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SOLUTION OF TWO-WEIGHT PROBLEMS FOR INTEGRAL TRANSFORMS WITH POSITIVE KERNELS

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ABSTRACT. Criteria for weak and strong two-weighted inequalities are obtained for integral transforms with positive kernels.

INTRODUCTION

In this paper we derive solutions of two-weight problems for integral transforms with a positive kernel. For weak type problems these transforms are assumed to be defined on general spaces with measure and a given quasi-metric, while a strong type problem is solved in the case of homogeneous type spaces. Similar problems have been investigated for some particular cases in [1]–[17].

Let (X, d, μ) be a topological space with complete measure μ and a given quasi-metric d, i.e., with a non-negative function $d : X \times X \to R^1$ satisfying the following conditions:

(1) $d(x, x) = 0$ for any $x \in X$;

(2) $d(x, y) > 0$ for any x and y from X;

(3) there exists a constant a_0 such that $d(x, y) \le a_0 d(y, x)$ for any x and y from X;

(4) there exists a constant a_1 such that $d(x, y) \le a_1(d(x, z) + d(z, y))$ for any x, y and z from X;

(5) the class of continuous functions with a compact support is dense everywhere in $L^1(X, \mu)$;

(6) it is assumed that each ball $B(x, r)$ is measurable and there exists a constant $b > 0$ such that $\mu B(x, 2r) \le b\mu B(x, r)$ for an arbitrary ball $B(x, r)$, i.e., the measure μ satisfies the doubling condition.

Spaces with conditions (1)–(6) are called homogeneous type spaces [18], [19].

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A locally integrable everywhere positive function $w : X \to R^1$ is called a weight function (a weight). For a μ-measurable set E we set

$$wE = \int_E w(x)d\mu.$$

The paper is organized as follows: In §1 the criteria of weak type two-weighted inequalities are established for integral transforms with a positive kernel defined on homogeneous type spaces. In §2 a simple procedure is described for obtaining the corresponding results for the upper space using the theorems formulated in the preceding section.

§3 is devoted to the solution of strong type two-weight problems for integral transforms with a positive kernel for homogeneous type spaces. The main idea underlying the solution of such problems consists in developing the method and conditions proposed in [1], [5], where similar problems were solved in Euclidean spaces using the familiar covering lemma of Besicovich which, as is known (see, for instance, [11]), does not hold in general in homogeneous type spaces. Nevertheless for the general case we have succeeded in finding criteria of two-weighted inequalities which are as simple and elegant as the ones given in [1], [5] for Euclidean spaces.

Note that results similar to ours were previously obtained in [11], [12] (see also [16]) only for spaces having a group structure.

1. A Criterion of a Weak Type Two-Weighted Inequality for integral Transforms with a Positive Kernel

In this section it will no longer be automatically assumed that μ satisfies the doubling condition. We derive a necessary and sufficient condition guaranteeing the validity of a weak type two-weighted inequality for the operator

$$T(f)(x) = \sup_{t\geq 0} \Big| \int_X k(x,y,t)f(y)d\mu \Big|,$$

where $k : X \times X \times [0,\infty) \to R^1$ is a positive measurable kernel.

For the general case we have the folowing sufficient criterion of a weak type two-weighted inequality.

Theorem 1.1. *Let $1 < p < q < \infty$, v and μ be arbitrary locally finite measures on X so that $\mu\{x\} = 0$ for any $x \in X$.*

If the condition

$$c_0 = \sup \big(vB(x,2N_0 r)\big)^{\frac{1}{q}} \times \\ \times \bigg(\int_{X\setminus B(x,r)} k^{p'}(x,y,t)w^{1-p'}(y)d\mu \bigg)^{\frac{1}{p'}} < \infty, \tag{1.1}$$

where the supremum is taken over all $t > 0$ *and balls* $B(x,r)$ *with* $\mu B(x,r) > 0$, $N_0 = a_1(1+2a_0)$, *the constants* a_0 *and* a_1 *are from the definition of a quasi-metric, is fulfilled, then there exists a constant* $c > 0$ *such that the inequality*

$$v\big\{x \in X : \, T(f)(x) > \lambda\big\} \le c\lambda^{-q}\bigg(\int\limits_X |f(x)|^p w(x)d\mu\bigg)^{\frac{q}{p}} \tag{1.2}$$

holds for any μ*-measurable nonnegative function* $f : X \to R^1$ *and arbitrary* $\lambda > 0$.

Before we proceed to proving this theorem, we will give a the familiar covering lemma.

Lemma A ([18], Lemma 2). *Let* $\mathcal{F}$ *be a family* $\{B(x,r)\}$ *of balls with bounded radii. Then there is a countable subfamily* $\{B(x_i,r_i)\}$ *consisting of pairwise disjoint balls such that each ball in* $\mathcal{F}$ *is contained within one of the balls* $B(x_i, ar_i)$, *where* $a = 3a_1^2 + 2a_0a_1$. *The constants* a_0, a_1 *are from the definition of the space* (X,d,μ).

This lemma holds for general spaces in the sense discussed at the beginning of the section.

Proof of Theorem 1.1. Let f be an arbitrary nonnegative function from $L^p(X, wd\mu)$ and $\lambda > 0$. Without loss of generality we assume that

$$(vX)^{-\frac{1}{q}}\bigg(\int\limits_X f^p(y)w(y)d\mu\bigg)^{\frac{1}{p}} < \frac{\lambda}{2c_0}\,, \tag{1.3}$$

since otherwise we will have $vX < \infty$ and

$$v\big\{x \in X : \, T(f)(x) > \lambda\big\} \le vX \le (2c_0)^q\lambda^{-q}\bigg(\int\limits_X f^p(x)w(x)d\mu\bigg)^{\frac{q}{p}}$$

and the theorem will be proved.

Let $x \in E_\lambda = \{x \in X : T(f)(x) > \lambda\}$. Then there exists $r > 0$ depending on x such that

$$\big(vB(x,N_0r)\big)^{-\frac{1}{q}}\bigg(\int\limits_X f^p(y)w(y)d\mu\bigg)^{\frac{1}{p}} \ge \frac{\lambda}{2c_0} \tag{1.4}$$

and

$$\big(vB(x,2N_0r)\big)^{-\frac{1}{q}}\bigg(\int\limits_X f^p(y)w(y)d\mu\bigg)^{\frac{1}{p}} < \frac{\lambda}{2c_0}\,. \tag{1.5}$$

One can easily verify that there exists $r > 0$ for which (1.5) is fulfilled. However (1.5) cannot hold for any $r > 0$, since in that case we will have

$$\int\limits_{X\setminus B(x,r)} f(y)k(x,y,t)d\mu \leq$$

$$\leq \Bigg(\int\limits_{X\setminus B(x,r)} f^p(y)w(y)d\mu\Bigg)^{\frac{1}{p}} \Bigg(\int\limits_{X\setminus B(x,r)} w^{1-p'}(y)k^{p'}(x,y,t)d\mu\Bigg)^{\frac{1}{p'}} \leq$$

$$\leq c_0\Bigg(\int\limits_X f^p(y)w(y)d\mu\Bigg)^{\frac{1}{p}} \big(vB(x,2N_0r)\big)^{-\frac{1}{q}} < \frac{\lambda}{2}\,.$$

If in the latter inequality we pass to the limit as $r \to 0$, then by virtue of the condition $\mu\{x\} = 0$ for arbitrary $x \in X$ we will obtain

$$T(f)(x) \leq \frac{\lambda}{2} < \lambda,$$

which contradicts the condition $x \in E_\lambda$.

The above arguments imply in particular that there exists $r > 0$ for which (1.4) holds. If we consider an exact upper bound of such numbers r, then we will find $r > 0$ for which both (1.4) and (1.5) are fulfilled.

For such numbers r we obviously have

$$\int\limits_{X\setminus B(x,r)} f(y)k(x,y,t)d\mu \leq c_0\Bigg(\int\limits_X f^p(y)w(y)d\mu\Bigg)^{\frac{1}{p}} \big(vB(x,2N_0r)\big)^{-\frac{1}{q}} < \frac{\lambda}{2}\,.$$

Hence for $x \in E_\lambda$ and the corresponding $r > 0$ we obtain

$$\sup_{t\geq 0} \int\limits_{B(x,r)} f(y)k(x,y,t)d\mu \geq \frac{\lambda}{2}\,. \tag{1.6}$$

Let us now construct a sequence $(r_k)_{k\geq 0}$ as follows: set $r_0 = r$ and choose r_k $(k = 1, 2, \dots)$ such that

$$vB(x, N_0r_k) \leq 2^{-k}vB(x, N_0r_0) \leq vB(x, 2N_0r_k). \tag{1.7}$$

Each r_k from r_{k-1} can be obtained by dividing the latter by half as many times as required. The sequence $(r_k)_{k\geq 0}$ thus chosen will be decreasing and tending to zero by virtue of the condition $\mu\{x\} = 0$.

Using condition (1.1), inequality (1.6), and the chain of inequalities (1.7) we obtain

$$\frac{\lambda}{2} \leq \sup_{t\geq 0} \int\limits_{B(x,r)} f(y)k(x,y,t)d\mu =$$

$$= \sup_{t\geq 0} \sum_{k=0}^{\infty} \int\limits_{B(x,r_k)\setminus B(x,r_{k+1})} f(y)k(x,y,t)d\mu \leq$$

$$\leq \sup_{t\geq 0} \sum_{k=0}^{\infty} \bigg(\int\limits_{B(x,r_k)} f^p(y)w(y)d\mu \bigg)^{\frac{1}{p}} \bigg(\int\limits_{X\setminus B(x,r_{k+1})} w^{1-p'}(y)k^{p'}(x,y,t)d\mu \bigg)^{\frac{1}{p'}} \leq$$

$$\leq c_0 \sum_{k=0}^{\infty} \bigg(\int\limits_{B(x,r_k)} f^p(y)w(y)d\mu \bigg)^{\frac{1}{p}} \big(vB(x,2N_0 r_{k+1})\big)^{-\frac{1}{q}} \leq$$

$$\leq c \sum_{k=0}^{\infty} \big(vB(x,N_0 r_k)\big)^{\frac{1}{p}-\frac{1}{q}} \bigg(\int\limits_{B(x,r_k)} f^p(y)w(y)d\mu \bigg)^{\frac{1}{p}} \big(vB(x,N_0 r_k)\big)^{-\frac{1}{p}} \leq$$

$$\leq c \sum_{k=0}^{\infty} 2^{-k(\frac{1}{p}-\frac{1}{q})} \big(vB(x,N_0 r_0)\big)^{\frac{1}{p}-\frac{1}{q}} \times$$

$$\times \bigg(\int\limits_{B(x,r_k)} f^p(y)w(y)d\mu \bigg)^{\frac{1}{p}} \big(vB(x,N_0 r_k)\big)^{-\frac{1}{p}}.$$

Since it is assumed that $1 < p < q < \infty$, we have

$$c_{pq} = \sum_{k=0}^{\infty} 2^{-k(\frac{1}{p}-\frac{1}{q})} < \infty.$$

The latter chain of inequalities implies

$$c_{pq}^{-1} \sum_{k=0}^{\infty} 2^{-k(\frac{1}{p}-\frac{1}{q})} \frac{\lambda}{2} \leq c \sum_{k=0}^{\infty} 2^{-k(\frac{1}{p}-\frac{1}{q})} \big(vB(x,N_0 r_0)\big)^{\frac{1}{p}-\frac{1}{q}} \times$$

$$\times \bigg(\int\limits_{B(x,r_k)} f^p(y)w(y)d\mu \bigg)^{\frac{1}{p}} \big(vB(x,N_0 r_k)\big)^{-\frac{1}{p}}.$$

Hence we conclude that there exist n_0 and $c > 0$ such that

$$\lambda \leq c\big(vB(x,N_0 r_0)\big)^{\frac{1}{p}-\frac{1}{q}} \bigg(\int\limits_{B(x,r_{n_0})} f^p(y)w(y)d\mu \bigg)^{\frac{1}{p}} \big(vB(x,N_0 r_{n_0})\big)^{-\frac{1}{p}}.$$

Taking into account (1.4) in the latter inequality we obtain

$$\lambda \leq c\lambda^{-\frac{q}{p}+1} \bigg(\int\limits_{X} f^p(y)w(y)d\mu \bigg)^{\frac{1}{p}(\frac{q}{p}-1)} \times$$

$$\times\Bigg(\int\limits_{B(x,r_{n_0})} f^p(y)w(y)d\mu\Bigg)^{\frac{1}{p}}\big(vB(x,N_0r_{n_0})\big)^{-\frac{1}{p}}.$$

which implies

$$vB(x,N_0r_{n_0})\le c\lambda^{-q}\Bigg(\int\limits_{B(x,r_{n_0})} f^p(y)w(y)d\mu\Bigg)\Bigg(\int\limits_X f^p(y)w(y)d\mu\Bigg)^{\frac{q}{p}-1}.$$

To summarize the obtained results, we conclude that for each $x\in E_\lambda$ there exists a ball B_x centered at the point x such that

$$v(N_0B_x)\le c\lambda^{-q}\int\limits_{B_x} f^p(y)w(y)d\mu\Bigg(\int\limits_X f^p(y)w(y)d\mu\Bigg)^{\frac{q}{p}-1} \tag{1.8}$$

Since inequality (1.8) is fulfilled for any $x\in E_\lambda$, the family E_λ covers the set $\{B_x\}_{x\in E_\lambda}$. Let B_0 be an arbitrary ball in X. Due to Lemma A we can choose from the family $\{B_x\}$ a sequence of nonintersecting balls $(B_k)_k$ such that

$$E_\lambda\cap B_0\subset\bigcup_{k\ge1}^{\infty} N_0B_k.$$

Therefore by virtue of (1.8) we obtain

$$v\big(E_\lambda\cap B_0\big)\le\sum_k v(N_0B_k)\le$$

$$\le c\lambda^{-q}\sum_{k\ge1}\Bigg(\int\limits_{B_k} f^p(x)w(x)dx\Bigg)\Bigg(\int\limits_X f^p(x)w(x)d\mu\Bigg)^{\frac{q}{p}-1}\le$$

$$\le c\lambda^{-q}\int\limits_X f^p(x)w(x)d\mu\Bigg(\int\limits_X f^p(x)w(x)d\mu\Bigg)^{\frac{q}{p}-1}.$$

Thus

$$v\big(E_\lambda\cap B_0\big)\le c\lambda^{-q}\Bigg(\int\limits_X f^p(x)w(x)d\mu\Bigg)^{\frac{q}{p}},$$

where the constant c does not depend on B_0, λ, and f. If in the latter inequality we pass to the limit as $\operatorname{rad}B_0\to\infty$, we will find that ineequality (1.2) is valid. $\square$

Theorem 1.2. *Let $1<p<q<\infty$, $\mu\{x\}=0$ for arbitrary $x\in X$. If there exists a constant c_1 such that*

$$k(a,y,t)\le c_1k(x,y,t) \tag{1.9}$$

for arbitrary $t \geq 0$, $a \in X$, $y \in X \backslash B(a,r)$, $r > 0$ and $x \in B(a, 2N_0 r)$, then conditions (1.1) *and* (1.2) *are equivalent.*

Proof. The implication (1.1) $\Rightarrow$ (1.2) follows from Theorem 1.1 without condition (1.9).

Let us prove the implication (1.2) $\Rightarrow$ (1.1). First it will be shown that for any $x \in X$, $r > 0$ and $t \geq 0$ we have

$$\int\limits_{X \backslash B(x,r)} k^{p'}(x,y,t) w^{1-p'}(y) d\mu < \infty. \tag{1.10}$$

Assume the contrary. Let for some $a \in X$, $r > 0$ and $t_0 \geq 0$

$$\int\limits_{X \backslash B(a,r)} k^{p'}(a,y,t_0) w^{1-p'}(y) d\mu = \infty.$$

Then there exists nonnegative $g : X \to R^1$ such that

$$\int\limits_{X \backslash B(a,r)} g^p(y) w(y) dy \leq 1 \tag{1.11}$$

and

$$\int\limits_{X \backslash B(a,r)} g(y) k(a,y,t_0) d\mu = +\infty.$$

On the other hand, by condition (1.9) we have

$$T(g)(x) \geq \int\limits_{X \backslash B(a,r)} g(y) k(x,y,t_0) d\mu \geq c' \int\limits_{X \backslash B(a,r)} g(y) k(a,y,t_0) d\mu = +\infty.$$

for arbitrary $x \in B(a,r)$. Therefore

$$B(a,r) \subset \big\{x \in X : T(g)(x) > \lambda\big\}$$

for arbitrary $\lambda > 0$. Thus by (1.2) we obtain

$$vB(a,r) \leq v\big\{x \in X : T(g)(x) > \lambda\big\} \leq c\lambda^{-q}.$$

Recalling that λ is an arbitrary positive number, from the latter inequality we conclude that $vB(a,r) = 0$, which is impossible. Therefore (1.10) holds. Now we can proceed to proving directly the implication (1.2) $\Rightarrow$ (1.1).

Let $B(x,r)$ be an arbitrary ball and $z \in B(x, 2N_0 r)$. By condition (1.9) we have

$$T(f)(z) \geq \int\limits_{X \backslash B(x,r)} f(y) k(z,y,t) d\mu \geq \frac{1}{c_1} \int\limits_{X \backslash B(x,r)} f(y) k(x,y,t) d\mu, \quad t \geq 0,$$

for an arbitrary nonnegative function $f : X \to R^1$. Therefore from (1.2) we derive the inequality

$$vB(x,2N_0r) \le v\Big\{z \in X : T(f)(z) > \frac{1}{2c_1} \int\limits_{X\setminus B(x,r)} f(y)k(x,y,t)d\mu\Big\} \le$$

$$\le (2c_1)^q c\Big(\int\limits_{X\setminus B(x,r)} f(y)k(x,y,t)d\mu\Big)^{-q}\Big(\int\limits_X f^p(y)w(y)d\mu\Big)^{\frac{q}{p}},$$

where the constants on the right-hand side do not depend on $x \in X$, $r > 0$, and $t \ge 0$. If in the latter inequality we write f as

$$f(y) = \chi_{X\setminus B(x,r)}(y)w^{1-p'}(y)k^{p'-1}(x,y,t),$$

this will give us

$$vB(x,2N_0r) \le c\Big(\int\limits_{X\setminus B(x,r)} w^{1-p'}(y)k^{p'}(x,y,t)d\mu\Big)^{-\frac{q}{p'}},$$

where the constant does not depend on $x \in X$, $r > 0$, and $t \ge 0$. The latter implies that condition (1.1) is fulfilled. □

In the theorems proved above our consideration is limited to spaces for which $\mu\{x\} = 0$ for any $x \in X$. Below we will treat a more general case.

Theorem 1.3. *Let $1 < p < q < \infty$. It is assumed that the condition*

$$\sup_{\substack{t>0\\ x\in X\\ \mu\{x\}>0}} \big(v\{x\}\big)^{\frac{1}{q}} k(x,y,t)w^{-\frac{1}{p}}(x)\big(\mu\{x\}\big)^{\frac{1}{p'}} < \infty \tag{1.12}$$

is fulfilled along with (1.2). *Then the conclusion of Theorem* 1.1 *is valid, i.e.,* (1.2) *holds.*

Proof. After analyzing Theorem 1 we find that in the general case inequality (1.8) holds for any $x \in E_\lambda$ for which (1.4) and (1.5) are fulfilled simultaneously. For the case $\mu\{x\} > 0$ inequality (1.5) may hold for arbitrary $r > 0$. Nevertheless it will be shown below that inequality (1.8) remains valid for the general case too. Let $x \in E_\lambda$, $\mu\{x\} > 0$ and $f(x)\mu\{x\}\sup_{t\ge0} k(x,x,t) < \frac{\lambda}{2}$. Since then we have

$$\lim_{r\to0} \int\limits_{B(x,r)} f(y)k(x,y,t)d\mu = f(x)k(x,x,t)\mu\{x\},$$

for each $t \geq 0$ there will exist r_t such that

$$\int\limits_{B(x,r_t)} f(y)k(x,y,t)d\mu < \frac{\lambda}{2}\,. \tag{1.13}$$

Let us show that in that case inequlity (1.5) cannot be fulfilled for arbitrary $r > 0$. Indeed, if this is so, then by virtue of (1.1), (1.5), and (1.13) we will have

$$\begin{aligned}
T(f)(x) &\leq \sup_{t\geq 0} \int\limits_{B(x,r_t)} f(y)k(x,y,t)d\mu + \sup_{t\geq 0} \int\limits_{X\setminus B(x,r_t)} f(y)k(x,y,t)d\mu \leq \\
&\leq \sup_{t\geq 0} \int\limits_{B(x,r_t)} f(y)k(x,y,t)d\mu + \\
&+ \sup_{t\geq 0} \bigg(\int\limits_{X\setminus B(x,r_t)} k^{p'}(x,y,t)w^{1-p'}(y)d\mu \bigg)^{\frac{1}{p'}} \bigg(\int\limits_X f^p(y)w(y)d\mu \bigg)^{\frac{1}{p}} \leq \\
&\leq \sup_{t\geq 0} \int\limits_{B(x,r_t)} f(y)k(x,y,t)d\mu + \\
&+ c_0 \sup_{t\geq 0} \big(vB(x,2N_0r_t)\big)^{-\frac{1}{q}} \bigg(\int\limits_X f^p(y)w(y)d\mu \bigg)^{\frac{1}{p}} \leq \\
&\leq \sup_{t\geq 0} \int\limits_{B(x,r_t)} f(y)k(x,y,t)d\mu + \frac{\lambda}{2} \leq \lambda
\end{aligned}$$

which contradicts $x \in E_\lambda$. The contradiction obtained shows that in the case under consideration (1.5) cannot be fulfilled for any $r > 0$ and therefore, in common with the proof of Theorem 1.1, we can find $r > 0$ such that (1.4) and (1.5) will be fulfilled simultaneously, which fact leads to (1.8).

Assuming now that $x \in E_\lambda$, $\mu\{x\} > 0$ and $f(x)\mu\{x\}\sup_{t\geq 0} k(x,y,t) \geq \frac{\lambda}{2}$, we obtain

$$\begin{aligned}
\lambda \leq 2f(x)\mu\{x\}\sup_{t\geq 0} k(x,x,t) &= 2f(x)\big(v\{x\}\big)^{-\frac{1}{q}}\big(w(x)\big)^{\frac{1}{p}}\big(\mu\{x\}\big)^{\frac{1}{p}}\big(v\{x\}\big)^{\frac{1}{q}} \times \\
&\times \sup_{t\geq 0} k(x,x,t)\big(w(x)\big)^{-\frac{1}{p}}\big(\mu\{x\}\big)^{\frac{1}{p'}}.
\end{aligned}$$

By virtue of condition (1.12) the latter inequality gives rise to

$$\big(v\{x\}\big)^{\frac{1}{q}} \leq c\lambda^{-1} f(x)\big(w(x)\big)^{\frac{1}{p}}\big(\mu\{x\}\big)^{\frac{1}{p}}. \tag{1.14}$$

Since $\mu\{x\} > 0$, the point x is isolated. Therefore there exists $r > 0$ such that $B(x, N_0 r) = \{x\}$. Now (1.14) can be rewritten as

$$\big(v(N_0 B_x)\big)^{\frac{1}{q}} \le c\lambda^{-1}\bigg(\int\limits_{B_x} f(y)w(y)d\mu\bigg)^{\frac{1}{p}} \le$$
$$\le c\lambda^{-1}\bigg(\int\limits_{B_x} f^p(y)w(y)d\mu\bigg)^{\frac{1}{q}}\bigg(\int\limits_{X} f^p(y)w(y)d\mu\bigg)^{\frac{1}{q}(\frac{q}{p}-1)}$$

which implies that (1.8) is valid. Thus for almost all $x \in E_\lambda$, $\mu\{x\} > 0$ there exists a ball B_x centered at the point x such that (1.8) is fulfilled. In common with the proof of Theorem 1.1 we conclude that (1.2) is valid. $\square$

Theorem 1.4. *Let* $1 < p < q < \infty$. *By condition* (1.9) *the two-weighted inequality* (1.2) *is equivalent to the set of conditions* (1.1) *and* (1.12).

Proof. As shown while proving the preceding theorem, conditions (1.1) and (1.12) imply inequality (1.2) without condition (1.9).

By virtue of the proof of Theorem 1.2 it remains to show that the implication (1.2) $\Rightarrow$ (1.12) holds.

Let $\mu\{x\} > 0$. It is easy to verify that

$$T(f)(x) > \frac{1}{2}f(x)k(x,x,t)\mu\{x\}, \quad t \ge 0.$$

Therefore

$$\{x\} \subset \Big\{y \in X:\ T(f)(y) > \frac{1}{2}f(x)k(x,x,t)\mu\{x\}\Big\}.$$

By the latter inclusion and condition (1.2) we obtain

$$v\{x\} \le v\Big\{y \in X:\ T(f)(y) > \frac{1}{2}f(x)k(x,x,t)\mu\{x\}\Big\} \le$$
$$\le 2c\big(f(x)k(x,x,t)\mu\{x\}\big)^{-q}\bigg(\int\limits_X f^p(x)w(x)d\mu\bigg)^{\frac{q}{p}}. \tag{1.15}$$

After substituting $f_t(y) = \chi_{\{x\}}(y)w^{1-p'}(y)k^{p'-1}(x,y,t)$ in (1.15), the latter takes the form

$$v\{x\} \le 2c\big(w^{1-p'}(x)k^{p'}(x,x,t)\mu\{x\}\big)^{-q}\big(w^{1-p'}(x)k^{(p'-1)p}(x,x,t)\mu\{x\}\big)^{\frac{q}{p}} =$$
$$= 2c\big(w^{\frac{1}{p}}(x)k^{-1}(x,x,t)(\mu\{x\})^{-\frac{1}{p'}}\big)^q.$$

which implies that

$$\big(v\{x\}\big)^{\frac{1}{q}}w^{-\frac{1}{p}}(x)k(x,x,t)\big(\mu\{x\}\big)^{\frac{1}{p'}} \le 2c.$$

Since the constant x does not depend on x and t, from the latter inequality we conclude that condition (1.12) is fulfilled. $\square$

Analysis of the above theorems gives rise to the following two remarks:

Remark 1.1. When $X = R^n$ and d is a Euclidean distance, we can take the constant N_0 in the above theorems equal to unity. This can be done because the Besicovitch covering lemma (see [20]) can be applied instead of Lemma A to Euclidean spaces.

Remark 1.2. When $X = R^n$ and measure v is such that $vB(x,r)$ is continuous with respect to r, we can replace $vB(x, 2N_0 r)$ in condition (1.1) by $vB(x,r)$ and, accordingly, weaken condition (1.9) in Theorem 1.2 as follows: there exists a constant $c_1 > 0$ such that

$$k(a,y,t) \le c_1 k(x,y,t)$$

for arbitrary $t \ge 0$, a, x, and y from X satisfying the condition $d(a,x) \le d(a,y)$.

Next we will consider the case where $k(x,y,t) \equiv k(x,y)$. It will again be assumed that the measure μ is locally finite. Let

$$\mathcal{K}(f)(x) = \int\limits_X k(x,y)f(y)d\mu,$$

$$\mathcal{K}^*(f)(x) = \int\limits_X k(y,x)f(y)d\mu.$$

Definition 1.1. A positive measurable kernel $k : X \times X \to R^1$ will be said to satisfy the condition (V) $(k \in V)$ if there exists a constant $c > 0$ such that $k(x,y) < ck(x',y)$ for arbitrary x, y, and x' from X satisfying the condition $d(x,x') < Nd(x,y)$, where $N = 2N_0$.

Theorem 1.5. *Let $1 < p < q < \infty$, μ be an arbitrary locally finite measure, w be a weight, and $k \in V$. Then the following conditions are equivalent:*

(i) *there exists a constant $c_1 > 0$ such that the inequality*

$$w^{1-p'}\big\{x \in X : \mathcal{K}(f)(x) > \lambda\big\} \le$$
$$\le c_1 \lambda^{-p'} \bigg(\int\limits_X |f(y)|^{q'} v^{\frac{1}{1-q}}(y)d\mu \bigg)^{\frac{p'}{q'}} \tag{1.16}$$

holds for arbitrary $\lambda > 0$ and nonnegative $f \in L^p(X, wd\mu)$;

(ii) *there exists a constant* $c_2 > 0$ *such that*

$$\Big(\int_X \big(\mathcal{K}^*(\chi_B w^{1-p'})(x)\big)^q v(x)d\mu\Big)^{\frac{1}{q}} \leq c_2\Big(\int_B w^{1-p'}(y)d\mu\Big)^{\frac{1}{p}}, \quad (1.17)$$

for an arbitrary ball $B \subset X$*;*

(iii)

$$\sup_{\substack{x\in X\\ r>0}} \big(w^{1-p'}(NB)\big)^{\frac{1}{p'}}\Big(\int_{X\setminus B(x,r)} k^q(x,y)v(y)d\mu\Big)^{\frac{1}{q}} < \infty. \quad (1.18)$$

Proof. The implication (i)⇔(iii) follows from Theorem 1.2. We will prove (ii)⇒(iii). Applying the condition for the kernel for any $y \in X\setminus B(x,r)$, we obtain

$$\mathcal{K}^*\big(\chi_{NB} w^{1-p'}\big)(y) = \int_{NB(x,r)} k(z,y)w^{1-p'}(z)d\mu \geq$$
$$\geq c^{-1}k(x,y)\int_{NB(x,r)} w^{1-p'}(y)d\mu.$$

By the latter inequality and (1.17) we conclude that (1.18) holds.

Finally, it will be shown that the implication (i)⇒(ii) is valid. We have

$$\Big(\int_X \big(\mathcal{K}^*(\chi_B w^{1-p'})(x)\big)^q v(x)d\mu\Big)^{\frac{1}{q}} =$$
$$= \sup \int_X \mathcal{K}^*\big(w^{1-p'}\chi_B\big)(x)g(x)d\mu, \quad (1.19)$$

where the exact upper bound is taken with respect to all g for which

$$\int_X |g(x)|^{q'} v^{\frac{1}{1-q}}(x)d\mu \leq 1.$$

By Fubini's theorem

$$\int_X \mathcal{K}^*\big(\chi_B w^{1-p'}\big)(x)g(x)d\mu = \int_B w^{1-p'}(y)\mathcal{K}(g)(y)d\mu.$$

Next, assuming $\sigma = w^{1-p'}$ and applying inequality (1.16) we obtain

$$\int_B w^{1-p'}(y)\mathcal{K}(g)(y)d\mu = \int_0^\infty w^{1-p'}\big\{x \in B: \ \mathcal{K}(g)(x) > \lambda\big\}d\lambda \leq$$

$$\leq \int_0^{(\sigma B)^{-\frac{1}{p'}}} (\sigma B)d\lambda + c_1 \int_{(\sigma B)^{-\frac{1}{p'}}}^{\infty} \lambda^{-p'} = c_2\big(w^{1-p'}B\big)^{\frac{1}{p}}$$

and thereby prove the implication (i)⇒(ii) and, accordingly, the theorem. □

Summarizing the proofs of the above theorems, we have the right to claim that the following result is valid.

Theorem 1.6. *Let* $1 < p < q < \infty$, μ *and* v *be locally finite measures,* $\mu\{x\} = 0$. *It is assumed that the kernel* $k \in V$. *Then the following conditions are equivalent:*

(i) *there exists a constant* $c_1 > 0$ *such that*

$$v\big\{x \in X : \mathcal{K}(f)(x) > \lambda\big\} \leq c\lambda^{-q}\bigg(\int_X |f(x)|^p d\mu\bigg)^{\frac{q}{p}}$$

for arbitrary $\lambda > 0$ *and a nonnegative measurable weight* f*;*

(ii) *there exists a constant* $c_2 > 0$ *such that*

$$\bigg(\int_X \big(\mathcal{K}^*(\chi_B dv)(x)\big)^{p'} d\mu\bigg)^{\frac{1}{p'}} \leq c_2(vB)^{\frac{1}{q'}}$$

for any ball B *from* X*;*

(iii)

$$\sup_{\substack{x\in X\\ r>0}} \big(vB(x,2N_0r)\big)^{\frac{1}{q}}\bigg(\int_{X\setminus B(x,r)} k^{p'}(x,y)d\mu(y)\bigg)^{\frac{1}{p'}} < \infty.$$

In [5] (see also [7], Theorem 6.1.1) the two-weight problem was solved for integral transforms with a positive kernel in Euclidean spaces, in particular for Riesz potentials. By virtue of the above theorems and Remark 1.2 we obtain, for instance, a solution of the problem for one-sided potentials.

Consider the Riemann–Liouville transform

$$R_\alpha(f)(x) = \int_0^x (x-t)^{\alpha-1} f(t)dt$$

and the Weyl transform

$$W_\alpha(f) = \int_x^\infty (t-x)^{\alpha-1} f(t)dt,$$

where $0 < \alpha < 1$, $x > 0$.

As a consequence of Theorems 1.1 and 1.2 we conclude that the following statements are valid.

Theorem 1.7. *Let $1 < p < q < \infty$, $0 < \alpha < 1$. For the validity of the inequality*

$$v\big\{x \in (0,\infty) : \ |R_\alpha(f)(x)| > \lambda\big\} \le c\lambda^{-q}\bigg(\int_0^\infty |f(x)|^p w(x)dx\bigg)^{\frac{q}{p}}$$

with the constant not depending on f it is necessary and sufficient that the condition

$$\sup_{\substack{a,h\\ 0<h<a}} \bigg(\int_{a-h}^{a+h} v(y)dy\bigg)^{\frac{1}{q}} \bigg(\int_0^{a-h} \frac{w^{1-p'}(y)}{(a-y)^{(1-\alpha)p'}}dy\bigg)^{\frac{1}{p'}} < \infty$$

be fulfilled.

Theorem 1.8. *Let $1 < p < q < \infty$, $0 < \alpha < 1$. Then the following two conditions are equivalent:*

(i) *there exists a constant $c_1 > 0$ such that the inequality*

$$v\big\{x \in (0,\infty) : \ |W_\alpha(f)(x)| > \lambda\big\} \le c_1\lambda^{-q}\bigg(\int_0^\infty |f(x)|^p w(x)dx\bigg)^{\frac{q}{p}}$$

holds for any $\lambda > 0$ and $f \in L^p(X, wd\mu)$;

(ii)

$$\sup_{\substack{a,h\\ 0<h<a}} \bigg(\int_{a-h}^{a+h} v(y)dy\bigg)^{\frac{1}{q}} \bigg(\int_{a+h}^{\infty} \frac{w^{1-p'}(y)}{(y-a)^{(1-\alpha)p'}}dy\bigg)^{\frac{1}{p'}} < \infty.$$

By Theorem 1.2 we obtain as a particular case a solution of a weak-type two-weight problem for more general operators, for instance, for

$$N^+_{\alpha,\beta}(f)(x) = \sup_{c>x} \frac{1}{(c-x)^\beta} \int_x^c \frac{|f(s)|}{(s-x)^{1-\alpha}}ds,$$

$$M^+_{\alpha,\beta}(f)(x) = \sup_{c>x} \frac{1}{(c-x)^\beta} \int_x^c \frac{|f(s)|}{(c-s)^{1-\alpha}}ds,$$

$$N^-_{\alpha,\beta}(f)(x) = \sup_{c<x} \frac{1}{(c-x)^\beta} \int_c^\infty \frac{|f(s)|}{(x-s)^{1-\alpha}}ds,$$

$$M^{-}_{\alpha,\beta}(f)(x) = \sup_{c<x} \frac{1}{(x-c)^{\beta}} \int\limits_{c}^{\infty} \frac{|f(s)|}{(s-c)^{1-\alpha}} ds.$$

These operators were investigated in [17]. As an example we will give one of the corollaries of Theorem 1.2.

Theorem (J. Martin-Reyes [17]). *Let $0 \le \beta \le \alpha \le 1$, $\alpha > 0$, $1 \le p < q < \infty$. Then the following conditions are equivalent:*

(i) *there exists a constant $c_1 > 0$ such that*

$$v\big\{x:\ M^{+}_{\alpha,\beta}(f)(x) > \lambda\big\} \le c_1 \lambda^{-q} \Big(\int\limits_{R^1} |f(x)|^p w(x) \Big)^{\frac{q}{p}}$$

for any $\lambda > 0$ and measurable f;

(ii) *there exists a constant $c_2 > 0$ such that*

$$\Big(\int\limits_{a}^{b} v(x)dx \Big)^{\frac{1}{q}} \Big(\int\limits_{b}^{c} \frac{w^{1-p'}(s)}{(c-s)^{(1-\alpha)p'}} ds \Big)^{\frac{1}{p'}} \le c_2 (c-a)^{\beta}$$

for arbitrary a, b, and c satisfying the condition $a < b < c$.

2. A Two-Weight Problem In Upper Half-Space

For a space (X, d, μ) with a given quasi-metric and complete measure μ (not necessarily satisfying the doubling condition) we will consider an upper half-space of the product space $X \times R$. We set $\widehat{X} = X \times [0, \infty)$ and give, in the space $\widehat{X}$, a quasi-metric

$$\widehat{d}\big((x,t), y, s)\big) = \max\big\{d(x,y),\ |s-t|\big\}$$

and a measure $d\widehat{\mu} = d\mu \oplus \delta_0$, where δ_0 is the Dirac measure concentrated at zero.

Let $k : X \times X \times [0, \infty)$ be a positive measurable kernel satisfying the condition: there exists a constant $c > 0$ such that

$$k(x, y, t) \le c k(x', y, t') \tag{2.1}$$

for x, x', y from X, $t \ge 0$, $t' \ge 0$, satisfying the condition $d(x, x') + t' \le 5N_0(d(x, y) + t)$.

Consider the integral operators

$$T(f)(x,t) = \int\limits_{X} k(x, y, t) f(y) d\mu$$

and

$$T^*(gdv)(y) = \int_{\widehat{X}} k(x,y,t)g(x,t)dv(x,t), \quad y \in X,$$

where v is some locally finite measure in $\widehat{X}$.

Consider the kernel $\widehat{k}$ defined on $\widehat{X} \times \widehat{X}$ as follows:

$$\widehat{k}\big((x,t),(y,s)\big) = k\big(x,y,|t-s|\big).$$

We readily obtain

$$T(f)(x,t) = \int_{\widehat{X}} \widehat{k}\big((x,t),(y,s)\big) f(y) d\widehat{\mu}(y) = \widehat{\mathcal{K}}(f)(x,t)$$

and

$$T^*(gdv)(y) = \int_{\widehat{X}} \widehat{k}\big((x,t),(y,s)\big) g(x,t) dv(x,y).$$

It is easy to verify that if the kernel satisfies condition (2.1), then $\widehat{k} \in V$. Therefore Theorem 1.6 may give rise to

Theorem 2.1. *Let $1 < p < q < \infty$, μ and v be locally finite measures on X and $\widehat{X}$, respectively. Further assume that the kernel k satisfies condition* (2.1)*. Then the following conditions are equivalent:*

(i) *there exists $c_1 > 0$ such that*

$$v\big\{(x,t) \in \widehat{X} : \big|T(f)(x,t)\big| > \lambda\big\} \le c_1 \lambda^{-q} \bigg(\int_X |f(x)|^p d\mu \bigg)^{\frac{q}{p}}$$

for any $\lambda > 0$ and f;

(ii) *there exists a constant $c_2 > 0$ such that*

$$\bigg(\int_X \Big(\int_{\widehat{B}} k(x,y,t) dv(x,t) \Big)^{p'} d\mu(y) \bigg)^{\frac{1}{p'}} \le c_2 (v\widehat{B})^{\frac{1}{q'}}$$

for any ball $\widehat{B}$ from $\widehat{X}$;

(iii)

$$\sup \big(v\widehat{B}(a,t)\big)^{\frac{1}{q}} \bigg(\int_{X \setminus B(a,t)} k^{p'}(a,y,t) d\mu(y) \bigg)^{\frac{1}{p'}} < \infty,$$

where $\widehat{B}(a,t) = B(a,t) \times [0,2t)$.

An idea similar to the one discussed in this section but in a slightly different interpretation was used in [12].

3. Solution of a Strong Type Two-Weight Problem for Integral Transforms Defined on Homogeneous Type Spaces

In this section X is assumed to be a homogeneous type space, which means that (X, d, μ) satisfies conditions (1)–(6) given in the introduction.

Let further $k : X \times X \to R^1$ be a positive measurable kernel. Our purpose is to give a complete description of kernels k and pairs of weights (v, w) which provide the validity of a strong type two-weighted inequality for the transform

$$\mathcal{K}(f)(x) = \int\limits_X k(x,y) f(y) d\mu$$

under certain assumptions for k.

In what follows it will be assumed that $k^*(x,y) = k(y,x)$.

Definition 3.1. A kernel k satisfies condition (V') $(k \in V')$ if there exists positive constant $c_1 > 1$ and $c_2 > 1$ such that

$$k(x,y) \le c_1 k(x',y) \tag{3.1}$$

for arbitrary x, x', and y from X satisfying the condition $d(x',y) \le c_2 d(x,y)$.

In what follows X will be assumed to be a space such that $B(x,R) \backslash B(x,r) \neq \varnothing$ for arbitrary r and R satisfying the condition $0 < r < R < \infty$.

Proposition 3.1. *Conditions (V) and (V') are equivalent.*

Proof. In the first place we note that condition (V') implies that for any $c_2' > 1$ there is $c_1' > 1$ such that

$$k(x,y) \le c_1' k(x',y) \tag{3.2}$$

for any x, x', and y from X satisfying the condition $d(x',y) \le c_2' d(x,y)$.

Let now $k \in V'$. Let x, x', y be arbitrary points from X which satisfy the condition $d(x,x') < N d(x,y)$. Obviously,

$$\begin{aligned} d(x',y) &\le a_1 d(x',x) + d(x,y) \le \\ &\le a_1 \big(N a_0 d(x,y) + d(x,y)\big) = a_1 (N a_0 + 1) d(x,y). \end{aligned}$$

For the number $c_2' = a_1(N a_0 + 1)$ there exists $c_1' > 0$ such that condition (3.2) will be fulfilled and therefore $k \in V$.

Let further $k \in V$. Choose a constant $c_2 = \frac{1+4a_0}{a_0}$, where $N = 2a_1(1 + 2a_0)$. Obviously, $c_2 > 1$. Let x, y, and x' from X be such that $d(x',y) \le c_2 d(x,y)$. Then we will have

$$\begin{aligned} d(x,x') &\le a_1 \big(d(x,y) + d(y,x')\big) \le \\ &\le a_1 \big(d(x,y) + a_0 d(x',y)\big) \le a_1 (1 + c_2 a_0) d(x,y). \end{aligned}$$

Therefore

$$d(x,x') \le Nd(x,y)$$

and condition (3.2) will be fulfilled by virtue of condition (V). We thus conclude that $k \in V'$. □

Theorem 3.1. *Let $1 < p < q < \infty$, k and k^* satisfy condition (V). Then for the inequality*

$$\Big(\int\limits_X |Kf(x)|^q v(x)d\mu\Big)^{\frac{1}{q}} \le c_3\Big(\int\limits_X |f(x)|^p w(x)d\mu\Big)^{\frac{1}{p}} \tag{3.3}$$

to hold, where the constant c_3 does not depend on f, it is necessary and sufficient that the following two conditions be fulfilled simultaneously:

$$\sup_{\substack{x\in X\\ r>0}} \big(vB(x,2N_0r)\big)^{\frac{1}{q}}\Big(\int\limits_{X\setminus B(x,r)} k^{p'}(x,y)w^{1-p'}(y)d\mu\Big)^{\frac{1}{p'}} < \infty, \tag{3.4}$$

$$\sup_{\substack{x\in X\\ r>0}} \big(w^{1-p'}B(x,2N_0r)\big)^{\frac{1}{p'}}\Big(\int\limits_{X\setminus B(x,r)} k^{q}(y,x)v(y)d\mu\Big)^{\frac{1}{q}} < \infty. \tag{3.5}$$

The proof of Theorem 3.1 will be based on Theorem 1.2 and the following result of Sawyer and Wheeden.

Theorem A ([11]). *Let $1 < p < q < \infty$, k and k^* satisfy condition (V). Then* (3.3) *holds if and only if the conditions*

$$\int\limits_X \big(\mathcal{K}(\chi_B w^{1-p'})(x)\big)^q v(x)d\mu \le c\Big(\int\limits_B w^{1-p'}(x)d\mu\Big)^{\frac{q}{p}} < \infty, \tag{3.6}$$

$$\int\limits_X \big(\mathcal{K}^*(\chi_B v)\big)^{p'} w^{1-p'}(x)d\mu \le c\Big(\int\limits_B v(x)d\mu\Big)^{\frac{p'}{q'}} \tag{3.7}$$

are fulfilled sumultaneously, where the constant c does not depend on B.

Proof of Theorem 3.1. By virtue of Theorem 1.1 conditions (3.4) and (3.5) imply the following weak type inequalities:

$$v\{x\in X:\ \big|\mathcal{K}(f)(x)\big| > \lambda\} \le c_4\lambda^{-q}\Big(\int\limits_X |f(x)|^p w(x)d\mu\Big)^{\frac{q}{p}} \tag{3.8}$$

and

$$w^{1-p'}\{x\in X:\ \big|\mathcal{K}^*(f)(x)\big| > \lambda\} \le$$

$$\leq c_5\lambda^{-p'}\bigg(\int\limits_X |f(x)|^{q'} v^{\frac{1}{1-q}}(x)d\mu\bigg)^{\frac{p'}{q'}}. \tag{3.9}$$

By Theorem 1.5 we obtain the implications (3.9) $\Rightarrow$ (3.6) and (3.8) $\Rightarrow$ (3.7). Therefore due to Theorem A we conclude that (3.3) holds.

Let now (3.3) be fulfilled. Then (3.8) will obviously be fulfilled too and so will (3.4) on account of Theorem 1.2. On the other hand, the validity of (3.3) implies

$$\bigg(\int\limits_X \big|\mathcal{K}^*(f)(x)\big|^{p'} w^{1-p'}(x)d\mu\bigg)^{\frac{1}{p'}} \leq c_6\bigg(\int\limits_X |f(x)|^{q'} v^{\frac{1}{1-q}}(x)d\mu\bigg)^{\frac{1}{q'}}. \tag{3.10}$$

Indeed,

$$\bigg(\int\limits_X \big|\mathcal{K}^*(f)(x)\big|^{p'} w^{1-p'}(x)d\mu\bigg)^{\frac{1}{p'}} \leq \sup\int\limits_X \mathcal{K}^*(|f|)(x)|g(x)|d\mu,$$

where the supremum is taken with respect to g for which

$$\int\limits_X |g(x)|^p w(x)dx \leq 1.$$

Next, applying the Hölder inequality and (3.3) we obtain

$$\int\limits_X \mathcal{K}^*(|f|)(x)|g(x)|d\mu \leq c_3\bigg(\int\limits_X |f(x)|^{q'} v^{\frac{1}{1-q}}(x)d\mu\bigg)^{\frac{1}{q}}.$$

Thus we have obtained the implication (3.3) $\Rightarrow$ (3.10). Inequality (3.10) further implies (3.9) and therefore we conclude by virtue of Theorem 1.2 that condition (3.5) is fulfilled. $\square$

In some particular cases the criteria for strong type two-weighted inequalities take a simpler form.

By definition, the measure ν satisfies the reverse doubling condition if there exist constants $\beta_1 > 1$ and $\beta_2 < 1$ such that

$$\nu B(x,r) \leq \beta_2 \nu B(x,\beta_1 r) \tag{3.11}$$

for arbitrary $x \in X$ and $r > 0$.

We set $h(B) = \sup\{k(x,y) : x,y \in B, d(x,y) \geq cr(B)\}$, where $r(B)$ is the radius of the ball B and c is a sufficiently small positive constant depending on a_1.

Theorem 3.2. *Let $1 < p < q < \infty$ and the measures v and $w^{1-p'}$ satisfy the reverse doubling condition. Then for inequality* (2.3) *to hold it is necessary and sufficient that the condition*

$$\sup_B h(B)\Big(\int_B v(x)d\mu\Big)^{\frac{1}{q}}\Big(\int_B w^{1-p'}(x)d\mu\Big)^{\frac{1}{p'}} < \infty \tag{3.12}$$

be fulfilled.

Proof. By the reverse doubling condition we have

$$vB(x,Nr) \le \beta_2 vB(x,\beta_1 Nr),$$

where $N = 2N_0$. Assuming that $\beta_1 > N$, we obtain

$$\big(vB(x,Nr)\big)^{\frac{1}{q}}\Big(\int_{X\setminus B(x,r)} w^{1-p'}(y)k^{p'}(x,y)d\mu\Big)^{\frac{1}{p'}} \le$$

$$\le \sum_{k=0}^{\infty}\big(vB(x,Nr)\big)^{\frac{1}{q}}\Big(\int_{\beta_1^k NB(x,r)\setminus\beta_1^{k-1}NB(x,r)} w^{1-p'}(y)k^{p'}(x,y)d\mu\Big)^{\frac{1}{p'}} \le$$

$$\le \sum_{k=0}^{\infty}\beta_2^{\frac{k}{q}}\big(vB(x,\beta_1^k Nr)\big)^{\frac{1}{q}}\big(w^{1-p'}B(x,\beta_1^k Nr)\big)^{\frac{1}{p'}} hB(x,\beta_1^k Nr) \le$$

$$\le c\sum_{k=0}^{\infty}\beta_2^{\frac{k}{q}} < \infty.$$

Hence in that case (3.12) implies (3.4). If the function $w^{1-p'}$ satisfies the reverse doubling condition, in a similar manner (3.12) implies (3.5). It remains to apply Theorem 3.1. □

Theorem 3.3. *Let $1 < p < q < \infty$ and there exists $r > 1$ such that*

$$\sup h(B)(\mu B)^{\frac{1}{q}+\frac{1}{p'}}\Big(\frac{1}{\mu B}v^r B\Big)^{\frac{1}{rq}}\Big(\frac{1}{\mu B}w^{(1-p')r}B\Big)^{\frac{1}{rp'}} < \infty. \tag{3.13}$$

Then (2.3) *holds.*

Proof. Using the Hölder inequality and (3.13) we obtain

$$\sup h(B)(\mu B)^{\frac{1}{q}}\Big(\frac{1}{\mu B}\int_B v^r(y)d\mu\Big)^{\frac{1}{rq}}\Big(\int_B w^{1-p'}(y)d\mu\Big)^{\frac{1}{p'}} < \infty \tag{3.14}$$

and

$$\sup h(B)(\mu B)^{\frac{1}{p'}}\left(\int\limits_B v(y)d\mu\right)^{\frac{1}{q}}\left(\frac{1}{\mu B}\int\limits_B w^{(1-p')r}(y)d\mu\right)^{\frac{1}{p'r}}<\infty. \quad (3.15)$$

Now we will show that (3.14) implies (3.4). Setting $X\backslash B(x,r) = \cup_k(B_k\backslash B_{k-1})$, $B_0 = B$, $B_1 \supset NB$, $B_k \subset B_{k-1}$, and $\mu B_k \le \frac{1}{2}\mu B_{k+1}$, we have

$$\begin{aligned}
&\left(\int\limits_{NB(x,r)} v(y)d\mu\right)^{\frac{1}{q}}\left(\int\limits_{X\backslash B(x,r)} w^{1-p'}(y)k^{p'}(x,y)d\mu\right)^{\frac{1}{p'}} \le \\
&\le \sum_{k=1}^{\infty}\left(\int\limits_{B_1} v(y)d\mu\right)^{\frac{1}{q}} h(B_k)\left(\int\limits_{B_k} w^{1-p'}(y)d\mu\right)^{\frac{1}{p'}} \le \\
&\le \sum_{k=1}^{\infty}(\mu B_1)^{\frac{1}{r'q}}\left(\int\limits_{B_1} v^r(y)d\mu\right)^{\frac{1}{rq}} h(B_k)\left(\int\limits_{B_k} w^{1-p'}(y)d\mu\right)^{\frac{1}{p'}} \le \\
&\le \sum_{k=1}^{\infty}\left(\frac{\mu B_1}{\mu B_k}\right)^{\frac{1}{r'q}} h(B_k)(\mu(B_k)^{\frac{1}{q}}\left(\frac{1}{\mu B_k}\int\limits_{B_k} v^r(y)d\mu\right)^{\frac{1}{rq}}\left(\int\limits_{B_k} w^{1-p'}(y)d\mu\right)^{\frac{1}{p'}} \le \\
&\le c\sum_{k=1}^{\infty}\left(\frac{1}{2^k}\right)^{\frac{1}{r'q}}.
\end{aligned}$$

In a similar manner (3.15) implies (3.5). The remaining proof follows from Theorem 3.1. □

Before obtaining by Theorem 3.1 a solution of strong type two-weight problems for a number of specific integral operators, we would like to make some remarks about conditions to be imposed on the kernel k which will be used in the theorems of this sections.

Definition 3.2. The kernel k will be said to satisfy condition (V_1) ($k \in V_1$) if there exists a constant $c > 0$ such that condition (3.1) is fulfilled for any x, y, and x' from X satisfying the condition $d(x,x') < d(x,y)$.

Definition 3.3. The kernel k satisfies condition (V_1') ($k \in V_1'$) if there exists a positive constant c_1 suuch that (3.1) holds for any x, y, and x' from X satisfying the condition $d(x',y) < d(x,y)$.

In contrast to conditions V and V', conditions V_1 and V_1' are the uncomparable ones. For instance, the kernel

$$k(x,y)=\begin{cases}(x-y)^{\gamma-1} & \text{for} \quad x>y,\\ 0 & \text{for} \quad x<y\end{cases}$$

satisfies condition V_1' but does not satisfy condition V_1.

For the kernel $k(x,y)=e^{-|x-y|}$ we have $k\in V_1$ but $k\notin V_1'$.

For R^1 these two conditions can be combined into one condition: $k\in\widetilde{V}$ if there exists a constant $c>1$ such that (3.1) holds for any x, y, and x' satisfying $|2x'-x-y|\le 3|x-y|$.

Note that by taking into consideration a simple geometrical character of the straight line and following the proof of Theorem A we can ascertain that it remains valid in R^1 also for kernels k satisfying the condition $k\in\widetilde{V}$, $k^*\in\widetilde{V}$.

Using further Remark 2 from §1 and the fact that the kernels to be considered below satisfy the above requirement we obtain the following statements.

Theorem 3.4. *Let $1>p<q<\infty$, $0<\alpha<1$. For the inequality*

$$\Big(\int\limits_0^\infty \big|R_\alpha(f)(x)\big|^q v(x)dx\Big)^{\frac1q}\le c_1\Big(\int\limits_0^\infty |f(x)|w(x)dx\Big)^{\frac1p}$$

with the constant c_1 not depending on f to hold it is necessary and sufficient that two conditions

$$\sup_{\substack{a,h\\0<h<a}}\Big(\int\limits_{a-h}^{a+h}v(y)dy\Big)^{\frac1q}\Big(\int\limits_0^{a-h}\frac{w^{1-p'}(y)}{(a-y)^{(1-\alpha)p'}}dy\Big)^{\frac1{p'}}<\infty,$$

$$\sup_{\substack{a,h\\0<h<a}}\Big(\int\limits_{a-h}^{a+h}w^{1-p'}(y)dy\Big)^{\frac1{p'}}\Big(\int\limits_{a+h}^{\infty}\frac{v(y)}{(a-y)^{(1-\alpha)q}}dy\Big)^{\frac1q}<\infty$$

be fulfilled simultaneously.

Theorem 3.5. *Let $1<p<q<\infty$, $0<\alpha<1$. For the inequality*

$$\Big(\int\limits_0^\infty \big|W_\alpha(f)(x)\big|^q v(x)dx\Big)^{\frac1q}\le c\Big(\int\limits_0^\infty |f(x)|w(x)dx\Big)^{\frac1p}$$

with the constant c not depending on f to hold it is necessary and sufficient that two conditions

$$\sup_{\substack{a,h\\0<h<a}} \Big(\int_{a-h}^{a+h} v(y)dy\Big)^{\frac{1}{q}} \Big(\int_{a+h}^{\infty} \frac{w^{1-p'}(y)}{(y-a)^{(1-\alpha)p'}}dy\Big)^{\frac{1}{p'}} < \infty,$$

$$\sup_{\substack{a,h\\0<h<a}} \Big(\int_{a-h}^{a+h} w^{1-p'}(y)dy\Big)^{\frac{1}{p'}} \Big(\int_{0}^{a+h} \frac{v(y)}{(y-a)^{(1-\alpha)q}}dy\Big)^{\frac{1}{q}} < \infty$$

be fulfilled simultaneously.

For the case $\alpha > 1$ the two-weight problem for R_α and W_α was previously solved by many authors while for $0 < \alpha < 1$ it remained open (see, for instance, [21], [22]) until this paper.

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