$$\Big(s_0 \operatorname{id}_{\nu+1} + C(s_2)\Big)^{-1} ds_0 + \frac{w_0}{w_2}\Big(s_0 \operatorname{id}_{\nu+1} + C(s_2)\Big)^{-1} C(s_2)\frac{ds_2}{s_2} = \\ = \frac{1}{s_0\phi}\Big(S ds_0 + \frac{w_0}{w_2}T\frac{ds_2}{s_2}\Big).$$

Lemma 3.4 implies that the (i,j)-element, $i \ne j$, of this matrix admits the expression

$$\frac{S_{i,j}}{w_2s_2}(w_2s_2ds_0 - w_0s_0ds_2) = \frac{S_{i,j}}{w_2s_2}\iota_\xi(\theta),$$

where

$$\xi = w_0s_0\frac{\partial}{\partial s_0} + w_2s_2\frac{\partial}{\partial s_2}, \qquad \theta = ds_0 \wedge ds_2.$$

Note that $S_{i,j},\ i \ne j$, are always divisible by s_2 from the concrete form of the matrix given in Lemma 3.3. By the same lemma the diagonal elements of this matrix, except the $(1,1)$-element equal to $1/s_0$, can be written as follows:

$$\frac{d\phi}{m\phi}, \quad \nu = 2m; \qquad\qquad \frac{d\phi}{\nu\phi}, \quad \nu = 2m+1.$$

To summarize, we have shown for the integrals $\mathbf{K} = (K_0, \ldots, K_\nu)$ that

$$d\mathbf{K} = L(A' ds_0/s_0 + B' d\phi/\phi + H' \iota_\xi(\theta)/s_0\phi)\mathbf{K},$$

where

$$\begin{aligned} A' &= \operatorname{diag}(1, 0, \ldots, 0), \\ B' &= \operatorname{diag}(0, 1/m, \ldots, 1/m) \quad when \;\; \nu = 2m, \\ B' &= \operatorname{diag}(0, 1/\nu, \ldots, 1/\nu) \quad when \;\; \nu = 2m+1, \end{aligned}$$

and H' is a matrix with the polynomial entries with the zero diagonal part. On making the transition in the variables from (s_0, s_2) to (t_1, t_2), we get a total differential system for the integrals $J_0(t), \ldots, J_\nu(t)$:

$$d\,\mathbf{J} = L\Big(2A' \frac{dt_2}{t_2} + B' \frac{d\varphi}{\varphi} + H \iota_\eta \frac{\omega}{t_2\varphi}\Big)\,\mathbf{J}. \tag{15}$$

In order to see that it is possible to write (15) as a system with a single denominator $t_2\varphi(t)$, we solve the equations below with respect to the matrices $A, H_1 \in \operatorname{End}(\mathbf{C}^{\nu+1}) \otimes \mathbf{C}[t]$:

$$2A' \frac{dt_2}{t_2} + B' \frac{d\varphi}{\varphi} + H \frac{\iota_\eta(\omega)}{t_2\varphi} = A \frac{d(t_2\varphi)}{t_2\varphi} + H_1 \frac{\iota_\eta(\omega)}{t_2\varphi}.$$

Calculation of the matrices A and H_1 gives the desired formula (9). □

4. Further Remarks and Problems

Consider the divisor

$$D = \big\{t \in \mathbf{C}^n : h(t) = h_1(t) \ldots h_m(t) = 0\big\}$$

where $h_1(t), \ldots, h_m(t) \in \mathbf{C}[t]$ are irreducible factors, i.e.,

$$D = \bigcup_{i=1}^{m} D_i, \; D_i = \big\{t \in \mathbf{C}^n : h_i(t) = 0\big\}.$$

In the case where D_i, $1 \le i \le m$, form a set of normal crossing divisors so that

(1) D_i intersects transversally D_j, $i \ne j$;

(2) $\dim D_i \cap D_j \cap D_k \le n-3 \; for \;\; i \ne j \ne k \ne i$,

it is known (see [13], (2.9)) that $\Omega^1_{\mathbf{C}^n}(\log D)$ is generated by

$$dh_1/h_1, \ldots, dh_m/h_m \tag{16}$$

as an $\mathcal{O}_{\mathbf{C}^n}$-module. In such a case, "a Pfaff system of the Fuchsian type" is defined in quite a natural manner for a set of unknown functions $I = (I_1, \dots, I_\mu)$:

$$d\,\mathbf{I} = \Big(\sum_{j=1}^{m} A_j \frac{dh_j}{h_j}\Big)\,\mathbf{I}, \tag{17}$$

where $A_j \in \mathrm{End}(\mathbf{C}^\mu)\otimes\mathcal{O}_{\mathbf{C}^n}$. Pfaff systems of this type were studied in [18]. Theorem 3.1, however, implies that when the components of the divisor do not intersect transversally, there arise logarithmic differential forms like $\iota_\eta(\omega)$ in (9), in addition to those of type (16). That is to say, it is natural to think of a class of systems

$$d\,\mathbf{I} = \Big(A\frac{dh}{h} + \sum_{j=1}^{n-1} B_j \frac{\omega_j}{h}\Big)\mathbf{I} \tag{18}$$

with $\omega_j \in Tors\,\Omega^1_D$ and $A, B_j \in \mathrm{End}(\mathbf{C}^\mu)\otimes\mathcal{O}_{\mathbf{C}^n}$ satisfying the integrability condition

$$dA\frac{dh}{h} + \sum_{j=1}^{n-1} dB_j\frac{\omega_j}{h} + B_j d\Big(\frac{\omega_j}{h}\Big) = \sum_{j=1}^{n-1}(B_jA - AB_j)\frac{\omega_j dh}{h^2}.$$

The above expression (18) is appropriate in describing a Pfaff system with $\Omega^1_{\mathbf{C}^n}(\log D)$ coefficients in view of the following exact sequence proved in [14]:

$$0 \longrightarrow \frac{dh}{h}\mathcal{O}_{\mathbf{C}^n} + \Omega^1_{\mathbf{C}^n} \longrightarrow \Omega^1_{\mathbf{C}^n}(\log\, D) \xrightarrow{\;h\;} Tors\,\Omega^1_D \longrightarrow 0.$$

Here $rank(Tors\,\Omega^1_D) = n-1$. Furthermore, when $\Omega^1_{\mathbf{C}^n}(\log\, D)$ is a free $\mathcal{O}_{\mathbf{C}^n}$-module, the *integral variety* defined by the dual free module $Der_{\mathbf{C}^n}(\log\, D)$ coincides with $D = \{t \in \mathbf{C}^n : h(t) = 0\}$ (see [13] (1.9)).

Note that in the case where the divisor D consists of normally crossing divisors D_i that satisfy conditions (1) and (2) mentioned above, the $\mathcal{O}_D$-module of torsion differentials $Tors\,\Omega^1_D$ is generated by $(m-1)$ differential 1-forms

$$hdh_1/h_1, \dots \widehat{hdh_i/h_i} \dots, hdh_m/h_m, \qquad 1 \le i \le m,$$

where the ith form hdh_i/h_i is omitted. Therefore, in this situation system (17) can be considered as a special case of (18). Thus one may regard a system of type (18) as a natural generalization of "a Pfaff system of the Fuchsian type" to the case of a divisor consisting of components that do not cross normally.

The following two questions concerning systems of type (18) were proposed by K. Aomoto.

Question 1. Let us consider an arbitrary representation $\rho \in \pi_1(\mathbf{C}^n \setminus D)$. Is it possible to find a system of form (18) such that its solutions induce ρ as their monodromy representation? In other words, is class (18) wide enough for the existence of solutions to the Riemann–Hilbert problem?

K. Aomoto gave positive answer (see [19]) to this problem in the case where ρ is contained in a unipotent subgroup of $GL(\mu, \mathbf{C})$.

Question 2. Describe the cases where there exists an appropriate finite covering space X over $\mathbf{C}^n$,

$$\begin{array}{cccc} \pi: & X & \to & \mathbf{C}^n \\ & \cup & & \cup \\ & \widetilde{D} & \to & D \end{array}$$

such that the preimage of a system of type (18) under π has the form

$$d\,\tilde{\mathbf{I}} = \Big(\sum_{j=1}^{m} A_j \frac{dh_j}{h_j}\Big)\,\tilde{\mathbf{I}},$$

where $A_j \in \mathrm{End}(\mathbf{C}^\mu) \otimes \mathcal{O}_X,\ h_j \in \mathcal{O}_X,\ j = 1, \dots, m$, and $\widetilde{D} = \cup_{j=1}^{m} \{z \in X : h_j(z) = 0\}$. As K. Aomoto pointed out, that system (9) turns out to be the case in question because its solutions are described by Pochhammer's hypergeometric functions that are interpreted as solutions to the following system [20]:

$$\frac{d}{dz}\,\mathbf{J} = \Big(\frac{A_1}{z} + \frac{A_2}{z-1}\Big)\,\mathbf{J},$$

where $A_1, A_2 \in \mathrm{End}(\mathbf{C}^\mu)$.

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References

1. L. Euler, Institutionum calculi integralis, volumen secundum. *Academia Imperialis Scientiarum, Petropolis,* 1769.

2. C. F. Gauss, Disquisitiones genrales circa seriem infinitam. *Comm. Gott.* **2**(1813), 123–161.

3. E. Brieskorn, Die Monodromie der isolierten Singularitäten von Hyperflächen. *Manuscripta Math.* **2**(1970), 103–161; **51**(1972), 189–217.

4. K. Saito, On the uniformization of complements of discriminant loci. *Hyperfunctions and Linear Partial Differential Equations, RIMS Kokyuroku* **287**(1977), 117–137.

5. S. Ishiura and M. Noumi, A calculus of the Gauss–Manin system of type A_ℓ, I, II. *Proc. Japan Acad. Ser. A, Math. Sci.* **58**(1982), 13–16, 62–65.

6. M. Noumi, Expansion of the solutions of a Gauss–Manin system at a point of infinity. *Tokyo J. Math.* **7**(1984), 1–60.

7. K. Aomoto, Les équations aux différences et les intégrales des fonctions multiformes. *J. Fac. Sci. Univ. Tokyo, Sec. IA Math.* **22**(1975), 271–297.

8. K. Aomoto, Configurations and invariant Gauss–Manin connections of integrals I. *Tokyo J. Math.* **5**(1982), 249–287.

9. I. M. Gelfand, V. A. Vasiliev, and A. V. Zelevinski, Generalized hypergeometric functions on complex Grassmannians. *Funct. Anal. Appl.* **21**(1987), No. 1, 19–31.

10. S. Guzev, Picard–Fuchs systems for deformations of simple complete intersections. *Unpublished notes,* 1983.

11. G.-M. Greuel, Der Gauß–Manin–Zusammenhang isolierter Singularitäten von vollständigen Durchschnitten. *Math. Ann.* **214**(1975), No. 3, 235–266.

12. K. Saito, On the periods of primitive integrals I. *Preprint, RIMS, Kyoto Univ.,* 1982.

13. K. Saito, Theory of logarithmic differential forms. *J. Fac. Sci. Univ. Tokyo, Ser. IA* **27**(1980), No. 2, 265–291.

14. A. G. Aleksandrov, Nonisolated hypersurface singularities. *Adv. Sov. Math.* **1**, 211–246; *Amer. Math. Soc., Providence, Rhode Island,* 1990.

15. M. Giusti, Classification des singularités isolées simples d'intersections completès. *Singularities, Proc. Symp. Pure Math.* **40**, Pt.1, 457–494; *Amer. Math. Soc., Providence, Rhode Island,* 1983.

16. A. H. M. Levelt, Hypergeometric functions. *Proc. Konikl. Nederlandse Akad. Wet., Ser. A, Math. Sci.* **64**(1961), 190–211.

17. W. Ebeling, The monodromy groups of isolated singularities of complete intersections. *Lecture Notes in Math.,* 1293, *Springer, Berlin,* 1987.

18. R. Gerard et A. H. M. Levelt, Étude d'une classe particulière de système de Pfaff du type de Fuchs sur l'espace projectif complexe. *J. Math. Pures Appl.* **51**(1972), 189–217.

19. K. Aomoto, Fonctions hyperlogarithmiques et groupes de monodromie unipotents. *J. Fac. Sci. Univ. Tokyo, Sec. IA Math.* **25**(1978), 149–156.

20. P. Appell and J. Kampé de Fériet, Fonctions hypergeometriques et hypersphériques. *Paris,* 1926.

21. Kh. M. Malikov, Over-determination of a system of differential equations for versal integrals of type A, D, E. (Russian) *Differentsial'nye Uravneniya* **18**(1982), No. 8, 1394–1400.

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