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ON THE MODE-CHANGE PROBLEM FOR RANDOM MEASURES

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ABSTRACT. The classical change-point problem in modern terms, i.e., the mode-change problem, is stated for sufficiently general set-indexed random processes, namely for random measures. A method is shown for solving this problem both in the general form and for the intensity of compound Poisson random measures. The results obtained are novel for the change-point problem, too.

1. INTRODUCTION

1. The goal of this paper is to state the classical change-point problem in modern terms, i.e., to formulate the mode-change problem for sufficiently general set-indexed random processes by which we mean random signed measures with realizations almost surely (a.s.) belonging to $\mathbf{D}(\mathcal{A})$ which is the space of set functions "outer continuous with inner limits" and where $\mathcal{A}$, a domain of set functions, is a family of Borel subsets of a d-dimensional compact ($d \geq 1$) under some entropy condition. An attempt is made to indicate one way of solving this problem in the general form and in a particular case, namely, for the intensity of compound Poisson random measures. The obtained results are novel for the classical change-point problem, too.

When studying various phenomena of nature and social life, one may come across a situation in which a certain observed process is flowing in time or evolving in space, or both flowing in time and evolving in space. In that case one is faced with the problem of understanding the variability character of this process in the sense whether it evolves in one and the same mode or there exists some moment (or several moments or some domain) after which a different mode of behavior sets in, that is to say, after which "the mode changes." For example, when studying chronologically arranged archaeological data collected from a specific locality, one is interested in

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finding out whether any abrupt change in the culture occurred there, and if it did, then after which moment. When studying a literary text, one is concerned with determining the identity of the author of the text or detecting "inserts" of other authors. The problem of determining the moment and place of the change in the seismic conditions in a certain area is an example of identifying the change in the mode both in time and in space.

The statistical "change-set" problem or, as otherwise termed, the "mode-change" problem serves as good model for describing and solving the above-mentioned problems. The mode-change problem consists of two parts:

(i) testing the hypotheses on the existence of a change-set and

(ii) constructing a "sufficiently good," say, consistent estimator for the change-set if the latter exists.

The exact formulation of this problem for random signed measures is given in Section 3 and for the intensity of compound Poisson random measures in Section 4.

Set-indexed empirical processes, partial-sum processes, Brownian processes, Lévy (infinitely divisible) processes, compound Poisson processes, compound point processes, etc., — all these are examples of random signed measures.

Here one of the ways for solving the mode-change problem is indicated using the results of the fundamental papers by Dudley [1–4], Adler et al. [5, 6], and Bass and Pyke [7–9] devoted to the investigation of the set-indexed random processes. These results are stated as preliminaries in the greater part of Section 2.

2. Preliminaries

Let $\mathcal{B}$ be the Borel σ-field formed by subsets of some compact $\Gamma \subset \mathbb{R}^d$, say, $\Gamma = I^d$, where $I^d = [0,1]^d$, $d \geq 1$, and let $\mathcal{A}$ be a subfamily of $\mathcal{B}$, $\mathcal{A} \subset \mathcal{B}$, satisfying the following conditions:

(a) $\mathcal{A}$ is a collection of closed subsets of Γ;

(b) $\mathcal{A}$ is closed with respect to the metric

$$d_L(A,B) = |A \triangle B|, \quad A, B \in \mathcal{A},$$

where $|\cdot|$ denotes the Lebesgue measure, $A \triangle B = (A \cap \overline{B}) \cup (\overline{A} \cap B)$ is the symmetric difference, and $\overline{A} = \Gamma \backslash A$;

(c) for each $\delta > 0$ there is a finite subset $\mathcal{A}_\delta$ (δ-net) of $\mathcal{A}$ such that whenever $A \in \mathcal{A}$ there exists $B \in \mathcal{A}_\delta$ with $A \subset B^0 \subset B \subset A^\delta$, where B^0 is the interior of B with respect to the relative topology on Γ and A^δ is the open δ-neighborhood of A;

(d) there are constants $K > 0$ and $r \in (0,1)$ such that for a sufficiently small $\delta > 0$ we have $H(\delta) \leq K\delta^{-r}$, where $H(\delta) = \ln(\sharp \mathcal{A}_\delta^*)$ is the entropy and $\sharp \mathcal{A}_\delta^*$ is the cardinality of the smallest δ-net $\mathcal{A}_\delta^*$ of $\mathcal{A}$.

The family of closed convex sets in I^d, $d > 1$, for which $r = (d-1)/2$ with $d = 2$, is an example of $\mathcal{A}$ covered by the above conditions (see [2]).

The conditions (a)–(c) imply that $\mathcal{A}$ is a compact separable and complete and, in particular, totally bounded under the metric d_L. The condition (d) will be discussed below (see Remark 2.5).

Definition 2.1. A function $X : \mathcal{A} \to \mathbb{R}$ is said to be outer continuous with inner limits at $A \in \mathcal{A}$ if

(i) $A \subset A_n \in \mathcal{A}$, $d_L(A, A_n) \to 0$ implies $X(A_n) \to X(A)$ as $n \to \infty$;

(ii) $A_n \in \mathcal{A}$, $A_n \subset A^0$, $d_L(A, A_n) \to 0$ implies the existence of $\lim_{n\to\infty} X(A_n)$.

Let $\mathbf{D}(\mathcal{A})=\{X : \mathcal{A} \to \mathbb{R}$ such that X is outer continuous with inner limits at each $A \in \mathcal{A}\}$.

Remark 2.1. The space $\mathbf{D}(\mathcal{A})$ is the intersection of the space $\mathbf{D}(\mathcal{A})$ from [8] and the space of cadlag functions from [6].

By analogy with [8] and [9] the metric on $\mathbf{D}(\mathcal{A})$ is defined by

$$d_{\mathbf{D}}(X, Y) = d_L\big(G(X), G(Y)\big),$$

where $G(X)$ is a graph of the function X, i.e., the closure of the set $\{(A, X(A)), A \in \mathcal{A}\}$ with respect to the metric

$$\rho\big((A_1, r_1), (A_2, r_2)\big) = d_L(A_1, A_2) + |r_1 - r_2|$$

on $\mathcal{A} \times \mathbb{R}$. The metric $d_{\mathbf{D}}$ is a pseudometric on $\mathbf{D}(\mathcal{A})$. We shall identify the functions X and Y for $d_{\mathbf{D}}(X, Y) = 0$. (Note, for example, that for $a \in [-1, 1]$ the functions $X_a(t) = \sin(\frac{1}{t})$, $t \in (0, 1]$, $X_a(0) = a$, are all identified by $d_{\mathbf{D}}$).

Remark 2.2. Since the metric d_L is weaker than the Hausdorff metric d_H considered in [8, 9], the above metric $d_{\mathbf{D}}$ is also weaker than its analog in the same papers.

For the space $\mathbf{D}(\mathcal{A})$ the σ-field $\mathcal{F} = G^{-1}(\mathcal{B}_{\mathcal{G}})$ is induced by the graph function $G(\cdot)$ and the Borel σ-field $\mathcal{B}_{\mathcal{G}}$ generated by the family $\mathcal{G}$ of closed subsets of $\mathcal{A} \times \mathbb{R}$.

A weak convergence of probabilities P_n on $(\mathbf{D}(\mathcal{A}), \mathcal{F})$ to the probability P is denoted by $P_n \xrightarrow{w} P$ and understood in the usual sense: if $\int f dP_n \to \int f dP$ for all functionals f continuous and bounded on $\mathbf{D}(\mathcal{A})$ with respect to the metric $d_{\mathbf{D}}$. If $\{X_n\}_{n\geq 0}$ is a sequence of set-indexed random processes whose sample functions are from the space $\mathbf{D}(\mathcal{A})$ a.s., then X_n is said to converge weakly to X_0 (denoted by $X_n \xrightarrow{w} X_0$) if the distribution laws of X_n converge weakly to those of X_0.

From [8] we have the following proposition:

Proposition 2.1. *$X_n \xrightarrow{w} X_0$ if:*

(i) *$P^{X_n}_{A_1,\dots,A_k}$ that are finite-dimensional distributions of X_n converge to those of X_0 for all finite k and $A_1,\dots,A_k \in \mathcal{A}^*$, where $\mathcal{A}^*$ is a dense subset of $\mathcal{A}$ with respect to d_L;*

(ii) *the sequence $\{X_n\}_{n\geq 1}$ is tight, that is to say, for all $\varepsilon>0$ there exists a compact subset F^ε of $\mathbf{D}(\mathcal{A})$ such that*

$$\inf_n P\{X_n \in F^\varepsilon\} \geq 1-\varepsilon.$$

Definition 2.2. The mapping $X : \mathcal{B}\times\Omega \to \mathbb{R}$ is a random measure (r.m.) if:

(i) $X(A,\cdot)$ is a random variable (r.v.) for each $A\in\mathcal{B}$;

(ii) $X(\cdot,\omega)$ is a finitely additive measure on $\mathcal{B}$ a.s.

Let the restriction of the r.m. X to $\mathcal{A}\subset\mathcal{B}$ be also denoted by X. Define var $(X(A), A\in\mathcal{A})$ as total variation of X with respect to $(\Gamma,\mathcal{B})$.

On account of Theorem 8.2 from [9] we have the following criterion for a sequence of r.m.'s to be tight:

Proposition 2.2. *A sequence of r.m.'s $\{X_n(A), A\in\mathcal{A}\}_{n\geq 1}$ is tight if for any $\varepsilon>0$ there exist positive numbers M_ε, δ_ε, an integer-valued function $N(\delta)$, and a positive-valued function $h(\delta)$ such that the conditions*

$$\sup_n P\big\{X_n \in \overline{F}{}^\varepsilon_k\big\} < \varepsilon/3, \quad k=1,2,3,$$

are satisfied for the following three subsets of $\mathbf{D}(\mathcal{A})$:

$$F_1^\varepsilon = \Big\{X:\ \mathit{var}\,\big(X(A), A\in\mathcal{A}\big) \leq M_\varepsilon\Big\},$$

$$F_2^\varepsilon = \Big\{X:\ \textit{every element of } G(X) \textit{ is within a distance} \leq \delta_\varepsilon$$
$$\textit{from some } (A_i, X(A_i)),\ i=1,\dots,N(\delta_\varepsilon),\ \textit{with respect to } \rho\Big\},$$

$$F_3^\varepsilon = \Big\{X:\ \sup_{A_i\subset B\subset A_i^{h(\delta_\varepsilon)}} |X(B)-X(A_i)| < \delta_\varepsilon,\ i=1,\dots,N(\delta_\varepsilon)\Big\}$$

($A_1,\dots,A_{N(\delta_\varepsilon)} \in \mathcal{A}$ is some δ_ε-net of $\mathcal{A}$ possibly depending on X).

Definition 2.3. A r.m. X is called a Lévy r.m. if:

(i) it has independent increments, i.e., r.v.'s $X(A_1),\dots,X(A_k)$ are independent for pairwise disjoint sets $A_1.\dots,A_k\in\mathcal{B}$;

(ii) it is stochastically continuous, i.e., for any sequence $A_n\in\mathcal{B}$ and all $\varepsilon>0$ we have $P\{|X(A_n)-X(A_0)|>\varepsilon\}\to 0$ whenever $d_L(A_n,A_0)\to 0$ for an arbitrary set $A_0\in\mathcal{B}$.

From [5–7] we have the following two propositions:

Proposition 2.3. *If a Lévy r.m. actually exists, then the logarithm of its characteristic functions can be expressed for each $A \in \mathcal{B}$ by*

$$\ln E\{e^{iuX(A)}\} = \Psi_A^{(1)}(u) + \Psi_A^{(2)}(u), \tag{2.1}$$

where

$$\Psi_A^{(1)}(u) = iuM(A) - (1/2)u^2V(A), \tag{2.2}$$

$$\Psi_A^{(2)}(u) = \int_{|x|<1}(e^{iux} - 1 - iux)\nu(A, dx) + + \int_{|x|\geq 1}(e^{iux} - 1)\nu(A, dx) \tag{2.3}$$

and M is a real-valued function on $\mathcal{B}$ continuous in the metric d_L, V is a finite nonnegative measure on Γ, absolutely continuous with respect to the Lebesgue measure; for all $A \in \mathcal{B}$, $\nu(A, \cdot)$ is a positive measure on $\mathbb{R}$ with $\nu(\Gamma, \{0\}) = 0$ and satisfying

$$\int (x^2 \wedge 1)\nu(\Gamma, dx) < \infty;$$

for every Borel set $B \subset \mathbb{R}$ with a positive distance to $\{0\}$, $\nu(\cdot, B)$ is a nonnegative finite Borel measure on Γ, absolutely continuous with respect to the Lebesgue measure.

Proposition 2.4. *There exists a set-indexed process with independent increments and logarithm of the characteristic function of form* (2.1). *Moreover, this process has a version which is a Lévy r.m. on $\mathcal{B}$.*

Remark 2.3. From (2.1)–(2.3) it is clear that every Lévy r.m. X can be expressed as the sum of the independent Gaussian r.m. X_G and the non-Gaussian Lévy r.m. $X_{\overline{G}}$, i.e., $X = X_G + X_{\overline{G}}$.

Definition 2.4. The Lévy r.m. is P-homogeneous with real numbers μ, $\sigma > 0$ and the Lévy measure $\nu(dx)$ and called the Lévy (P, μ, σ, ν)-r.m. if for each $A \in \mathcal{B}$ the logarithm of its characteristic function (2.1) has the form

$$\ln E\{e^{iuX(A)}\} = P(A)\Big\{iu\mu - \sigma^2u^2/2 + + \int_{|x|<1}(e^{iux} - 1 - iux)\nu(dx) + \int_{|x|\geq 1}(e^{iux} - 1)\nu(dx)\Big\}, \tag{2.4}$$

where P is some probability measure on $\mathcal{B}$, absolutely continuous with respect to the Lebesgue measure.

According to Remark 2.3 the Lévy (P, μ, σ)-r.m. is the Gaussian r.m. if $\nu \equiv 0$ and it is the Brownian P-r.m. if, additionally, $\mu = 0$, $\sigma \neq 0$. If $\nu \not\equiv 0$ and $\mu = \sigma = 0$, then it is the non-Gaussian (P, ν)-r.m.

Definition 2.5. For $\alpha \in (0, 2)$ the non-Gaussian (P, ν)-r.m. is called the Lévy (P, α)-stable r.m. if ν is the stable Lévy measure of the exponent α having the form

$$\nu(dx) = \begin{cases} c_1 x^{-(1+\alpha)} dx, & x > 0, \\ c_2 |x|^{-(1+\alpha)} dx, & x < 0, \end{cases} \tag{2.5}$$

where c_1, c_2 are positive finite constants.

If $c_1 = c_2 = c$, then we say that the Lévy (P, α)-stable r.m. X is symmetric with logarithm of characteristic function having the form

$$\ln E\{e^{iuX(A)}\} = -cP(A)|u|^\alpha, \quad \alpha \in (0, 2). \tag{2.6}$$

If $\alpha \in (1, 2)$, then $EX(A)$ exists, and for the symmetric Lévy (P, α)-stable r.m. $EX(A) \equiv 0$. Note that the converse statement is also valid, that is to say, if $EX(A) \equiv 0$, then the Lévy (P, α)-stable r.m. is symmetric.

If the Lévy measure ν is concentrated at a single point x_0, i.e., if

$$\nu(dx) = \lambda d\delta(x - x_0), \tag{2.7}$$

where $\lambda > 0$ and $\delta(\cdot)$ is the Dirac measure, then the non-Gaussian (P, ν)-r.m. is the Poisson P-r.m. π with the intensity $J(A) = E\pi(A) = \lambda P(A)$.

Let $\{x_n\}_{n \geq 1}$ be a sequence of independent identically distributed (i.i.d.) r.v.'s not depending on the sequence of Poisson P-r.m.'s $\{\pi_n(A), A \in \mathcal{A}\}$, $n \geq 1$, with $\lambda = n$ for each $n = 1, 2, \ldots$. Then

$$K_n(A) = \sum_{i \leq \pi_n(A)} x_i \tag{2.8}$$

is called the compound Poisson P-r.m. Its intensity is equal to

$$J_n(A) = EK_n(A) = naP(A), \tag{2.9}$$

if $Ex_i = a$. Note that the compound Poisson P-r.m. can also be written in the form

$$K_n(A) = \sum_{i \leq \pi_n(\Gamma)} x_i I_A(\xi_i), \tag{2.10}$$

where ξ_i are d-dimensional r.v.'s not depending on x_i and $\pi_n(\Gamma)$ and having the distribution P and $I_A(\xi_i) = I\{\xi_i \in A\}$, where $I\{\cdot\}$ is the indicator function.

Under certain conditions compound Poisson r.m.'s can be approximated by compound point r.m.'s, namely, it is easy to prove

Proposition 2.5. *If the conditions $(a)-(d)$ are applied to the subfamily $\mathcal{A}$ and r.v.'s x_i lie in the domain of normal attraction of the stable law of the index $\alpha \in (1,2)$, then the three normalized r.m.'s*

$$\big(\pi_n(\Gamma)\big)^{-1/\alpha} \sum_{i \le \pi_n(\Gamma)} (x_i - a) I_A(\xi_i),$$

$$n^{-1/\alpha} \sum_{i \le \pi_n(\Gamma)} (x_i - a) I_A(\xi_i), \quad n^{-1/\alpha} \sum_{i \le n} (x_i - a) I_A(\xi_i), \quad A \in \mathcal{A},$$

converge weakly to the same limit as $n \to \infty$.

When the domain of the Lévy (P, μ, σ, ν)-r.m. X is the whole Borel σ-algebra, then its sample functions are not all bounded as a rule. A question arises here how its domain $\mathcal{A} \subset \mathcal{B}$ should be restricted for its sample functions to be bounded and continuous or to be at least elements of the space $\mathbf{D}(\mathcal{A})$ a.s. The following propositions from [1, 2, 7–9] give the answer to this question.

Proposition 2.6. *Let the entropy H of the subfamily $\mathcal{A} \subset \mathcal{B}$ satisfy the condition*

$$\int_0^1 \big(H(x)/x\big)^{1/2} dx < \infty. \tag{2.11}$$

Then the sample functions of the Lévy (P, μ, σ, ν)-r.m. $X(A)$, $A \in \mathcal{A}$, are continuous with respect to the metric d_L a.s. if and only if it is the Gaussian (P, μ, σ)-r.m.

Therefore the sample functions of the non-Gaussian (P, ν)-r.m. are discontinuous a.s.

Remark 2.4. If condition (2.11) is not satisfied for the entropy H of $\mathcal{A}$, then according to [4] sample functions of the Gaussian (P, μ, σ)-r.m. $X(A)$, $A \in \mathcal{A}$, may not even be bounded.

Proposition 2.7. *If the Lévy measure ν is such that*

$$\int \big(|x| \wedge 1\big) \nu(dx) < \infty, \tag{2.12}$$

then the sample functions of the non-Gaussian (P, ν)-r.m. belong to the space $\mathbf{D}(\mathcal{B})$ a.s. (i.e., no entropy condition is required).

Thus according to (2.5) and (2.7) the sample functions of the Lévy (P, α)-stable r.m. $X(A)$ with $\alpha < 1$ and of the Poisson P-r.m. $\pi(A)$, $A \in \mathcal{B}$, are elements of the space $\mathbf{D}(\mathcal{B})$ a.s.

Proposition 2.8. *Let the entropy H of the subfamily $\mathcal{A} \subset \mathcal{B}$ satisfy the condition*

$$\int_0^1 \big(H(x)/x\big)^{1-1/\alpha} dx < \infty, \tag{2.13}$$

where $\alpha \in (1,2)$; then the sample functions of the Lévy (P,α)-stable r.m. $X(A)$, $A \in \mathcal{A}$, belong to the space $\mathbf{D}(\mathcal{A})$ a.s.

Proposition 2.9. *Let there be positive constants K, r such that for the entropy H of the subfamily $\mathcal{A} \subset \mathcal{B}$ the condition*

$$H(x) \leq Kx^{-r} \tag{2.14}$$

is satisfied for sufficiently small $x > 0$. Then the sample functions of the non-Gaussian (P,ν)-r.m. $X(A)$, $A \in \mathcal{A}$, belong to the space $\mathbf{D}(\mathcal{A})$ a.s.

Remark 2.5. If in condition (2.14) we additionally require that $r < 1$, then conditions (2.11) and (2.13), too, will be fulfilled for the entropy H. Thus according to Remarks 2.3 and 2.4 the condition (d) above ensures the belonging of the sample functions of the Lévy (P,μ,σ,ν)-r.m. $X(A)$, $A \in \mathcal{A}$, to the space $\mathbf{D}(\mathcal{A})$ a.s.

Let us assume that in the sequel the subfamily $\mathcal{A} \subset \mathcal{B}$ will always be subjected to the conditions (a)–(d) above. We shall consider the Lévy (P,μ,σ,ν)-r.m. $L(A)$, $A \in \mathcal{A}$, with $EL \equiv 0$. By analogy with Dudley [3] we introduce

Definition 2.6. The r.m.

$$\mathcal{L}(A) = L(A) - P(A)L(\Gamma), \quad A \in \mathcal{A}, \tag{2.15}$$

is called the Lévy (P,μ,σ,ν)-bridge.

If L is the Gaussian $(P,0,\sigma)$-r.m., then $\mathcal{L}$ is the Brownian $(P,\sigma,)$-bridge and denoted by $\mathcal{L}_G$. If L is the non-Gaussian (P,ν)-r.m., then due to the fact that its sample functions are discontinuous a.s. the r.m. (2.15) is called the Lévy broken (P,ν)-bridge and denoted by $\mathcal{L}_b$. Thus, generally speaking, $\mathcal{L} = \mathcal{L}_G + \mathcal{L}_b$, where $\mathcal{L}_G$ and $\mathcal{L}_b$ are independent of each other.

It is clear that $\mathcal{L}$ is a stochastically continuous r.m. but without independent increments. Note that $\mathcal{L}(\varnothing) = \mathcal{L}(\Gamma) = 0$.

One can easily prove the following two propositions:

Proposition 2.10. *Let $\{L_n\}_{n\geq 1}$ be the sequence of Lévy (P,μ,σ,ν)-r.m.'s with $P_n \xrightarrow{w} P_0$, $\mu_n \to \mu_0$, $\sigma_n \to \sigma_0$, $\nu_n \xrightarrow{w} \nu_0$ as $n \to \infty$, and let L_0 be the Lévy $(P_0,\mu_0,\sigma_0,\nu_0)$-r.m.; then $L_n \xrightarrow{w} L_0$.*

Remark 2.6. When $\Gamma = [0,1]$, $\mathcal{A} = \{[0,t], 0 \leq t \leq 1\}$, and P_n is the Lebesgue measure, a similar proposition was proved by Skorokhod [10].

Let $\{X_n\}_{n\geq 1}$ be the sequence of r.m.'s with $EX_n \equiv 0$, and $\{\Delta_n(A) - E\Delta_n(A), A \in \mathcal{A}\}_{n\geq 1}$, be the sequence of Lévy $(P_n, \mu_n, \sigma_n, \nu_n)$-r.m.'s with $P_n \xrightarrow{w} P_0$, $\mu_n \to \mu_0$, $\sigma_n \to \sigma_0$, $\nu_n \xrightarrow{w} \nu_0$, and nonnegative $E\Delta_n$ be of order n^β as $n \to \infty$ $(\beta > 0)$ and $E\Delta_n(\cdot)/E\Delta_n(\Gamma) \xrightarrow{w} Q(\cdot)$, where Q is some probability measure on Γ. These two sequences are independent of each other. Consider the sequence of r.m.'s $\{Y_n\}_{n\geq 1}$ where $Y_n = X_n + \varepsilon_n \Delta_n$ and $\varepsilon_n \searrow 0$ as $n \to \infty$.

Proposition 2.11. *If $X_n \xrightarrow{w} L$, where L is the Lévy (P_0, μ, σ, ν)-r.m. with $EL \equiv 0$ and ε_n is asymptotically equivalent to $\{E\Delta_n(\Gamma)\}^{-1}$, then*

$$X_n(\cdot) - P_n(\cdot)X_n(\Gamma) \xrightarrow{w} \mathcal{L}(\cdot)$$

and

$$Y_n(\cdot) - P_n(\cdot)Y_n(\Gamma) \xrightarrow{w} \mathcal{L}(\cdot) + \mathcal{M}(\cdot),$$

where $\mathcal{L}$ is the Lévy (P_0, μ, σ, ν)-bridge and $\mathcal{M} = Q - P_0$.

3. The Mode-Change Problem in a General Form

Let $\{X_n(A), A \in \mathcal{A}\}_{n\geq 1}$ and $\{\Delta_n(A), A \in \mathcal{A}\}_{n\geq 1}$, be two independent sequences of r.m.'s with sample functions from the space $\mathbf{D}(\mathcal{A})$ a.s. Consider two probability spaces $(\mathbf{D}(\mathcal{A}),\ F,\ P_{kn})$, $k = 0, 1$, where P_{0n} and P_{1n} are induced by X_n and $X_n + \varepsilon_n \Delta_n$, respectively, with $\varepsilon_n \searrow 0$ as $n \to \infty$.

The mode-change problem can be stated in a general form as follows: On account of the realizations of some observed r.m.'s $\widetilde{X}_n(A)$, $A \in \mathcal{A}$, we have:

(i) to test the null hypotheses

$H_{0n}:\ \widetilde{X}_n(A) \stackrel{d}{=} X_n(A)$

against the alternatives

$H_{1n}:$ there exists some change-set $C \in \mathcal{A}$, such that $\widetilde{X}_n(A) \stackrel{d}{=} X_n(A) + \varepsilon_n \Delta_n(A \cap C)$

($\stackrel{d}{=}$ denotes the equality of the distribution laws);

(ii) to construct a consistent estimator $\widehat{C}_n$ for the change-set C.

The consistency of the estimator is understood in the sense that $d_L(\widehat{C}_n, C) \to 0$ in probability as $n \to \infty$. The mode-change problem reduces to the classical change-point one when $\Gamma = [0, 1]$ and $\mathcal{A} = \{[0, t], 0 \leq t \leq 1\}$, $C = (t_0, 1]$, where t_0 is the unknown change-point.

Remark 3.1. One can formulate the mode-change problem with C not necessarily from $\mathcal{A}$, but in that case it is not quite clear how the consistency of the estimator should be determined for the change-set. In such a situation one should evidently take into consideration each time the specific character of the class $\mathcal{A}$. In the classical change-point problem this question is set aside, since here the structure of the class $\mathcal{A}$ does not affect the understanding of the consistency of the estimator for the change-point.

Remark 3.2. The mode-change problem reduces to that of testing the hypotheses when the change-set $C = \Gamma$ and to that of goodness of fit when $C = \varnothing$.

From our point of view, to solve the mode-change problem we have to study the asymptotic behavior of an object such that it will be possible not only to distinguish between the hypotheses H_{0n} and H_{1n} but also to construct a consistent estimator for the change-set C.

In the case of the existence of $E_0 X_n = \lambda_n \not\equiv 0$ and $E_1 \Delta_n = \delta_n \not\equiv 0$, where E_k denotes the mathematical expectation when the hypotheses H_{kn}, $k = 0, 1$, are valid, we think that such an object is the observed r.m. $\widetilde{X}_n$ which is normalized in an appropriate manner and is centered by the conditional expectation, i.e., we have

$$T_n(A) = n^{-\gamma}\Big\{\widetilde{X}_n(A) - E_0\big[\widetilde{X}_n(A)\big|\widetilde{X}_n(\Gamma)\big]\Big\}, \quad A \in \mathcal{A}, \quad \gamma > 0. \tag{3.1}$$

Then for our purpose it appears sufficient to center by the linear regression $a_n(A)\widetilde{X}_n(\Gamma)$ where the set-function $a_n(A)$ is determined from the condition

$$E_0\big[a_n(A)\widetilde{X}_n(\Gamma)\big] = E_0\Big[E_0\big[\widetilde{X}_n(A)\big|\widetilde{X}_n(\Gamma)\big]\Big].$$

Therefore

$$a_n(A) = \lambda_n(A)/\lambda_n(\Gamma),$$

and instead of the r.m. T_n from (3.1) we can consider

$$Z_n(A) = n^{-\gamma}\Big\{\widetilde{X}_n(A) - \big(\lambda_n(A)/\lambda_n(\Gamma)\big)\widetilde{X}_n(\Gamma)\Big\}, \quad A \in \mathcal{A}. \tag{3.2}$$

Obviously, $E_0 Z_n \equiv 0$ and $E_1 Z_n(A) = n^{-\gamma}\varepsilon_n \delta_n(C)\mathcal{M}_n(A)$, where $\mathcal{M}_n(A) = \delta_n(A \cap C)/\delta_n(C) - \lambda_n(A)/\lambda_n(\Gamma)$.

If δ_n and λ_n are nonnegative or nonpositive measures on $\mathcal{A}$ and for the normalized measures $P_n(A) = \lambda_n(A)/\lambda_n(\Gamma)$ and $Q_n(A \cap C) = \delta_n(A \cap C)/\delta_n(C)$ the condition

$$Q_n(A \cap C) > P_n(A \cap C)$$

is fulfilled for all $A \in \mathcal{A}$ for which $A \cap C \neq \varnothing$, then the set-function $\mathcal{M}_n(A) = Q_n(A \cap C) - P_n(A)$ has the unique maximum which is attained only on the change-set C. Indeed, for all $A \neq C$ we have

$$\begin{aligned} &\mathcal{M}_n(C) - \mathcal{M}_n(A) = \\ &= Q_n\big(C \backslash (A \cap C)\big) - P_n\big(C \backslash (A \cap C)\big) + P_n(A \cap \overline{C}) > 0. \end{aligned} \tag{3.3}$$

By the same argument it is clear that $\mathcal{M}_n$ attains its minimum only on the complement to the change-set C and $\mathcal{M}_n(C) = -\mathcal{M}_n(\overline{C})$. Therefore the form of $E_1 Z_n(A)$ is the key to constructing a consistent estimator for the change-set C.

Remark 3.3. From the above reasoning it is clear why the observed r.m. $\widetilde{X}_n$ should not be centered by $E_0\widetilde{X}_n$. Though for the r.m.

$$Y_n = n^{-\gamma}\big(\widetilde{X}_n - E_0\widetilde{X}_n\big) \tag{3.4}$$

we have $E_0 Y_n \equiv 0$ and $E_1 Y_n(A) = n^{-\gamma}\varepsilon_n\delta_n(C)Q_n(A\cap C)$, the change-set C is not the unique set of attaining the maximum for $Q_n(A\cap C)$. Any set $A \in \mathcal{A}$ containing C is also a set of attaining the maximum.

Let $E_0 X_n = \lambda_n$ and $E_1\Delta_n = \delta_n$ exist, λ_n and δ_n being nonnegative or nonpositive measures on $\mathcal{A}$, both being of order n^β ($\beta > 0$) as $n\to\infty$. The normalized measures $P_n \xrightarrow{w} P$, $Q_n \xrightarrow{w} Q$ where the probability measures P and Q are absolutely continuous with respect to the Lebesgue measure and

$$Q(A\cap C) > P(A\cap C) \tag{3.5}$$

for all $A\in\mathcal{A}$ for which $A\cap C \neq \varnothing$.

Further assume that the sequence of r.m.'s $\{n^{-\gamma}(X_n-\lambda_n)\}_{n\geq 1}$ ($0<\gamma<\beta$) satisfies the conditions of Proposition 2.2 and for all finite $k\geq 1$ and $A_1,\dots,A_k\in\mathcal{A}^*$ ($\mathcal{A}^*$ is a dense subset of $\mathcal{A}$) their finite-dimensional distributions $P_n^{A_1,\dots,A_k}$ converge to those of some Lévy (P,μ,σ,ν)-r.m. L with $EL\equiv 0$. The r.m.'s $n^{-\gamma}(\Delta_n-\delta_n)$ are the Lévy $(P_n,\mu_n,\sigma_n,\nu_n)$-r.m.'s with $P_n\xrightarrow{w} P$, $\mu_n\to\mu_0$, $\sigma_n\to\sigma_0$, $\nu_n\xrightarrow{w}\nu_0$, as $n\to\infty$, where $\sigma_0>0$, μ_0 are real numbers and ν_0 is some Lévy measure.

Under the above assumptions one way of solving the mode-change problem is via the following two theorems:

Theorem 3.1. *If* $n^{-\gamma}\varepsilon_n|\delta_n(C)|\to 1$ *as* $n\to\infty$, *then under the hypotheses* H_{0n}*:*

$$Y_n \xrightarrow{w} L,\quad Z_n\xrightarrow{w}\mathcal{L}$$

and under the alternatives H_{1n}*:*

$$Y_n \xrightarrow{w} L+m,\quad Z_n\xrightarrow{w}\mathcal{L}+\mathcal{M}\quad \textit{in the case } \delta_n>0$$

and

$$Y_n \xrightarrow{w} L-m,\quad Z_n\xrightarrow{w}\mathcal{L}-\mathcal{M}\quad \textit{in the case } \delta_n<0,$$

where $\mathcal{L}$ *is the Lévy* (P,μ,σ,ν)*-bridge and*

$$m(A) = Q(A\cap C),\quad \mathcal{M}(A) = Q(A\cap C)-P(A),\quad A\in\mathcal{A}.$$

Theorem 3.2. *If $n^{\beta-\gamma}\varepsilon_n \to \infty$ as $n \to \infty$, then the estimators*

$$\widehat{C}_n = \begin{cases} \arg\max_{A\in\mathcal{A}} Z_n(A) & \text{if } \delta_n > 0, \\ \arg\min_{A\in\mathcal{A}} Z_n(A) & \text{if } \delta_n < 0 \end{cases}$$

and

$$C'_n = \arg\max_{A\in\mathcal{A}} |Z_n(A)| \quad (\text{if the sign of } \delta_n \text{ is unknown})$$

are consistent estimators for the change-set C.

Example. Consider the classical change-point problem:

Let $x_1, \dots, x_n$ be independent observations on the r.v. ξ distributed with the density $\widetilde{f}(x)$. On account of these observations we have:

(i) to test the null hypothesis

$H_0 :\ \widetilde{f}(x) = f(x)$

against the alternative:

H_1 : there exists a change-point $1 < r_0 < n$ or $t_0 = r_0/n$ such that $x_1, \dots, x_{r_0}$ are observations on the r.v. distributed with the density $\widetilde{f}(x) = f(x)$ while $x_{r_0+1}, \dots, x_n$ are observations on the r.v. distributed with the density $\widetilde{f}(x) = f_n(x) = f(x) + \varepsilon_n h_n(x)$ where $\varepsilon_n \downarrow 0$ and $h_n \to h_0$ in some sense as $n \to \infty$;

(ii) to construct a consistent estimator for the change-point if the latter exists.

This problem can be reduced to the mode-change problem for random measures provided that we take into account the following reasoning:

The probability density of the sum $\xi + \varepsilon_n \Delta_n$ of independent r.v'.s can be written in the form $f + \varepsilon_n \mathbf{A} g_n$, where

$$\mathbf{A}g(x) = -\int \left[y \int_0^1 f'(x - \varepsilon_n y t) dt \right] g(y) dy,$$

f and g_n are the probability densities of ξ and Δ_n, respectively.

When g runs through a closed set $\mathbf{G}$ of probability densities with finite first moments, the functions $\mathbf{A}g$ will run through the closed family $\mathbf{H}^* \subset \mathbf{H}$, where $\mathbf{H}$ is a family of functions h with $\int h(x)dx = 0$ and finite $\int xh(x)dx$. It is obvious that $h_n \in \mathbf{H}$.

For any given h_n let us consider

$$\mathbf{H}_n^* = \left\{ h :\ h \in \mathbf{H}^*,\ \int xh(x)dx = \int xh_n(x)dx \right\},$$

and find the function h_n^* such that $\varrho(h_n, h_n^*) = \inf_{h\in\mathbf{H}_n^*} \varrho(h_n, h)$, where ϱ is the metric in whose sense the convergence $h_n \to h_0$ is considered.

Instead of $f_n = f + \varepsilon_n h_n$ we can consider $f_n^* = f + \varepsilon_n h_n^*$, where $h_n^* \to h_0^*$ in the metric ϱ and f_n^* is the convolution of f with some $g_n \in \mathbf{G}$, which is a quasisolution for the following Fredholm integral equation of the first kind:

$$\mathbf{A}g = h_n. \tag{3.6}$$

However, from the asymptoptic viewpoint there is no difference whether we consider f_n or f_n^* as an alternative.

In this example $\Gamma = [0,1]$, $\mathcal{A} = [0,t]$, $0 \leq t \leq 1$, and $\mathbf{D}(\mathcal{A})$ is in fact the Skorokhod space $\mathbf{D}_{[0,1]}$. (In $\mathbf{D}(\mathcal{A})$ the metric $d_{\mathbf{D}}$ is stronger than the Skorokhod metric.) Using the observations $x_1, \dots, x_n$ let us construct the random measures, i.e., the processes

$$\widetilde{X}_n(t) = \widetilde{X}_n([0,t]) = \sum_{i \leq nt} x_i, \quad t \in [0,1].$$

On account of these processes we have:

(i) to test the null hypotheses

$H_{0n}:\ \widetilde{X}_n([0,t]) \overset{d}{=} X_n([0,t])$

against the alternatives

$H_{1n}:$ there exists some change-set $(t_0, 1]$ such that

$$\widetilde{X}_n([0,t]) \overset{d}{=} X_n([0,t]) + \varepsilon_n \Delta_n'\big([0,t] \cap (t_0,1]\big),$$

where

$$X_n(t) = X_n([0,t]) = \sum_{i \leq nt} \xi_i, \quad \Delta_n'\big([0,t] \cap (t_0,1]\big) = \sum_{nt_0 < i \leq n} \Delta_n^{(i)},$$

where $\xi_1, \dots, \xi_n, \Delta_n^{(1)}, \dots, \Delta_n^{(n)}$ is the sequence of independent r.v.'s with $\xi_1, \dots, \xi_n$ being distributed identically with the density f and $\Delta_n^{(1)}, \dots, \Delta_n^{(n)}$ being distributed identically with the density g_n, which is a quasisolution of equation (3.6);

(ii) to construct a consistent estimator for the change-set provided that the latter exists.

If $\int x f(x)dx = a \neq 0$, $\int x h_n(x)dx = b_n$, $n = 0, 1, 2, \dots$, and $b_n \to b_0 \neq 0$ as $n \to \infty$, then

$$\begin{aligned} \lambda_n\big([0,t]\big) &= nat, \\ \delta_n\big([0,t] \cap C\big) &= nb_n(t - t_0) I\{t > t_0\}, \quad t \in [0,1], \end{aligned}$$

and

$$\delta_n(C) = nb_n(1 - t_0).$$

Therefore

$$P_n([0,t]) = P([0,t]) = t, \quad P([0,t] \cap C) = I\{t > t_0\}(t - t_0),$$
$$Q_n([0,t] \cap C) = Q([0,t] \cap C) = I\{t > t_0\}(t - t_0)/(1 - t_0).$$

Since $t_0 \in (0,1)$, condition (3.5) is fulfilled automatically. All other required conditions are also fulfilled here.

If the variances of r.v.'s x_1 and $\Delta_n^{(1)}$ are finite, then $\gamma = 1/2$, and processes (3.4) and (3.2) take, respectively, the form

$$Y_n^{(1)}(t) = n^{-1/2} \sum_{i \le nt} (x_i - a),$$
$$Z_n^{(1)}(t) = n^{-1/2} \Big(\sum_{i \le nt} x_i - t \sum_{i \le n} x_i \Big) = n^{-1/2} \sum_{l \le nt} (x_i - \overline{x}_n),$$

where $\overline{x}_n = n^{-1} \sum_{i \le n} x_i$.

If $\varepsilon_n = O(n^{-1/2})$, then by Theorem 3.1, under the hypotheses H_{0n},

$$Y_n^{(1)} \xrightarrow{w} L_G, \quad Z_n^{(1)} \xrightarrow{w} \mathcal{L}_G$$

and, under the alternatives H_{1n},

$$Y_n^{(1)} \xrightarrow{w} L_G + b_0 m, \quad Z_n^{(1)} \xrightarrow{w} \mathcal{L}_G + b_0 \mathcal{M},$$

where L_G is the Wiener process, $\mathcal{L}_G$ is the Brownian bridge, and

$$m(t) = (t_0 - t) I\{t > t_0\},$$
$$\mathcal{M}(t) = I\{t > t_0\}(t - t_0)/(1 - t_0) - t, \quad t \in [0,1].$$

The function $\mathcal{M}$ has the unique minimum only at the change-point t_0, while the maximum of the function m is attained all over the interval $[0, t_0]$. (Note that $\mathcal{M}(t_0) = \mathcal{M}([0, t_0].)$ The set $[0, t_0]$ is a complement to the change-set $(t_0, 1]$, and on which the function $\mathcal{M}$ has its minimum.)

If $n^{1/2} \varepsilon_n \to \infty$, then by Theorem 3.2 the estimators

$$\widehat{t}_n = \begin{cases} \arg\max_{t \in [0,1]} Z_n^{(1)}(t) & \text{if } \ b_0 < 0, \\ \arg\min_{t \in [0,1]} Z_n^{(1)}(t) & \text{if } \ b_0 > 0 \end{cases}$$

and

$$t'_n = \arg\max_{t \in [0,1]} |Z_n^{(1)}(t)|$$

are the consistent estimators for the change-point t_0.

Remark 3.4. The statistics based on $Y_n^{(1)}$ and $Z_n^{(1)}$ are discussed in [11].

If the variances of r.v.'s x_1 and $\Delta_n^{(1)}$ are infinite and these r.v.'s lie in the domain of normal attraction of the stable law of an index $\alpha \in (1,2)$, then $\gamma = 1/\alpha$ and processes (3.4) and (3.2) take, respectively, the form

$$Y_n^{(2)}(t) = n^{-1/\alpha} \sum_{i \le nt} (x_i - a),$$
$$Z_n^{(2)}(t) = n^{-1/\alpha} \sum_{i \le nt} (x_i - \overline{x}_n), \quad t \in [0,1].$$

Here if $\varepsilon_n = O(n^{-(1-1/\alpha)})$, then according to Theorem 3.1, under the hypotheses H_{0n},

$$Y_n^{(2)} \xrightarrow{w} L_\alpha, \quad Z_n^{(2)} \xrightarrow{w} \mathcal{L}_b$$

and, under the alternatives H_{1n},

$$Y_n^{(2)} \xrightarrow{w} L_\alpha + b_0 m, \quad Z_n^{(2)} \xrightarrow{w} \mathcal{L}_b + b_0 \mathcal{M},$$

where L_α is the symmetric Lévy (P,α)-stable random process with P being the Lebesgue measure on $[0,1]$, $\mathcal{L}_b$ is the broken bridge corresponding to L_α, and the functions m and $\mathcal{M}$ are as above.

If $n^{1-1/\alpha}\varepsilon_n \to \infty$, then by Theorem 3.2 the consistent estimators have the same form as $\widehat{t_n}$ and t_n'.

Remark 3.5. These last results are novel for the change-point problem, too.

Proof of Theorem 3.1. Under the hypotheses H_{0n}

$$Y_n(A) \overset{d}{=} n^{-\gamma}\{X_n(A) - \lambda_n(A)\},$$
$$Z_n(A) \overset{d}{=} n^{-\gamma}\{\big(X_n(A) - \lambda_n(A)\big) - P_n(A)\big(X_n(\Gamma) - \lambda_n(\Gamma)\big)\}$$

and under the alternatives H_{1n}

$$\begin{aligned} Y_n(A) &\overset{d}{=} n^{-\gamma}\{X_n(A) - \lambda_n(A)\} + \varepsilon_n n^{-\gamma}\delta_n(C) Q_n(A \cap C), \\ Z_n(A) &\overset{d}{=} n^{-\gamma}\{\big(X_n(A) - \lambda_n(A)\big) - P_n(A)\big(X_n(\Gamma) - \lambda_n(\Gamma)\big)\} + \\ &\quad + \varepsilon_n n^{-\gamma}\{\big(\Delta_n(A \cap C) - \delta_n(A \cap C)\big) - P_n(A)\big(\Delta_n(C) - \delta_n(C)\big)\} + \\ &\quad + \varepsilon_n n^{-\gamma}\delta_n(C)\big(Q_n(A \cap C) - P_n(A)\big). \end{aligned}$$

Both under the hypotheses H_{0n} and under the alternatives H_{1n} the weak convergence of Y_n and Z_n follows from Propositions 2.1, 2.2, 2.10, 2.11, and by Remark 2.5 we have

$$\varepsilon_n n^{-\gamma}\{\big(\Delta_n(A \cap C) - \delta_n(A \cap C)\big) - P_n(A)\big(\Delta_n(C) - \delta_n(C)\big)\} \to 0$$

in probability uniformly on $\mathcal{A}$. $\square$

Proof of Theorem 3.2. Assume that $\delta_n > 0$. If $n^{\beta-\gamma}\varepsilon_n \to \infty$, then because of the fact that the Lévy bridge $\mathcal{L}$ is bounded a.s. we have

$$Z_n/\varepsilon_n n^{\gamma}\delta_n(C) \xrightarrow{w} \mathcal{M},$$

$\mathcal{M}$ is a continuous set-function on $\mathcal{A}$ and due to condition (3.5) it is proved by analogy with (3.3) that $\mathcal{M}$ has the unique maximum at the change-set C. Therefore by the fact that then $\arg\max_{A\in\mathcal{A}}(Z_n(A)/\varepsilon_n n^{-\gamma}\delta_n(C))$ is a continuous functional with respect to the metric $d_{\mathbf{D}}$ in the space $\mathbf{D}(\mathcal{A})$ it turns out that $d_L(\widehat{C}_n, C) \to 0$ in probability. For $\delta_n < 0$ the proof is similar. Since $\mathcal{M}$ has the unique minimum only on the complement to the change-set C and $\mathcal{M}(C) = -\mathcal{M}(\overline{C})$ with $\overline{C} \notin \mathcal{A}$, there follows the consistency of the estimator C_n'. □

4. The Mode-Change Problem for the Intensity of Compound Poisson Random Measures

Due to (2.8) and (2.9) the mode-change problem in terms of the mean values for compound Poisson P-r.m.'s can be stated as follows:

On the basis of realizations of the observed compound Poisson P_n-r.m.'s $\widetilde{K}_n(A)$, $A \in \mathcal{A}$, we have

(i) to test the null hypotheses

H_{0n}' : $P_n(A) = P_0(A)$, $A \in \mathcal{A}$,

against the alternatives

H_{1n}' : there exists some change-set $C \in \mathcal{A}$ such that

$$P_n(A) = P_0(A) + \varepsilon_n\delta_{1n}(A\cap C) + o(\varepsilon_n)\delta_{2n}(A\cap C),$$

where $\varepsilon_n \searrow 0$ as $n \to \infty$ and $\{\delta_{kn}\}_{m\geq 1}$, $k = 1, 2$ are the sequences of measures on $\mathcal{A}$, which are absolutely continuous with respect to the Lebesgue measure. The measures δ_{1n} are nonnegative or nonpositive and have a weak limit δ_1, while δ_{2n} are bounded;

(ii) to construct a consistent estimator $\widehat{C}_n$ for the change-set C.

Remark 4.1. Keeping in mind the representation of compound Poisson P_n-r.m.'s in form (2.10), the foregoing statement of the mode-change problem concerns only the change occurring in the distribution of r.v.'s ξ_i.

Let the r.v.'s x_i in the representation of the observed compound Poisson P_n-r.m.'s in form (2.8) or (2.10) be in the domain of normal attraction of the stable law of an index $\alpha \in (1, 2)$ with $Ex_i = a \neq 0$. Consider the objects

$$Y_n' = \big(\pi_n(\Gamma)\big)^{-1/\alpha}\big(\widetilde{K}_n - E_0\widetilde{K}_n\big) \tag{4.1}$$

and

$$Z_n'(A) = \big(\pi_n(\Gamma)\big)^{-1/\alpha}\big\{\widetilde{K}_n(A) - P_0(A)\widetilde{K}_n(\Gamma)\big\}, \quad A \in \mathcal{A}. \tag{4.2}$$

According to (2.9) we have

$$E_0\widetilde{K}_n(A) = J_{0n}(A) = naP_0(A),$$
$$E_1\widetilde{K}_n(A) = J_{1n}(A) = na\big[P_0(A) + \varepsilon_n\delta_1(A\cap C) + o(\varepsilon_n)\delta_{2n}(A\cap C)\big].$$

Hence $\gamma = 1/\alpha$ and $\beta = 1$ (compare with the general formulation in Section 3). Since we are interested in the asymptotic behavior of r.m.'s (4.1) and (4.2), by Proposition 2.5 it is sufficient to investigate the asymptotic behavior of the r.m.'s

$$Y_n''(A) = n^{-1/\alpha}\Big[\sum_{i\le n} x_i I_A(\xi_i) - naP_0(A)\Big] \tag{4.3}$$

and

$$Z_n''(A) = n^{-1/\alpha}\Big[\sum_{i\le n} x_i I_A(\xi_i) - P_0(A)\sum_{i\le n} x_i\Big]. \tag{4.4}$$

These equalities can be rewritten as

$$Y_n'' = L_n + n^{1/2-1/\alpha}aV_n \tag{4.5}$$

and

$$Z_n''(A) = Y_n''(A) - P_0(A)Y_n''(\Gamma), \tag{4.6}$$

where

$$L_n(A) = n^{-1/\alpha}\sum_{i\le n}(x_i - a)I_A(\xi_i),$$
$$V_n(A) = n^{-1/2}\sum_{i\le n}\big[I_A(\xi_i) - P_0(A)\big].$$

Since the subfamily $\mathcal{A}$ satisfies the conditions (a)–(d), according to [3], under the hypotheses H_{0n}', we have $V_n \xrightarrow{w} \mathcal{L}_G$, where $\mathcal{L}_G$ is the Brownian P_0-bridge. Because of the fact that $\mathcal{L}_G$ has bounded realizations a.s.

$$an^{1/2-1/\alpha}\sup_{A\in\mathcal{A}}|V_n(A)| \to 0 \quad \text{in probability}. \tag{4.7}$$

If $\varepsilon_n n^{1-1/\alpha}|a\delta_{1n}(C)| \to 1$, then under the alternatives H_{1n}', $n^{1/2-1/\alpha}aV_n$ converges in probability to m' or $-m'$ depending on whether $a\delta_{1n}(C)$ is greater or smaller than 0, where the set-function $m'(A) = \delta_1(A\cap C)/\delta_1(C)$.

Both under the hypotheses H_{0n}' and under the alternatives H_{1n}'

$$L_n \xrightarrow{w} L_\alpha, \tag{4.8}$$

where L_α is the Lévy (P_0,α)-stable symmetric r.m. This fact is proved using Proposition 2.1. The convergence of finite-dimensional distributions of L_n to those of L_α and the tightness of the sequence $\{L_n\}_{n\ge1}$ both under the hypotheses H'_{0n} and under the alternatives H'_{1n} is proved quite similarly to the proof of Proposition 5.1 in [8] and to the proof of the tightness of the sequence of random processes from Example 6.3 in [9], respectively. The only difference consists in that instead of the Lebesgue measure of sets $A \in \mathcal{A}$ we should consider the measure P_0 under the hypotheses H'_{0n} and the measure $P_n(A) = P_0(A) + \varepsilon_n\delta_1(A \cap C) + o(\varepsilon_n)\delta_{2n}(A \cap C)$ under the alternatives H'_{1n}.

Thus by (4.5)–(4.8) under the hypotheses H'_{0n} we have

$$Y''_n \xrightarrow{w} L_\alpha, \quad Z''_n \xrightarrow{w} \mathcal{L}_b$$

and therefore by Proposition 2.6

$$Y'_n \xrightarrow{w} L_\alpha, \quad Z'_n \xrightarrow{w} \mathcal{L}_b,$$

where $\mathcal{L}_b$ is the Lévy broken (P_0, α)-bridge.

Under the alternatives H'_{1n} we have

$$Y''_n \xrightarrow{w} L_\alpha + m', \quad Z''_n \xrightarrow{w} \mathcal{L}_b + \mathcal{M}' \quad \text{for} \quad a\delta_1 > 0$$

and

$$Y''_n \xrightarrow{w} L_\alpha - m', \quad Z''_n \xrightarrow{w} \mathcal{L}_b - \mathcal{M}' \quad \text{for} \quad a\delta_1 < 0,$$

where the set-function $\mathcal{M}'(A) = \delta_1(A \cap C)/\delta_1(C) - P_0(A)$. Therefore by Proposition 2.5 the r.m.'s Y'_n and Z'_n will have the same asymptotic behavior as Y''_n and Z''_n, respectively.

If

$$\delta_1(A \cap C)/\delta_1(C) > P_0(A) \tag{4.9}$$

for all $A \in \mathcal{A}$ with $A \cap C \neq \varnothing$, then, as shown above, the unique maximum of $\mathcal{M}'$ is attained only on the change-set C.

Thus, in common with Theorems 3.1 and 3.2, the following two theorems are valid.

Theorem 4.1. *If $\varepsilon_n n^{1-1/\alpha}|a\delta_{1n}(C)| \to 1$ for $\alpha \in (1,2)$ as $n \to \infty$, then under the hypotheses H'_{0n}*

$$Y'_n \xrightarrow{w} L_\alpha, \quad Z'_n \xrightarrow{w} \mathcal{L}_b$$

and under the alternatives H'_{1n}

$$Y'_n \xrightarrow{w} L_\alpha + m', \quad Z'_n \xrightarrow{w} \mathcal{L}_b + \mathcal{M}' \quad \textit{for} \quad a\delta_1 > 0$$

and

$$Y'_n \xrightarrow{w} L_\alpha - m', \quad Z'_n \xrightarrow{w} \mathcal{L}_b - \mathcal{M}' \quad \textit{for} \quad a\delta_1 < 0.$$

Theorem 4.2. *If $n^{1-1/\alpha}\varepsilon_n \to \infty$ for $\alpha \in (1,2)$ as $n \to \infty$, then by condition* (4.9) *the estimators*

$$\widehat{C}_n = \begin{cases} \arg\max_{A\in\mathcal{A}} Z_n'(A) & \text{if } a\delta_1 > 0, \\ \arg\min_{A\in\mathcal{A}} Z_n'(A) & \text{if } a\delta_1 < 0 \end{cases}$$

and

$$C_n' = \arg\max_{A\in\mathcal{A}} |Z_n'(A)| \quad (\textit{if the sign of } a\delta_1 \ \textit{ is unknown})$$

are consistent estimators for the change-set C.

Remark 4.2. If P_0 is unknown, we should use the processes

$$\widehat{Z}_n(A) = \pi_n(\Gamma))^{-1/\alpha}\Big[\sum_{i\le\pi_n(\Gamma)} x_i I_A(\xi) - \widehat{P}_n(A)\sum_{i\le\pi_n(\Gamma)} x_i\Big],$$

where $\widehat{P}_n(A) = \pi_n(A)/\pi_n(\Gamma)$. The processes $\widehat{Z}_n$ have the same asymptotic behavior as the processes $\widehat{Z}_n$. Thus, the estimator

$$C_n'' = \arg\max_{A\in\mathcal{A}} |\widehat{Z}_n(A)|$$

will also be consistent.

Remark 4.3. If x_i are degenerate r.v.'s, in particular, $x_i \equiv 1$, then $\widetilde{K}_n(A)$ are Poisson P_n-r.m.'s $\widetilde{\pi}_n(A)$. The objects

$$n^{-1/2}\big[\widetilde{\pi}_n(A) - nP_0(A)\big] \ \text{ and } \ n^{-1/2}\big[\widetilde{\pi}_n(A) - P_0(A)\widetilde{\pi}_n(\Gamma)\big], \ \ A\in\mathcal{A},$$

weakly converge under the hypotheses H_{0n}' to the Brownian P_0-r.m. L_G and to the Brownian P_0-bridge $\mathcal{L}_G$, respectively, and under the alternatives H_{1n}' to $L_G \pm m'$ and $\mathcal{L}_G \pm \mathcal{M}'$, respectively.

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References

1. R. M. Dudley, Sample functions of the Gaussian process. *Ann. Probab.* **1**(1973), 66–103.

2. R. M. Dudley, Metric entropy of some classes of sets with differential boundaries. *J. Approx. Theory* **10**(1974), 227–236.

3. R. M. Dudley, Central limit theorems for empirical measures. *Ann. Probab.* **6**(1978), 899–929.

4. R. M. Dudley, Lower layers in $\mathbb{R}^2$ and convex sets in $\mathbb{R}^3$ are not GB classes. *Lecture Notes in Math.* **709**(1979), 97–102.

5. R. Y. Adler, D. Monrad, R. H. Scissors, and R. Y. Wilson, Representations, decompositions, and sample function continuity of random fields with independent increments. *Stochastic Process. Appl.* **15**(1983), 3–30.

6. R. Y. Adler and P. D. Feigin, On the cadlaguity of random measures. *Ann. Probab.* **12**(1984), No. 2, 615–630.

7. R. F. Bass and R. Pyke, The existence of set-indexed Lévy processes. *Z. Wahrsch. verw. Gebiete* **66**(1984), 157–172.

8. R. F. Bass and R. Pyke, The space $\mathbf{D}(\mathcal{A})$ and weak convergence for set-indexed processes. *Ann. Probab.* **13**(1985), No. 3, 860–884.

9. R. F. Bass and R. Pyke, A central limit theorem for $\mathbf{D}(\mathcal{A})$-valued processes. *Stochastic Process. Appl.* **24**(1987), 109–131.

10. A. V. Skorokhod, Limit theorems for stochastic processes with independent increments. *Theor. Probab. Appl.* **2**(1957), 138–171.

11. S. Zacks, Survey of classical and Bayesian approaches to the changepoint problem: Fixed sample and sequential procedures of testing and estimation. *Recent advances in statistics. Papers in honor of Herman Chernoff on his sixtieth birthday,* 245–269, *Academic Press, New-York,* 1983.

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