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ESTIMATION OF THE INTEGRAL MODULUS OF SMOOTHNESS OF AN EVEN FUNCTION OF SEVERAL VARIABLES WITH QUASICONVEX FOURIER COEFFICIENTS

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Abstract. The estimate of the modulus of smoothness of an even function of several variables with quasiconvex Fourier coefficients obtained in this paper extends one result of S. A. Telyakovski.

1. Let $(a_{i,j})_{i,j\geq 0}$ be a double numerical sequence. Denote by $\Delta^2 a_{i,j}$ the expression $\Delta_{12}(\Delta_{12}a_{i,j})$, where

$$\Delta_{12}a_{i,j}=\Delta_1(\Delta_2 a_{i,j})=\Delta_1(a_{i,j}-a_{i,j+1})=a_{i,j}-a_{i+1,j}-a_{i,j+1}+a_{i+1,j+1}.$$

Definition 1. The double sequence $(a_{i,j})_{i,j\geq 0}$ will be called quasiconvex if the series

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left[(i+1)(j+1)|\Delta^2 a_{i,j}|+(i+1)|\Delta_1(\Delta_{12}a_{ij})|+(j+1)|\Delta_2(\Delta_{12}a_{ij})|\right]$$

converges.

As can be easily shown, if the sequence $(a_{i,j})_{i,j\geq 0}$ is quasiconvex and

$$\lim_{i+j\to\infty} a_{i,j}=0,$$

then the series

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\lambda_{i,j}a_{i,j}\cos ix\cos jy, \tag{$*$}$$

where $\lambda_{0,0}=\frac{1}{4}$, $\lambda_{0,j}=\lambda_{i,0}=\frac{1}{2}$, $i,j=1,2,\dots,$ $\lambda_{i,j}=1$ for $i,j>0$, converges on $(0,2\pi)^2$ to some function $f\in L(T^2)$ and is its Fourier series with $T^2=[0,2\pi]^2$.

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The expression

$$\omega_{m,n}(\delta,\rho;f)_1 =$$
$$= \sup_{\substack{|h|\le\delta \\ |\eta|\le\rho}} \int_{T^2} \Big| \sum_{\mu=0}^{m}\sum_{\nu=0}^{n}(-1)^{\mu+\nu}\binom{m}{\mu}\binom{n}{\nu} f[x+(m-2\mu)h, y+(n-2\nu)\eta]\Big| dx\,dy$$

is usually called an integral modulus of smoothness of order (m,n) of a function $f\in L(T^2)$. Here and in what follows it is assumed that $\delta\in[0,\pi]$, $\rho\in[0,\pi]$.

Aljančič and Tomič [1], [2] considered an estimate of the integral modulus of continuity in terms of Fourier coefficients of the function f for some classes of sequences in the one-dimensional case. In 1963 M. and S. Izumi [3] proved that if the Fourier coefficients $a_n\to 0$, $n\to\infty$, form a quasiconvex sequence, then the estimate

$$\omega(\delta;f)_1 \le c\delta \sum_{i\le\frac{1}{\delta}} i^2|\Delta_1^2 a_i| + \sum_{i>\frac{1}{\delta}} i|\Delta_1^2 a_i|^1$$

holds for the integral modulus of continuity $\omega(\delta;f)_1$.

This result was later generalized by Telyakovski [4] who proved a theorem giving an estimate of an integral modulus of smoothness of order m, $m\in\mathbb{N}$.

In this paper an estimate is obtained for the integral modulus of smoothness for a function of several variables which is even with respect to each variable.

2. For a further discussion we need

Lemma 1. *Let $2mh\le t\le\pi$. Then*

$$|T_p^m| = \Big|\sum_{i=0}^{m}(-1)^i\binom{m}{i}K_p[t+(m-2i)h]\Big| \le c(m)h^m p^{m-1}t^{-2},$$
$$|T_p^m|\le c(m)h^m p^{m+1},$$

where $K_p(t)$ is the Fejer kernel, $p\in\mathbb{N}$.

Proof. It is easy to show that

$$T_p^m = \sum_{i=0}^{m}(-1)^i\binom{m}{i}K_p[t+(m-2i)h] =$$
$$= \sum_{i=0}^{m-1}(-1)^i\binom{m-1}{i}\big\{K_p[t+(m-2i)h]-K_p[t+(m-2(i+1))h]\big\} =$$

[1]Here and in what follows, c, $c(m)$, $c(m,n),\dots$ denote, generally speaking, various positive constants depending only on the parameters indicated in the brackets.

$$= -2h\sum_{i=0}^{m-1}(-1)^i\binom{m-1}{i}K_p'[t+(m-2(i+1-Q_1))h] = \cdots =$$
$$= (-1)^m h^m K_p^{(m)}(t+c(m)h),$$

where $c(m) = m-2(m-Q_1-\cdots-Q_m)$, $0<Q_i<1$, $i=1,\dots,m$.

Since $-m<c(m)<m$, we have

$$K_p^{(m)}(t+c(m)h) = \frac{1}{p+1}\sum_{i=0}^{p}\mathcal{D}_i^{(m)}(t+c(m)h) =$$
$$= \pm\frac{1}{p+1}\sum_{i=0}^{p}\sum_{j=1}^{i}j^m\begin{cases}\cos j(t+c(m)h), & 2|m,\\ \sin j(t+c(m)h), & 2\nmid m,\end{cases}$$

where $\mathcal{D}_i(t)$ is the Dirichlet kernel.

Let $2|m$ (the case $2\nmid m$ is considered similarly). Then applying twice the Abel transformation and taking into account the estimates

$$K_p(t) \le \frac{c}{pt^2}, \quad 0<|t|\le\pi,$$
$$K_p(t) \le p, \quad |t|\le\pi,$$

we find

$$K_p^{(m)}(t+c(m)h) = \frac{1}{p+1}\sum_{i=0}^{p}\sum_{j=1}^{i}\left|j^m-2(j+1)^m+\right.$$
$$\left.+(j+2)^m\right|jK_j(t+c(m)h)+$$
$$+\frac{1}{p+1}\sum_{i=0}^{p}\left|(i-1)^m-i^m\right|iK_i(t+c(m)h)+$$
$$+\frac{1}{p+1}\sum_{i=0}^{p}\left|i^m-(i+1)^m\right|iK_i(t+c(m)h)+$$
$$+\frac{1}{p+1}\cdot p^{m+1}K_p(t+c(m)h)\le$$
$$\le\begin{cases}c(m)p^{m-1}t^{-2}, & 2mh\le t\le\pi,\\ c(m)p^{m+1}, & -\pi\le t\le\pi.\end{cases}$$

Therefore

$$|T_p^m| \le c(m)h^m p^{m-1}t^{-2}, \quad 2mh\le t\le\pi,$$
$$|T_p^m| \le c(m)h^m p^{m+1}, \quad -\pi\le t\le\pi. \qquad \square$$

The two-dimensional analogue of Lemma 1 given below is proved similarly.

Lemma 2. *Let $2mh \le x \le \pi$, $2n\eta \le y \le \pi$. Then*

$$|T^{m,n}_{p,q}| = \Big|\sum_{\mu=0}^{m}\sum_{\nu=0}^{n}(-1)^{\mu+\nu}\binom{m}{\mu}\binom{n}{\nu}K_p[x+(m-2\mu)h]K_q[y+(n-2\nu)\eta]\Big| \le$$
$$\le c(m,n)\,\frac{h^m\eta^n p^{m-1}q^{n-1}}{(xy)^2},$$
$$|T^{m,n}_{p,q}| \le c(m,n)h^m\eta^n p^{m+1}q^{n+1}, \quad (x,y)\in T^2.$$

Lemma 3 ([5]). *Let $2mh \le x \le \pi$. Then*

$$\Big|\sum_{i=0}^{m}(-1)^i\binom{m}{i}\mathcal{D}_p[x+(m-2i)h]\Big| \le c(m)h^m p^m x^{-1},$$

where $\mathcal{D}_p$ is the Dirichlet kernel, $p\in\mathbb{N}$.

Theorem 1. *Let a double sequence $(a_{i,j})_{i,j\ge 0}$ be quasiconvex and*

$$\lim_{i+j\to\infty} a_{i,j} = 0.$$

Then for the sum f of the series $(*)$ we have

$$\omega_{m,n}(h,\eta;f)_1 \le c(m,n)\Big\{h^m\eta^n\sum_{i=1}^{[\frac{1}{2mh}]}\sum_{j=1}^{[\frac{1}{2n\eta}]} i^{m+1}j^{n+1}|\Delta^2 a_{i,j}| +$$
$$+ h^m\sum_{i=1}^{[\frac{1}{2mh}]}\sum_{j=[\frac{1}{2n\eta}]+1}^{\infty} i^{m+1}j|\Delta^2 a_{i,j}| +$$
$$+ \eta^n\sum_{i=[\frac{1}{2mh}]+1}^{\infty}\sum_{j=1}^{[\frac{1}{2n\eta}]} ij^{n+1}|\Delta^2 a_{i,j}| +$$
$$+ \sum_{i=[\frac{1}{2mh}]+1}^{\infty}\sum_{j=[\frac{1}{2n\eta}]+1}^{\infty} ij|\Delta^2 a_{i,j}|\Big\} =$$
$$= \sum_{s=1}^{4}\mathcal{P}_s(m,n,h,\eta).$$

Proof. We write

$$A(m,n,f) = 4\int_0^\pi\int_0^\pi |\Delta^{m,n}f|\,dx\,dy = 4\Big\{\int_0^{2mh}\int_0^{2n\eta} + \int_{2mh}^{\pi}\int_0^{2n\eta} + \int_0^{2mh}\int_{2n\eta}^{\pi} +$$

$$+\int\limits_{2mh}^{\pi}\int\limits_{2n\eta}^{\pi}\Big\}|\Delta^{m,n}f|\,dx\,dy\equiv\sum_{s=1}^{4}A_s(m,n,f),$$

where

$$\Delta^{m,n}f=\sum_{\mu=0}^{m}\sum_{\nu=0}^{n}(-1)^{\mu+\nu}\binom{m}{\mu}\binom{n}{\nu}f[x+(m-2\mu)h,y+(n-2\nu)\eta]=$$

$$=\sum_{\mu=0}^{m}\sum_{\nu=0}^{n}(-1)^{\mu+\nu}\binom{m}{\mu}\binom{n}{\nu}\Big\{\sum_{i=1}^{[\frac{1}{x}]}\sum_{j=1}^{[\frac{1}{y}]}+\sum_{i=[\frac{1}{x}]+1}^{\infty}\sum_{j=1}^{[\frac{1}{y}]}+\sum_{i=1}^{[\frac{1}{x}]}\sum_{j=[\frac{1}{y}]+1}^{\infty}+$$

$$+\sum_{i=[\frac{1}{x}]+1}^{\infty}\sum_{j=[\frac{1}{y}]+1}^{\infty}\Big\}a_{i,j}\cos i[x+(m-2\mu)h]\cos[y+(n-2\nu)\eta]\equiv$$

$$\equiv\sum_{s=1}^{4}\mathcal{T}_s(m,n).$$

It is easy to show that

$$\mathcal{T}_1\equiv\mathcal{T}_1(m,n)=2^{m+n}\sum_{i=1}^{[\frac{1}{x}]}\sum_{j=1}^{[\frac{1}{y}]}a_{i,j}\sin^m ih\sin^n j\eta\times$$

$$\times\begin{cases}(-1)^{\frac{m+n}{2}}\cos ix\cos jy, & 2|m,\ \ 2|n,\\ (-1)^{\frac{m+n-1}{2}}\sin ix\cos jy, & 2\nmid m,\ \ 2|n,\\ (-1)^{\frac{m+n-1}{2}}\cos ix\sin jy, & 2|m,\ \ 2\nmid n,\\ (-1)^{\frac{m+n-2}{2}}\sin ix\sin jy, & 2\nmid|m,\ \ 2\nmid n.\end{cases}$$

Further,

$$\int\limits_{2mh}^{\pi}\int\limits_{2n\eta}^{\pi}|\mathcal{T}_1|dx\,dy\le c(m,n)h^m\eta^n\int\limits_{2mh}^{\pi}\int\limits_{2n\eta}^{\pi}\sum_{i=1}^{[\frac{1}{x}]}\sum_{j=1}^{[\frac{1}{y}]}i^mj^n|a_{i,j}|dx\,dy\le$$

$$\le c(m,n)h^m\eta^n\int\limits_{1}^{2mh}\int\limits_{1}^{2n\eta}(xy)^{-2}\sum_{i=1}^{[x]}\sum_{j=1}^{[y]}i^mj^n|a_{i,j}|dx\,dy\le$$

$$\le c(m,n)h^m\eta^n\sum_{r=1}^{[\frac{1}{2mh}]}\sum_{s=1}^{[\frac{1}{2n\eta}]}(rs)^{-2}\sum_{i=1}^{r}\sum_{j=1}^{s}i^mj^n|a_{i,j}|\le$$

$$\le c(m,n)h^m\eta^n\sum_{i=1}^{[\frac{1}{2mh}]}\sum_{j=1}^{[\frac{1}{2n\eta}]}i^mj^n|a_{i,j}|\sum_{r=i}^{[\frac{1}{2mh}]}\sum_{s=j}^{[\frac{1}{2n\eta}]}(rs)^{-2}\le$$

$$\leq c(m,n)h^m\eta^n\sum_{i=1}^{[\frac{1}{2mh}]}\sum_{j=1}^{[\frac{1}{2n\eta}]} i^{m-1}j^{n-1}|a_{i,j}|. \tag{2.1}$$

Applying twice the Hardy transformation [6], $p=[\frac{1}{2mh}]$, $q=[\frac{1}{2n\eta}]$, which is a two-dimensional analog of the Abel transformation, we find

$$\sum_{i=1}^{p}\sum_{j=1}^{q} i^{m-1}j^{n-1}|a_{i,j}| = \sum_{i=1}^{p} i^{m-1}\bigg\{\sum_{j=1}^{q-1}\big(|a_{i,j}|-|a_{i,j+1}|\big)\sum_{s=1}^{j}s^{n-1} +$$
$$+\sum_{s=1}^{q}s^{n-1}|a_{i,q}|\bigg\} \leq \sum_{j=1}^{q-1} j^n\bigg\{\sum_{i=1}^{p} i^{m-1}|\Delta_2 a_{i,j}| + \sum_{i=1}^{p} i^{m-1}q^n|a_{i,q}|\bigg\} =$$
$$= \sum_{j=1}^{q-1} j^n\bigg\{\sum_{i=1}^{p-1}\big(|\Delta_2 a_{i,j}|-|\Delta_2 a_{i+1,j}|\big)\sum_{r=1}^{i}r^{m-1} +$$
$$+\sum_{i=1}^{p} i^{m-1}|\Delta_2 a_{p,j}|\bigg\} + q^n\sum_{i=1}^{p-1}\big(|a_{i,q}|-|a_{i+1,q}|\big)\sum_{r=1}^{i}r^{m-1} +$$
$$+q^n|a_{p,q}|\sum_{i=1}^{p} i^{m-1} \leq \sum_{i=1}^{p-1}\sum_{j=1}^{q-1}|\Delta_{12}a_{i,j}|i^m j^n +$$
$$+p^m\sum_{j=1}^{q-1}j^n|\Delta_2 a_{p,j}| + q^n\sum_{i=1}^{p-1}i^m|\Delta_1 a_{i,q}| + p^m q^n|a_{p,q}| \leq$$
$$\leq \sum_{i=1}^{p-1}\sum_{j=1}^{q-1} i^{m+1}j^{n+1}|\Delta^2 a_{i,j}| + q^{n+1}\sum_{i=1}^{p-2}i^{m+1}|\Delta_{12}a_{i,q}-\Delta_{12}a_{i+1,q}| +$$
$$+p^{m+1}\sum_{j=1}^{q-2}j^{n+1}|\Delta_{12}a_{p,j}-\Delta_{12}a_{p,j+1}| + p^{m+1}q^{n+1}|\Delta_{12}a_{p,q}| +$$
$$+q^n\sum_{i=1}^{p-2}i^{m+1}|\Delta_1(a_{i,q}-a_{i+1,q})| + p^m\sum_{j=1}^{q-2}j^{n+1}|\Delta_2(a_{p,j}-a_{p,j+1})| +$$
$$+q^n p^{m+1}|\Delta_1 a_{p,q}| + p^m q^{n+1}|\Delta_2 a_{p,q}| + p^m q^n|a_{p,q}| \equiv$$
$$\equiv \sum_{\alpha=1}^{9} I_\alpha(m,n,p,q). \tag{2.2}$$

For I_2 we obtain

$$I_2 = q^n\sum_{i=1}^{p-2}i^{m+1}q|\Delta_1(\Delta_{12}a_{i,q})| = q^n\sum_{i=1}^{p-2}i^{m+1}q\Big|\sum_{j=q}^{\infty}\Delta^2 a_{i,j}\Big| \leq$$

$$\le q^n \sum_{i=1}^{p-2} i^{m+1} \sum_{j=q}^{\infty} j|\Delta^2 a_{i,j}|. \tag{2.3}$$

Similarly,

$$I_3 \le p^m \sum_{i=p}^{\infty} \sum_{j=1}^{q-2} i j^{n+1} |\Delta^2 a_{i,j}|. \tag{2.4}$$

Let us now estimate I_5. We have

$$I_5 \le q^n \sum_{i=1}^{p-2} i^{m+1} |\Delta_1^2 a_{i,q}|,$$

where

$$\Delta_1^2 a_{i,q} = a_{i,q} - a_{i+1,q} + a_{i+2,q}.$$

It is easy to show that

$$\Delta_1^2 a_{i,q} = \sum_{s=q}^{\infty} \Delta_2(\Delta_1^2 a_{i,s}) = \sum_{j=q}^{\infty} \Delta_2\Big(\sum_{s=j}^{\infty} \Delta_2(\Delta_2 a_{i,s})\Big) = \sum_{j=q}^{\infty}\sum_{s=j}^{\infty} \Delta^2 a_{i,s}.$$

Therefore

$$|\Delta_1^2 a_{i,q}| \le \sum_{j=q}^{\infty}\sum_{s=j}^{\infty} |\Delta^2 a_{i,s}| \le \sum_{s=q}^{\infty} s|\Delta^2 a_{i,s}|. \tag{2.5}$$

Thus

$$I_5 \le q^n \sum_{i=1}^{p-2} \sum_{s=q}^{\infty} i^{m+1} s |\Delta^2 a_{i,s}|. \tag{2.6}$$

Similarly,

$$I_6 \le p^m \sum_{i=p}^{\infty} \sum_{j=1}^{q-2} i j^{n+1} |\Delta^2 a_{i,j}|. \tag{2.7}$$

Next,

$$I_7 = p^m q^n \cdot p|\Delta_1 a_{p,q}|.$$

Since

$$p|\Delta_1 a_{p,q}| = p\Big|\sum_{i=p}^{\infty} \Delta_1^2 a_{i,q}\Big| \le \sum_{i=p}^{\infty} i|\Delta_1^2 a_{i,q}|,$$

by repeating the arguments used for I_5 we obtain

$$I_7 \le p^m q^n \sum_{i=p}^{\infty}\sum_{j=q}^{\infty} ij|\Delta^2 a_{i,j}|. \tag{2.8}$$

The same estimate holds for I_8. For I_4 we have

$$I_4 \le p^m q^n \sum_{i=p}^{\infty}\sum_{j=q}^{\infty} ij|\Delta^2 a_{i,j}|. \tag{2.9}$$

In the same way,

$$I_9 \le p^m q^n \Big| \sum_{i=p}^{\infty}\sum_{j=q}^{\infty}\sum_{r=i}^{\infty}\sum_{s=j}^{\infty} \Delta^2 a_{r,s}\Big| \le p^m q^n \sum_{r=p}^{\infty}\sum_{s=q}^{\infty} rs|\Delta^2 a_{r,s}|. \tag{2.10}$$

Using (2.2)–(2.10), we find

$$\int\limits_{2mh}^{\pi}\int\limits_{2n\eta}^{\pi} |\mathcal{T}_1|\,dx\,dy \le \sum_{\alpha=1}^{4} \mathcal{P}_\alpha(m,n,h,\eta). \tag{2.11}$$

If $p=[\frac{1}{x}]+1$, $q=[\frac{1}{y}]+1$, we again apply twice the Hardy transformation for $\mathcal{T}_4$ and obtain

$$\begin{aligned}
\mathcal{T}_4 = {} & \sum_{\mu=0}^{m}\sum_{\nu=0}^{n}(-1)^{\mu+\nu}\binom{m}{\mu}\binom{n}{\nu}\Big\{a_{p,q}\mathcal{D}_p[x+(m-2\mu)h]\mathcal{D}_q[y+(n-2\nu)\eta]- \\
& -\mathcal{D}_p[x+(m-2\mu)h]\Delta_2 a_{p,q}(q+1)K_q[y+(n-2\nu)\eta]- \\
& -\mathcal{D}_q[y+(n-2\nu)\eta]\Delta_1 a_{p,q}(p+1)K_p[x+(m-2\mu)h]+ \\
& +\mathcal{D}_p[x+(m-2\mu)h]\sum_{j=q}^{\infty}\Delta_2^2 a_{p,j}(j+1)K_j[y+(n-2\nu)\eta]+ \\
& +\mathcal{D}_q[y+(n-2\nu)\eta]\sum_{i=p}^{\infty}\Delta_1^2 a_{i,q}(i+1)K_i[x+(m-2\mu)h]+ \\
& +\Delta_{12}^2 a_{p,q}(p+1)(q+1)K_p[x+(m-2\mu)h]K_q[y+(n-2\nu)\eta]- \\
& -(p+1)K_p[x+(m-2\mu)h]\sum_{j=q}^{\infty}\Delta_2(\Delta_{12}a_{p,j})(j+1)K_j[y+(n-2\nu)\eta]- \\
& -(q+1)K_q[y+(n-2\nu)\eta]\sum_{i=p}^{\infty}\Delta_1(\Delta_{12}a_{i,q})(i+1)K_i[x+(m-2\mu)h]+ \\
& +\sum_{i=p}^{\infty}\sum_{j=q}^{\infty}\Delta^2 a_{i,j}(i+1)(j+1)K_i[x+(m-2\mu)h]K_j[y+(n-2\nu)\eta]\Big\}= \\
& =\sum_{\alpha=1}^{9}\mathcal{T}_4^{(\alpha)}(m,n,p,q).
\end{aligned} \tag{2.12}$$

By Lemma 2 (see (2.1)) we have

$$\int_{2mh}^{\pi}\int_{2n\eta}^{\pi}|\mathcal{T}_4^{(1)}|\,dx\,dy \le c(m,n)h^m\eta^n \int_1^{\frac{1}{2mh}}\int_1^{\frac{1}{2n\eta}} |a_{[x],[y]}|x^{m-1}y^{n-1}\,dx\,dy \le$$

$$\le c(m,n)h^m\eta^n \sum_{i=1}^{[\frac{1}{2mh}]}\sum_{j=1}^{[\frac{1}{2n\eta}]} i^{m-1}j^{n-1}|a_{i,j}|\,dx\,dy \le$$

$$\le \sum_{s=1}^{4}\mathcal{P}_s(m,n,h,\eta). \tag{2.13}$$

Applying Lemmas 3 and 1 for $\mathcal{T}_4^{(2)}$ we find

$$\int_{2mh}^{\pi}\int_{2n\eta}^{\pi}|\mathcal{T}_4^{(2)}|\,dx\,dy \le c(m,n)h^m\eta^n \int_1^{\frac{1}{2mh}}\int_1^{\frac{1}{2n\eta}} x^{m-1}y^n|\Delta_2 a_{[x]+1,[y]+1}|\,dx\,dy \le$$

$$\le c(m,n)h^m\eta^n \sum_{i=1}^{[\frac{1}{2mh}]}\sum_{j=1}^{[\frac{1}{2n\eta}]} i^{m-1}j^n|\Delta_2 a_{i,j}|.$$

Using the Abel transformation leads to

$$\int_{2mh}^{\pi}\int_{2n\eta}^{\pi}|\mathcal{T}_4^{(2)}|\,dx\,dy \le c(m,n)\Big\{h^m\eta^n \sum_{j=1}^{[\frac{1}{2n\eta}]} j^{n+1}\sum_{i=1}^{[\frac{1}{2mh}]} i^{m-1}|\Delta_2^2 a_{i,j}| +$$

$$+ h^m\eta^{-1}\sum_{i=1}^{[\frac{1}{2mh}]} i^{m-1}|\Delta_2 a_{i,[\frac{1}{2n\eta}]}|\Big\}.$$

Again applying twice the Abel transformation we obtain

$$\int_{2mh}^{\pi}\int_{2n\eta}^{\pi}|\mathcal{T}_4^{(2)}|\,dx\,dy \le c(m,n)\Big\{h^m\eta^n \sum_{i=1}^{[\frac{1}{2mh}]}\sum_{j=1}^{[\frac{1}{2n\eta}]} i^{m+1}j^{n+1}|\Delta^2 a_{i,j}| +$$

$$+\eta^n\sum_{j=1}^{[\frac{1}{2n\eta}]} j^{n+1}h^{-1}\Big|\sum_{i=[\frac{1}{2mh}]}^{\infty}\Delta_1\big(\Delta_1(\Delta_2^2 a_{i,j})\big)\Big| +$$

$$+\eta^n\sum_{j=1}^{[\frac{1}{2n\eta}]} j^{n+1}\Big|\sum_{r=[\frac{1}{2mh}]}^{\infty}\sum_{i=r}^{\infty}\Delta_1\big(\Delta_1(\Delta_2^2 a_{i,j})\big)\Big| +$$

$$+h^m \sum_{i=1}^{[\frac{1}{2mh}]} i^{m+1}\eta^{-1}\Big|\sum_{j=[\frac{1}{2n\eta}]}^{\infty} \Delta^2 a_{i,j}\Big| + \eta^{-1}\Big|\sum_{j=q}^{\infty}\Delta_2^2 a_{[\frac{1}{2mh}],j}\Big|\Big\} \le$$

$$\le c(m,n)\Big\{h^m\eta^n \sum_{i=1}^{[\frac{1}{2mh}]}\sum_{j=1}^{[\frac{1}{2n\eta}]} i^{m+1}j^{n+1}|\Delta^2 a_{i,j}| +$$

$$+\eta^n \sum_{j=1}^{[\frac{1}{2n\eta}]} j^{n+1} \sum_{i=[\frac{1}{2mh}]+1}^{\infty} i|\Delta^2 a_{i,j}| + \eta^n \sum_{j=1}^{[\frac{1}{2n\eta}]} j^{n+1} \sum_{i=[\frac{1}{2mh}]+1}^{\infty} i|\Delta^2 a_{i,j}| +$$

$$+h^m \sum_{i=1}^{[\frac{1}{2mh}]} \sum_{j=[\frac{1}{2n\eta}]+1}^{\infty} i^{m+1}j|\Delta^2 a_{i,j}| + \sum_{i=[\frac{1}{2mh}]+1}^{\infty}\sum_{j=[\frac{1}{2n\eta}]+1}^{\infty} ij|\Delta^2 a_{i,j}|\Big\} \le$$

$$\le \sum_{s=1}^{4} \mathcal{P}_s(m,n,h,\eta). \tag{2.14}$$

One can estimate $\mathcal{T}_4^{(3)}$ quite similarly.

Let us now consider $\mathcal{T}_4^{(4)}$. We have

$$\mathcal{T}_4^{(4)} = \sum_{\mu=0}^{m}(-1)^\mu\binom{m}{\mu}\mathcal{D}_{[\frac{1}{x}]}[x+(m-2\mu)h]\sum_{\nu=0}^{n}(-1)^\nu\binom{n}{\nu}\times$$

$$\times\Bigg(\sum_{j=[\frac{1}{y}]+1}^{[\frac{1}{2n\eta}]} + \sum_{j=[\frac{1}{2n\eta}]+1}^{\infty}\Bigg)(j+1)\Delta_2^2 a_{[\frac{1}{x}],j}K_j[y+(n-2\nu)\eta] =$$

$$= \mathcal{T}_4^{(4)}(1) + \mathcal{T}_4^{(4)}(2). \tag{2.15}$$

Since

$$|\mathcal{T}_4^{(4)}(1)| \le c(m,n)h^m\eta^n x^{-m-1}y^{-2} \sum_{j=[\frac{1}{y}]+1}^{[\frac{1}{2n\eta}]} j^n|\Delta_2^2 a_{[\frac{1}{x}],j}|,$$

we obtain

$$\int_{2mh}^{\pi}\int_{2n\eta}^{\pi} |\mathcal{T}_4^{(4)}(1)|\,dx\,dy \le c(m,n)h^m\eta^n \sum_{i=1}^{[\frac{1}{2mh}]} i^{m-1} \sum_{s=1}^{[\frac{1}{2n\eta}]}\sum_{j=s+1}^{[\frac{1}{2n\eta}]} j^n|\Delta_2^2 a_{i,j}| \le$$

$$\le c(m,n)h^m\eta^n \sum_{j=1}^{[\frac{1}{2n\eta}]} j^{n+1} \sum_{i=1}^{[\frac{1}{2mh}]} i^{m-1}|\Delta_2^2 a_{i,j}|.$$

Repeating the arguments used in estimating $\mathcal{T}_4^{(2)}$, we find

$$\int\limits_{2mh}^{\pi}\int\limits_{2n\eta}^{\pi}|\mathcal{T}_4^{(4)}(1)|\,dx\,dy \le \mathcal{P}_1(m,n,h,\eta)+\mathcal{P}_3(m,n,h,\eta). \tag{2.16}$$

Similarly,

$$\int\limits_{2mh}^{\pi}\int\limits_{2n\eta}^{\pi}|\mathcal{T}_4^{(4)}(2)|\,dx\,dy \le c(m,n)h^m\eta^{-1}\sum_{i=1}^{[\frac{1}{2mh}]} i^{m-1}\sum_{j=[\frac{1}{2n\eta}]+1}^{\infty}|\Delta_2^2 a_{i,j}| \le$$

$$\le \sum_{s=1}^{4}\mathcal{P}_s(m,n,h,\eta). \tag{2.17}$$

By (2.16) and (2.17) we conclude that

$$\int\limits_{2mh}^{\pi}\int\limits_{2n\eta}^{\pi}|\mathcal{T}_4^{(4)}|\,dx\,dy \le \sum_{s=1}^{4}\mathcal{P}_s(m,n,h,\eta). \tag{2.18}$$

Quite similarly we obtain

$$\int\limits_{2mh}^{\pi}\int\limits_{2n\eta}^{\pi}|\mathcal{T}_4^{(5)}|\,dx\,dy \le \sum_{s=1}^{4}\mathcal{P}_s(m,n,h,\eta). \tag{2.19}$$

By Lemma 1 and analysis of the arguments used in estimating I_2 we have

$$\int\limits_{2mh}^{\pi}\int\limits_{2n\eta}^{\pi}|\mathcal{T}_4^{(6)}|\,dx\,dy \le c(m,n)h^m\eta^n\sum_{i=1}^{[\frac{1}{2mh}]}\sum_{j=1}^{[\frac{1}{2n\eta}]} i^m j^n|\Delta_{12}a_{i,j}| \le$$

$$\le c(m,n)h^m\eta^n\Bigg\{\sum_{i=1}^{[\frac{1}{2mh}]}\sum_{j=1}^{[\frac{1}{2n\eta}]} i^{m+1}j^{n+1}|\Delta^2 a_{i,j}| +$$

$$+h^{-m-1}\sum_{j=1}^{[\frac{1}{2n\eta}]} j^{n+1}|\Delta_2(\Delta_{12}a_{[\frac{1}{2mh}],j})| +$$

$$+\eta^{-n-1}\sum_{i=1}^{[\frac{1}{2mh}]} i^{m+1}|\Delta_1(\Delta_{12}a_{i,[\frac{1}{2n\eta}]}| + h^{-m-1}\eta^{-n-1}|\Delta_{12}a_{[\frac{1}{2mh}],[\frac{1}{2n\eta}]}| \le$$

$$\le c(m,n)\Bigg\{h^m\eta^n\sum_{i=1}^{[\frac{1}{2mh}]}\sum_{j=1}^{[\frac{1}{2n\eta}]} i^{m+1}j^{n+1}|\Delta^2 a_{i,j}| +$$

$$+h^m\eta^{-1}\sum_{i=[\frac{1}{2mh}]+1}^{\infty} i^{m+1}|\Delta_1(\Delta_{12}a_{i,[\frac{1}{2n\eta}]})|+$$

$$+h^{-1}\eta^n\sum_{j=1}^{[\frac{1}{2n\eta}]} j^{n+1}|\Delta_2(\Delta_{12}a_{[\frac{1}{2mh}],j})|+$$

$$+(h\eta)^{-1}|\Delta_{12}a_{[\frac{1}{2mh}],\frac{1}{2n\eta}]}| \le\le \sum_{s=1}^{4}\mathcal{P}_s(m,n,h,\eta). \qquad (2.20)$$

For $\mathcal{T}_4^{(8)}$ we obtain

$$\int_{2mh}^{\pi}\int_{2n\eta}^{\pi}|\mathcal{T}_4^{(8)}|\,dx\,dy \le c(m,n)h^m\eta^n\int_1^{\frac{1}{2mh}}\int_1^{\frac{1}{2n\eta}} y^n\sum_{i=[x]+1}^{[\frac{1}{2mh}]} i^m|\Delta_1(\Delta_{12}a_{i,[y]})|dxdy+$$

$$+c(m,n)\eta^n\int_1^{\frac{1}{2n\eta}} y^n\int_{mh}^{\pi+mh} x^{-2}\sum_{i=[\frac{1}{2mh}]+1}^{\infty}|\Delta_1(\Delta_{12}a_{i,[y]})|dxdy\le$$

$$\le c(m,n)\Big\{h^m\eta^n\sum_{r=1}^{[\frac{1}{2mh}]}\sum_{j=1}^{[\frac{1}{2n\eta}]} j^n\sum_{i=r+1}^{[\frac{1}{2mh}]} i^m|\Delta_1(\Delta_{12}a_{i,j})|+$$

$$+\eta^n\sum_{j=1}^{[\frac{1}{2n\eta}]} j^n\sum_{r=1}^{[\frac{1}{mh}]}\sum_{i=[\frac{1}{2mh}]+1}^{\infty}|\Delta_1(\Delta_{12}a_{i,j})|\Big\}\le$$

$$\le c(m,n)\Big\{h^m\eta^n\sum_{j=1}^{[\frac{1}{2n\eta}]} j^n\sum_{r=1}^{[\frac{1}{2mh}]} i^{m+1}|\Delta_1(\Delta_{12}a_{i,j})|+$$

$$+\eta^n\sum_{i=[\frac{1}{2mh}]+1}^{\infty}\sum_{j=1}^{[\frac{1}{2n\eta}]} ij^n|\Delta_1(\Delta_{12}a_{i,j})|\Big\}=$$

$$\le c(m,n)\Big\{h^m\eta^n\sum_{r=1}^{[\frac{1}{2mh}]} i^{m+1}\sum_{j=1}^{[\frac{1}{2n\eta}]} j^n|\Delta_1(\Delta_{12}a_{i,j})|+$$

$$+\eta^n\sum_{i=[\frac{1}{2mh}]+1}^{\infty} i\sum_{j=1}^{[\frac{1}{2n\eta}]} j^n|\Delta_1(\Delta_{12}a_{i,j})|\Big\}.$$

Using the Abel transformation leads to

$$\sum_{j=1}^{[\frac{1}{2n\eta}]} j^n|\Delta_1(\Delta_{12}a_{i,j})| \leq \sum_{j=1}^{[\frac{1}{2n\eta}]} j^{n+1}|\Delta^2 a_{i,j}| + \eta^{-n-1}|\Delta_1(\Delta_{12}a_{i,[\frac{1}{2n\eta}]})|.$$

Hence (see (2.8)) it is easy to show that

$$\int_{2mh}^{\pi}\int_{2n\eta}^{\pi} |\mathcal{T}_4^{(8)}|\,dx\,dy \leq \sum_{\alpha=1}^{4} \mathcal{P}_\alpha(m,n,h,\eta). \tag{2.21}$$

Similarly,

$$\int_{2mh}^{\pi}\int_{2n\eta}^{\pi} |\mathcal{T}_4^{(7)}|\,dx\,dy \leq \sum_{\alpha=1}^{4} \mathcal{P}_\alpha(m,n,h,\eta). \tag{2.22}$$

Further,

$$\mathcal{T}_4^{(9)} = \sum_{\mu=0}^{m}\sum_{\nu=0}^{n}(-1)^{\mu+\nu}\binom{m}{\mu}\binom{n}{\nu}\Bigg\{ \sum_{i=[\frac{1}{x}]+1}^{[\frac{1}{2mh}]}\ \sum_{j=[\frac{1}{y}]+1}^{[\frac{1}{2n\eta}]} + \sum_{i=[\frac{1}{2mh}]+1}^{\infty}\ \sum_{j=[\frac{1}{y}]}^{[\frac{1}{2n\eta}]} +$$
$$+ \sum_{i=[\frac{1}{x}]+1}^{[\frac{1}{2mh}]}\ \sum_{j=[\frac{1}{2n\eta}]+1}^{\infty}\ \sum_{i=[\frac{1}{2mh}]+1}^{\infty}\ \sum_{j=[\frac{1}{2n\eta}]+1}^{\infty}\Bigg\}(i+1)(j+1)\times$$
$$\times K_i[x+(m-2\mu)h]K_j[y+(n-2\nu)\eta] = \sum_{s=1}^{4}\mathcal{T}_4^{(9)}(s). \tag{2.23}$$

By Lemma 2 we have

$$|\mathcal{T}_4^{(9)}(1)| \leq c(m,n)h^m\eta^n \sum_{i=[\frac{1}{x}]+1}^{[\frac{1}{2mh}]}\ \sum_{j=[\frac{1}{y}]+1}^{[\frac{1}{2n\eta}]} i^m j^n (xy)^{-2}|\Delta^2 a_{i,j}|.$$

Therefore

$$\int_{2mh}^{\pi}\int_{2n\eta}^{\pi} |\mathcal{T}_4^{(9)}(1)|dxdy \leq c(m,n)h^m\eta^n \sum_{r=1}^{[\frac{1}{2mh}]}\sum_{s=1}^{[\frac{1}{2n\eta}]}\sum_{i=r}^{[\frac{1}{2mh}]}\sum_{j=s}^{[\frac{1}{2n\eta}]} i^m j^n|\Delta^2 a_{i,j}| \leq$$
$$\leq c(m,n)h^m\eta^n \sum_{i=1}^{[\frac{1}{2mh}]}\sum_{j=1}^{[\frac{1}{2n\eta}]} i^{m+1}j^{n+1}|\Delta^2 a_{i,j}|. \tag{2.24}$$

The remaining terms of (2.22) are estimated by similar arguments. Thus we conclude that

$$\int\limits_{2mh}^{\pi}\int\limits_{2n\eta}^{\pi}|\mathcal{T}_4^{(9)}|\,dx\,dy\le\sum_{\alpha=1}^{4}\mathcal{P}_\alpha(m,n,h,\eta). \tag{2.25}$$

Taking into account (2.13), (2.14), (2.18)–(2.22), and (2.25) we obtain

$$\int\limits_{2mh}^{\pi}\int\limits_{2n\eta}^{\pi}|\mathcal{T}_4|\,dx\,dy\le\sum_{\alpha=1}^{4}\mathcal{P}_\alpha(m,n,h,\eta). \tag{2.26}$$

For $\mathcal{T}_2$ we have

$$|\mathcal{T}_2|\le c(n)\eta^n\sum_{j=1}^{[\frac{1}{y}]}j^n\bigg|\sum_{\mu=0}^{m}(-1)^\mu\binom{m}{\mu}\sum_{i=[\frac{1}{x}]+1}^{\infty}a_{i,j}\cos i[x+(m-2\mu)h]\bigg|.$$

Applying twice the Abel transformation, we find

$$\begin{aligned}&\sum_{\mu=0}^{m}(-1)^\mu\binom{m}{\mu}\sum_{i=[\frac{1}{x}]+1}^{\infty}a_{i,j}\cos i[x+(m-2\mu)h]=\\&=\sum_{\mu=0}^{m}(-1)^\mu\binom{m}{\mu}\bigg\{\sum_{i=[\frac{1}{x}]+1}^{\infty}(i+1)\Delta_1^2a_{i,j}K_i[x+(m-2\mu)h]-\\&-\Delta_1a_{[\frac{1}{x}],j}\Big[\frac{1}{x}\Big]K_{[\frac{1}{x}]}[x+(m-2\mu)h]-a_{[\frac{1}{x}],j}\mathcal{D}_{[\frac{1}{x}]}[x+(m-2\mu)h]\bigg\}.\end{aligned}$$

By virtue of Lemmas 3 and 1 we obtain

$$\begin{aligned}|\mathcal{T}_2|\le c(m,n)h^m\eta^n&\bigg\{\sum_{j=1}^{[\frac{1}{y}]}j^n\bigg(\sum_{i=[\frac{1}{x}]+1}^{[\frac{1}{2mh}]}i^m|\Delta_1^2a_{i,j}|x^{-2}+\\&+x^{-m-2}|\Delta_1a_{[\frac{1}{x}],j}|+|a_{[\frac{1}{x}],j}|x^{-m-1}\bigg)\bigg\}+\\&+c(n)\eta^n\sum_{j=1}^{[\frac{1}{y}]}j^n\bigg(\sum_{i=[\frac{1}{2mh}]}^{\infty}i|\Delta_1^2a_{i,j}|\times\\&\times\Big|\sum_{\mu=0}^{m}(-1)^\mu\binom{m}{\mu}K_i[x+(m-2\mu)h]\Big|.\end{aligned}$$

Hence

$$\sigma_{m,n} \equiv \int\limits_{2mh}^{\pi} \int\limits_{2n\eta}^{\pi} |\mathcal{T}_2| \, dx \, dy \leq$$

$$\leq c(m,n) h^m \eta^n \int\limits_{2mh}^{\pi} x^{-2} \int\limits_{2n\eta}^{\pi} \sum_{j=1}^{[\frac{1}{y}]} j^n \sum_{i=[\frac{1}{x}]+1}^{[\frac{1}{2mh}]} i^m |\Delta_1^2 a_{i,j}| \, dx \, dy +$$

$$+ c(m,n) h^m \eta^n \int\limits_{2mh}^{\pi} x^{-m-2} \int\limits_{2n\eta}^{\pi} \sum_{j=1}^{[\frac{1}{y}]} j^n |\Delta_1 a_{[\frac{1}{x}],j}| \, dx \, dy +$$

$$+ c(m,n) h^m \eta^n \int\limits_{2mh}^{\pi} x^{-m-1} \int\limits_{2n\eta}^{\pi} \sum_{j=1}^{[\frac{1}{y}]} j^n |a_{[\frac{1}{x}],j}| \, dx \, dy +$$

$$+ c(n) \eta^n \int\limits_{2mh}^{\pi} \int\limits_{2n\eta}^{\pi} \sum_{j=1}^{[\frac{1}{y}]} j^n \sum_{i=[\frac{1}{2mh}]+1}^{\infty} i |\Delta_1^2 a_{i,j}| \times$$

$$\times \Big| \sum_{\mu=0}^{m} (-1)^{\mu} \binom{m}{\mu} K_i[x + (m-2\mu)h] \Big| \, dx \, dy \leq$$

$$\leq c(m,n) h^m \eta^n \Bigg\{ \int\limits_{1}^{\frac{1}{2mh}} \int\limits_{1}^{\frac{1}{2n\eta}} y^{-2} \sum_{j=1}^{[y]} j^n \sum_{i=[x]+1}^{[\frac{1}{2mh}]} i^m |\Delta_1^2 a_{i,j}| \, dx \, dy +$$

$$+ \int\limits_{1}^{\frac{1}{2mh}} x^m \int\limits_{1}^{\frac{1}{2n\eta}} y^{-2} \sum_{j=1}^{[y]} j^n |\Delta_1 a_{[x],j}| \, dx \, dy +$$

$$+ \int\limits_{1}^{\frac{1}{2mh}} x^{m-1} \int\limits_{1}^{\frac{1}{2n\eta}} y^{-2} \sum_{j=1}^{[y]} j^n |a_{[x],j}| \, dx \, dy \Bigg\} +$$

$$+ c(m,n) \eta^n \int\limits_{1}^{\frac{1}{2mh}} \int\limits_{1}^{\frac{1}{2n\eta}} y^{-2} \sum_{j=1}^{[y]} j^n \sum_{i=[\frac{1}{2mh}]+1}^{\infty} |\Delta_1^2 a_{i,j}| \, dx \, dy.$$

Next, as is easy to show,

$$\sigma_{m,n} \leq c(m,n) \Bigg\{ h^m \eta^n \sum_{r=1}^{[\frac{1}{2mh}]} \sum_{s=1}^{[\frac{1}{2n\eta}]} s^{-2} \sum_{j=1}^{s} j^n \sum_{i=r+1}^{[\frac{1}{2mh}]} i^m |\Delta_1^2 a_{i,j}| +$$

$$+ h^m\eta^n \sum_{r=1}^{[\frac{1}{2mh}]} r^m \sum_{s=1}^{[\frac{1}{2n\eta}]} s^{-2} \sum_{j=1}^{s} j^n |\Delta_1 a_{r,j}| +$$

$$+ h^m\eta^n \sum_{r=1}^{[\frac{1}{2mh}]} r^{m-1} \sum_{s=1}^{[\frac{1}{2n\eta}]} s^{-2} \sum_{j=1}^{s} j^n |a_{r,j}| +$$

$$+ \eta^n h^{-1} \sum_{s=1}^{[\frac{1}{2n\eta}]} s^{-2} \sum_{j=1}^{s} j^n \sum_{i=[\frac{1}{2mh}]+1}^{\infty} |\Delta_1^2 a_{i,j}| \Big\}.$$

Therefore, after simple calculations, we find

$$\sigma_{m,n} \le c(m,n)h^m\eta^n \Big\{ \sum_{i=1}^{[\frac{1}{2mh}]} i^{m+1} \sum_{j=1}^{[\frac{1}{2n\eta}]} j^{n-1} |\Delta_1^2 a_{i,j}| +$$

$$+ \sum_{r=1}^{[\frac{1}{2mh}]} r^m \sum_{j=1}^{[\frac{1}{2n\eta}]} j^{n-1} |\Delta_1 a_{r,j}| + \sum_{r=1}^{[\frac{1}{2mh}]} \sum_{j=1}^{[\frac{1}{2n\eta}]} r^{m-1} j^{n-1} |a_{r,j}| \Big\} +$$

$$+ c_{m,n}\eta^n \sum_{i=[\frac{1}{2mh}]+1}^{\infty} i \sum_{j=1}^{[\frac{1}{2n\eta}]} j^{n-1} |\Delta_1^2 a_{i,j}|.$$

The analysis of estimates (2.14) and (2.18) gives

$$\int_{2mh}^{\pi} \int_{2n\eta}^{\pi} |\mathcal{T}_2| \, dx \, dy \le \sum_{\alpha=1}^{4} \mathcal{P}_\alpha(m,n,h,\eta). \tag{2.27}$$

Similarly,

$$\int_{2mh}^{\pi} \int_{2n\eta}^{\pi} |\mathcal{T}_3| \, dx \, dy \le \sum_{\alpha=1}^{4} \mathcal{P}_\alpha(m,n,h,\eta). \tag{2.28}$$

Therefore by virtue of (2.11), (2.26), (2.27) and (2.28) we obtain

$$A_4(m,n;f) \le \sum_{\alpha=1}^{4} \mathcal{P}_\alpha(m,n,h,\eta). \tag{2.29}$$

For $A_1(m,n;f)$ we have

$$A_1(m,n;f) = \int_0^{2mh} \int_0^{2n\eta} |\Delta^{m,n} f| \, dx \, dy =$$

$$= \int_0^{2mh}\int_0^{2n\eta}\Bigg|\Bigg(\sum_{i=1}^{r}\sum_{j=1}^{s} + \sum_{i=r+1}^{\infty}\sum_{j=1}^{s} + \sum_{i=1}^{r}\sum_{j=s+1}^{\infty} + $$
$$+ \sum_{i=r+1}^{\infty}\sum_{j=s+1}^{\infty}\Bigg)(i+1)(j+1)\Delta^2 a_{i,j} \times$$
$$\times \sum_{\mu=0}^{m}\sum_{\nu=0}^{n}(-1)^{\mu+\nu}\binom{m}{\mu}\binom{n}{\nu}K_i[x+(m-2\mu)h] \times$$
$$\times K_j[y+(n-2\nu)\eta]\Bigg|dx\,dy = \sum_{\alpha=1}^{4} A_1^{(\alpha)}(m,n,h,\eta), \qquad (2.30)$$

where $r = [\frac{1}{2mh}]$, $s = [\frac{1}{2n\eta}]$. By Lemma 2 we obtain

$$A_1^{(1)} \le c(m,n)h^m\eta^n \int_0^{2mh}\int_0^{2n\eta}\sum_{i=1}^{r}\sum_{j=1}^{s} i^{m+2}j^{n+2}|\Delta^2 a_{i,j}|dx\,dy \le$$
$$\le c(m,n)h^m\eta^n \sum_{i=1}^{[\frac{1}{2mh}]}\sum_{j=1}^{[\frac{1}{2n\eta}]} i^{m+1}j^{n+1}|\Delta^2 a_{i,j}|. \qquad (2.31)$$

Further,

$$A_1^{(4)} = \int_0^{2mh}\int_0^{2n\eta}\Bigg|\sum_{i=r+1}^{\infty}\sum_{j=s+1}^{\infty}(i+1)(j+1)\Delta^2 a_{i,j}\Bigg|\Bigg(\sum_{m>2\mu}\sum_{n>2\nu} +$$
$$+ \sum_{m\le 2\mu}\sum_{n>2\nu} + \sum_{m>2\mu}\sum_{n\le 2\nu} + \sum_{m\le 2\mu}\sum_{n\le 2\nu}\Bigg)(-1)^{\mu+\nu}\binom{m}{\mu}\binom{n}{\nu} \times$$
$$\times K_i[x+(m-2\mu)h]K_j[y+(n-2\nu)\eta]\Bigg|\,dx\,dy = \sum_{\alpha=1}^{4} A_1^{(4)}(\alpha).$$

Obviously, it is sufficient to estimate $A_1^{(4)}(4)$. For the term of this sum $A_{1;\mu,\nu}^{(4)}(4)$, where μ ($[\frac{m}{2}] \le \mu \le m$) and ν ($[\frac{n}{2}] \le \nu \le n$) are fixed, $a = 2\mu - m$, $b = 2\nu - n$, we obtain, after passing to the variables $t = x - ah$, $\tau = y - b\eta$,

$$A_{i;\mu,\nu}^{(4)}(4) \le c(m,n)\sum_{i=[\frac{1}{2mh}]+1}^{\infty}\ \sum_{j=[\frac{1}{2n\eta}]+1}^{\infty}|\Delta^2 a_{i,j}|\sum_{r=1}^{3}\sum_{s=1}^{3}F_{r,s},$$

where

$$F_{r,s} = \int_{u_r}\int_{v_s}(i+1)(j+1)K_i(t)K_j(\tau)\,dt\,d\tau,$$

$$u_1=\Big[-ah,-\frac{1}{i}\Big],\quad u_2=\Big[-\frac{1}{i},\frac{1}{i}\Big],\quad u_3=\Big[\frac{1}{i},(2m-a)h\Big],$$
$$v_1=\Big[-b\eta,-\frac{1}{j}\Big],\quad v_2=\Big[-\frac{1}{j},\frac{1}{j}\Big],\quad v_3=\Big[\frac{1}{j},(2n-b)\eta\Big],$$
$$[-ah,-(2m-a)h]\times[-b\eta,(2n-b)\eta]=\bigcup_{r,s=1}^{3}u_r\times v_s.$$

Taking into account the properties of Fejer kernels in the respective intervals, we have

$$F_{1,1}\le ij\int\limits_{u_1}\int\limits_{v_1}\frac{dt\,d\tau}{ij(t\tau)^2},\quad F_{2,1}\le i^2j^2\,\frac{2}{i}\int\limits_{v_1}\frac{d\tau}{j\tau^2},$$
$$F_{3,1}\le ij\int\limits_{u_3}\int\limits_{v_1}\frac{dt\,d\tau}{ij(t\tau)^2},\quad F_{1,2}\le ij\,\frac{2}{j}\int\limits_{u_1}\frac{dt}{it^2},$$
$$F_{2,2}\le i^2j^2\,\frac{4}{ij},\quad F_{3,2}\le ij^2\,\frac{2}{j}\int\limits_{u_3}\frac{dt}{it^2},$$
$$F_{1,3}\le ij\int\limits_{u_1}\int\limits_{v_3}\frac{dt\,d\tau}{ij(t\tau)^2},\quad F_{2,3}\le i^2j\,\frac{2}{i}\int\limits_{v_3}\frac{d\tau}{j\tau^2},$$
$$F_{3,3}\le ij\int\limits_{u_3}\int\limits_{v_3}\frac{dt\,d\tau}{ij(t\tau)^2}.$$

Therefore

$$A_1^{(4)}(4)\le c(m,n)\sum_{i=[\frac{1}{2mh}]+1}^{\infty}\;\sum_{j=[\frac{1}{2n\eta}]+1}^{\infty}ij|\Delta^2a_{i,j}|. \tag{2.32}$$

$A_1^{(2)}$ and $A_1^{(3)}$ are estimated by the same scheme as used in deriving (2.31) and (2.32). Therefore for A_1 we have

$$A_1\le\sum_{s=1}^{4}\mathcal{P}_s(m,n,h,\eta). \tag{2.33}$$

It remains to consider $A_2(m,n;f)$ (A_3 is estimated similarly). We have

$$A_2=\int\limits_{2mh}^{\pi}\int\limits_{0}^{2n\eta}|\Delta^{m,n}f|\,dx\,dy=\int\limits_{2mh}^{\pi}\int\limits_{0}^{2n\eta}\bigg|\bigg\{\sum_{i=0}^{[\frac{1}{x}]}\sum_{j=0}^{[\frac{1}{2n\eta}]}+\sum_{i=[\frac{1}{r}]+1}^{\infty}\sum_{j=0}^{[\frac{1}{2n\eta}]}+$$
$$+\sum_{i=0}^{[\frac{1}{x}]}\sum_{j=[\frac{1}{2n\eta}]+1}^{\infty}+\sum_{i=[\frac{1}{x}]+1}^{\infty}\sum_{j=[\frac{1}{2n\eta}]+1}^{\infty}\bigg\}(i+1)(j+1)\Delta^2a_{i,j}\times$$

$$\times \sum_{\mu=0}^{m}\sum_{\nu=0}^{n}(-1)^{\mu+\nu}\binom{m}{\mu}\binom{n}{\nu}K_i[x+(m-2\mu)h]\times$$

$$\times K_j[y+(n-2\nu)\eta]\Bigg|\,dx\,dy=\sum_{\alpha=1}^{4}A_2^{(\alpha)}(m,n,h,\eta). \tag{2.34}$$

Again, by Lemma 2 we obtain

$$A_2^{(1)}\le c(m,n)h^m\eta^n\int\limits_{2mh}^{\pi}\int\limits_{0}^{2n\eta}\sum_{i=1}^{[\frac{1}{x}]}\sum_{j=1}^{[\frac{1}{2n\eta}]}i^{m+2}j^{n+2}|\Delta^2 a_{i,j}|dx\,dy\le$$

$$\le c(m,n)h^m\eta^n\int\limits_{1}^{\frac{1}{2mh}}x^{-2}\sum_{i=1}^{[x]}i^{m+2}\sum_{j=1}^{[\frac{1}{2n\eta}]}j^{n+1}|\Delta^2 a_{i,j}|dx\le$$

$$\le c(m,n)h^m\eta^n\sum_{i=1}^{[\frac{1}{2mh}]}\sum_{j=1}^{[\frac{1}{2n\eta}]}i^{m+1}j^{n+1}|\Delta^2 a_{i,j}|. \tag{2.35}$$

For $A_2^{(2)}$ we have

$$A_2^{(2)}=\int\limits_{2mh}^{\pi}\int\limits_{0}^{2n\eta}\Bigg|\Bigg(\sum_{i=[\frac{1}{x}]+1}^{[\frac{1}{2mh}]}+\sum_{i=[\frac{1}{2mh}]+1}^{\infty}\Bigg)\sum_{j=0}^{[\frac{1}{2n\eta}]}(i+1)(j+1)|\Delta^2 a_{i,j}|\times$$

$$\times\sum_{\mu=0}^{m}\sum_{\nu=0}^{n}(-1)^{\mu+\nu}\binom{m}{\mu}\binom{n}{\nu}K_i[x+(m-2\mu)h]\times$$

$$\times K_j[y+(n-2\nu)\eta]\Bigg|\,dx\,dy=A_2^{(2)}(1)+A_2^{(2)}(2). \tag{2.36}$$

Further,

$$A_2^{(2)}(1)\le c(m,n)h^m\eta^n\sum_{j=1}^{[\frac{1}{2n\eta}]}j^{n+1}\int\limits_{1}^{\frac{1}{2mh}}x^{-2}\sum_{i=[x]+1}^{[\frac{1}{2mh}]}i^{m+2}|\Delta^2 a_{i,j}|dx\,dy\le$$

$$\le c(m,n)h^m\eta^n\sum_{j=1}^{[\frac{1}{2n\eta}]}j^{n+1}\sum_{r=1}^{[\frac{1}{2mh}]}r^{-2}\sum_{i=r+1}^{[\frac{1}{2mh}]}i^{m+2}|\Delta^2 a_{i,j}|dx\le$$

$$\le c(m,n)h^m\eta^n\sum_{i=1}^{[\frac{1}{2mh}]}\sum_{j=1}^{[\frac{1}{2n\eta}]}i^{m+1}j^{n+1}|\Delta^2 a_{i,j}|. \tag{2.37}$$

Similarly (see (2.32)) we find

$$A_2^{(2)}(2) \leq c(m,n)\eta^n \sum_{i=[\frac{1}{2mh}]+1}^{\infty} \sum_{j=1}^{[\frac{1}{2n\eta}]} ij^{n+1}|\Delta^2 a_{i,j}|. \tag{2.38}$$

By (2.35)–(2.37) we obtain

$$A_2^{(2)} \leq \mathcal{P}_1(m,n,h,\eta) + \mathcal{P}_3(m,n,h,\eta). \tag{2.39}$$

It is likewise easy to show that

$$A_2^{(3)} \leq \mathcal{P}_2(m,n,h,\eta) + \mathcal{P}_4(m,n,h,\eta), \quad A_2^{(4)} \leq \mathcal{P}_4(m,n,h,\eta). \tag{2.40}$$

Keeping in mind (2.33), (2.34), (2.38)–(2.40), we find

$$A_2 \leq \sum_{s=1}^{4} \mathcal{P}_s(m,n,h,\eta). \tag{2.41}$$

With regard to (2.29), (2.33), and (2.41) we conclude that

$$A \leq \sum_{s=1}^{4} \mathcal{P}_s(m,n,h,\eta). \qquad \square$$

3. Let now $\mathbb{R}^k$ be a k-dimensional Euclidean space of points $\mathbf{x}=(x_1,\dots,x_k)$ with ordinary linear operations and the Euclidean norm $\|\mathbf{x}\|$. The product of two vectors $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^k$ is understood componentwise and so are the inequalities $\mathbf{x} < \mathbf{y}$, $\mathbf{x} \leq \mathbf{y}$. It is assumed that $T^k = [0,2\pi]^k$.

If $\mathbf{n} = (n_1,\dots,n_k)$ is a multi-index with non-negative integral components, $\mathbf{0} = (0,\dots,0)$, $\mathbf{1} = (1,\dots,1)$, and $(a_{\mathbf{n}})_{\mathbf{n}\geq 0}$ is a k-multiple numerical sequence, then

$$\Delta_i a_{\mathbf{n}} = a_{\mathbf{n}} - a_{\mathbf{n}+(\delta_{i1},\dots,\delta_{ik})},$$

where δ_{ij} is the Kronecker symbol, $i,j \in M = \{1,\dots,k\}$.

We introduce the notation

$$\Delta_M a_{\mathbf{n}} \equiv \Delta_{1\cdots k} a_{\mathbf{n}} = \Delta_1(\Delta_2(\cdots(\Delta_k a_{\mathbf{n}})\cdots)),$$

for each B, $\varnothing \neq B \subset M$, $\Delta_B a_{\mathbf{n}}$ is defined similarly, and assume that $\lambda(\mathbf{n})$ is the number of zero coordinates of the vector $\mathbf{n}$ (see [7]). We will also use the notation $B' = M \setminus B$, where $B \subset M$.

Definition 2. A sequence $(a_{\mathbf{n}})_{\mathbf{n}\geq 0}$ will be called quasiconvex if the series

$$\sum_{\mathbf{n}\geq\mathbf{0}} \sum_{B\subset M,\, B\neq\varnothing} \prod_{i\in B} (n_i+1)|\Delta_B(\Delta_M a_{\mathbf{n}})|$$

converges.

If the sequence $(a_{\mathbf{n}})_{\mathbf{n}\geq 0}$ is quasiconvex and

$$\lim_{\|n\|\to\infty} a_{\mathbf{n}} = 0,$$

then the series

$$\sum_{\mathbf{n}\geq\mathbf{0}} 2^{-\lambda(\mathbf{n})} a_{\mathbf{n}} \prod_{i=1}^{k} \cos n_i x_i \qquad (**)$$

converges on $(0, 2\pi)^k$ to some function $f : \mathbb{R}^k \to \mathbb{R}$ summable on T^k and is its Fourier series.

Let $\mathbf{m}$, $\mathbf{i}$, δ, $\mathbf{h} \in \mathbb{R}^k$, where $\mathbf{m}$ and $\mathbf{i}$ have non-negative integral components, while δ has positive components. Consider

$$\omega_{\mathbf{m}}(\delta; f)_1 = \sup_{-\delta\leq\mathbf{h}\leq\delta} \int_{T^k} \Big| \sum_{\mathbf{0}\leq\mathbf{i}\leq\mathbf{m}} (-1)^{\sum_{j=1}^{k} i_j} \prod_{j=1}^{k} \binom{m_j}{i_j} f[\mathbf{x} + (\mathbf{m} - 2\mathbf{i})\mathbf{h}] \Big| \, dx$$

as an integral modulus of smoothness of order $\mathbf{m}$ of the function $f \in L(T^k)$.

The validity of the following analogue of Lemmas 1 and 2 is obvious.

Lemma 4. *Let $\mathbf{p} \in \mathbb{R}^k$ have natural components, $\mathbf{h} > \mathbf{0}$ and $2\mathbf{mh} \leq \mathbf{x} \leq \pi\mathbf{1}$. Then*

$$|T_{\mathbf{p}}^{\mathbf{m}}| = \Big| \sum_{\mathbf{0}\leq\mathbf{i}\leq\mathbf{m}} (-1)^{\sum_{j=1}^{k} i_j} \prod_{j=1}^{k} \Big[\binom{m_j}{i_j} K_{p_j}[x_j + (m_j - 2i_j)h_j] \Big] \Big| \leq$$

$$\leq c(\mathbf{m}) \prod_{j=1}^{k} x_j^{-2} h_j^{m_j} P_j^{m_j-1}, \quad |T_{\mathbf{p}}^{\mathbf{m}}| \leq c(\mathbf{m}) \prod_{j=1}^{k} h_j^{m_j} P_j^{m_j+1}.$$

We have

Theorem 2. *Let a sequence $(a_{\mathbf{n}})_{\mathbf{n}\geq 0}$ be quasiconvex and $\lim_{\|\mathbf{n}\|\to\infty} a_{\mathbf{n}} = 0$. Then for the sum f of the series $(**)$ we have*

$$\omega_1^{(\mathbf{m})}(\delta; f) \leq c(\mathbf{m}) \sum_{B:B\subset M} \Bigg\{ \sum_{\substack{1\leq n_\nu\leq N_\nu \\ \nu\in B}} \sum_{\substack{N_\mu+1\leq n_\mu\leq\infty \\ \mu\in B'}} \times$$

$$\times \prod_{\nu\in B} N_\nu^{-m_\nu} i_\nu^{m_\nu+1} \prod_{\mu\in B'} i_\mu |\Delta_B(\Delta_M a_{\mathbf{n}})| \Bigg\}$$

where $N_\nu = [\frac{1}{\delta_\nu}]$, $\nu \in M$.

(As usual, the empty product is assumed to be zero.)

To prove the theorem note that the Hardy transformation is defined in $\mathbb{R}^k$ for $k > 2$ too and its structure becomes more complicated as the dimension increases. Nevertheless, the estimates from the proof of Theorem 1 can hold for the case $k > 2$ as well.

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