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COMBINATORIAL INVARIANCE OF STANLEY–REISNER RINGS

W. BRUNS AND J. GUBELADZE

Abstract. In this short note we show that Stanley–Reisner rings of simplicial complexes, which have had a "dramatic application" in combinatorics [2, p. 41], possess a rigidity property in the sense that they determine their underlying simplicial complexes.

For convenience we recall the notion of a Stanley–Reisner ring (for more information the reader is referred to [1, Ch. 5]). Let V be a finite set to be called below a vertex set. A system Δ of subsets of V is called an abstract simplicial complex (on the vertex set V) if the following conditions hold:

(a) $\{v\} \in \Delta$ for any element $v \in V$,

(b) $\sigma' \in \Delta$ whenever $\sigma' \subset \sigma$ for some $\sigma \in \Delta$.

Elements of Δ will be called faces.

Now assume we are given a field k and an abstract simplicial complex Δ on a vertex set V. The *Stanley–Reisner ring* corresponding to these data is defined as the quotient ring of the polynomial ring $k[v_1, \dots, v_n]/I$, where $n = \#(V)$, the v_i are the elements of V, and the ideal I is generated by the set of monomials $\{v_{i_1} \cdots v_{i_k} | \{v_{i_1}, \dots, v_{i_k}\} \notin \Delta\}$. This k-algebra will be denoted by $k[\Delta]$ and called the Stanley–Reisner ring of Δ. Further, the image of v_i in it will again be denoted by v_i (they are all different!) and hence will again be thought of as elements of V.

Theorem. *Let k be a field, and Δ and Δ' be two abstract simplicial complexes defined on the vertex sets $V = \{v_1, \dots, v_n\}$ and $U = \{u_1, \dots, u_m\}$ respectively. Suppose $k[\Delta]$ and $k[\Delta']$ are isomorphic as k-algebras. Then there exists a bijective mapping $\Psi : V \to U$ which induces an isomorphism between Δ and Δ'.*

Proof. Let $f : k[\Delta] \to k[\Delta']$ be a k-isomorphism. By scalar extension we may assume k is algebraically closed. Let us first show that without loss of

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generality we may also assume f is an isomorphism of augmented k-algebras, where $k[\Delta]$ is endowed with an augmented k-algebra structure induced by $v_i \mapsto 0$, and similarly for $k[\Delta']$. Indeed, if v_i is a zero-divisor in $k[\Delta]$ for some $i \in [1, n]$, then its image in $k[\Delta']$ cannot have a nonzero constant term (with respect to the uniquely determined canonical expansion). So the deviation from "being augmented" for f can appear only at the elements $v_i \in V$ which are not zero-divisors. It is easy to observe that $v_i \in V$ is not a zero-divisor in $k[\Delta]$ if and only if it is a variable for $k[\Delta]$, i.e., $k[\Delta] = k[\Delta^i][v_i]$, where Δ^i is a simplicial subcomplex of Δ consisting of those faces which do not contain v_i, and this v_i on the right-hand side is understood as a variable. Let $\{v_{i_1}, \dots, v_{i_k}\}$ be the set of all nonzero-divisor vertices and $\{c_{i_1}, \dots, c_{i_k}\}$ be the set of constant terms in the canonical expansions of $f(v_{i_1}), \dots, f(v_{i_k})$ respectively. Consider the elements $w_{i_1} = v_{i_1} - c_{i_1}, \dots, w_{i_k} - c_{i_k} \in k[\Delta]$. Clearly, they are all different. Let $W = \{w_1, \dots, w_n\}$ be the set obtained from V by substituting w_{i_j} by v_{i_j}, respectively, and let Δ° be the abstract simplicial complex on the vertex set W induced by the natural bijection between V and W. Since all of the v_{i_j} are (independent!) variables for $k[\Delta]$ (as remarked above), we conclude that $k[\Delta] = k[\Delta^\circ]$ and f is an augmented isomorphism between $k[\Delta^\circ]$, considered as an augmented k-algebra with respect to $w_i \mapsto 0$, and $k[\Delta']$. So from the very beginning we can assume f is augmented. □

Next we pass to the corresponding graded isomorphism (with respect to the augmentation ideals)

$$\mathrm{gr}(f) : \mathrm{gr}(k[\Delta]) \to \mathrm{gr}(k[\Delta']).$$

But $\mathrm{gr}(k[\Delta]) = k[\Delta]$ and $\mathrm{gr}(k[\Delta']) = k[\Delta']$. This means that we may also assume f is a graded k-isomorphism of graded k-algebras $k[\Delta]$ and $k[\Delta']$ where $\deg(v_1) = \dots = \deg(v_n) = \deg(u_1) = \dots = \deg(u_m) = 1$. Now passing to the geometrical picture (i.e., to the closed points of the corresponding affine schemes) we obtain the following situation: we are given two k-linear spaces

$$k^n = \mathrm{maxSpec}(k[v_1, \dots, v_n])$$

and

$$k^m = \mathrm{maxSpec}(k[u_1, \dots, u_m])$$

(the v_i and u_j are considered as variables) and two arrangments of k-linear coordinate subspaces (of appropriate dimensions)

$$\Delta^* = \mathrm{maxSpec}(k[\Delta]) \subset k^n,$$
$$(\Delta')^* = \mathrm{maxSpec}(k[\Delta']) \subset k^m.$$

More precisely, Δ^* consists of those coordinate subspaces of k^n which are spanned by the coordinate directions of $v_{i_1}, \dots, v_{i_k}$ whenever $\{v_{i_1}, \dots, v_{i_k}\} \in \Delta$, and similarly for $(\Delta')^*$. This claim follows directly from the equality

$$\Delta^* = \bigcap_{\{v_{i_1},\dots,v_{i_k}\} \notin \Delta} (v_{i_1}^\circ \cup \dots \cup v_{i_k}^\circ),$$

where $v_{i_j}^\circ$ denotes the coordinate hyperplane of dimension $n-1$ avoiding v_{i_j}, and the similar one for $(\Delta')^*$.

So for each maximal face (with respect to the inclusion) $\sigma \in \Delta$ we have the corresponding coordinate linear subspace $L_\sigma \subset k^n$ and

$$\Delta^* = \bigcup_{\sigma \text{ a maximal face of } \Delta} L_\sigma.$$

Similarly, for each maximal face $\sigma' \in \Delta'$ we have the corresponding coordinate linear subspace $M_{\sigma'} \subset k^m$ and

$$(\Delta')^* = \bigcup_{\sigma' \text{ a maximal face of } \Delta'} M_{\sigma'}.$$

The corresponding algebraic map

$$f^* : (\Delta')^* \to \Delta^*$$

will be the restriction of the k-linear isomorphism

$$F^* : k^m \to k^n$$

contravariantly corresponding to the (uniquely determined) graded k-isomorphism F from the commutative square

$$\begin{array}{ccc} k[v_1, \dots, v_n] & \xrightarrow{F} & k[u_1, \dots, u_m] \\ \downarrow & & \downarrow \\ k[\Delta] & \xrightarrow{f} & k[\Delta']. \end{array}$$

This gives rise to the well defined bijective map

$$\Phi : (\text{maximal faces of } \Delta') \to (\text{maximal faces of } \Delta).$$

Namely, $\Phi(\sigma') = $ (the maximal face σ of Δ for which $L_\sigma = f^*(M_{\sigma'})$).

After this "linear" interpretation it becomes obvious that $m = n$ and $\#\sigma' = \#\Phi(\sigma')$ for each maximal $\sigma' \in \Delta'$. Moreover,

$$\#(\sigma'_1 \cap \ldots \cap \sigma'_t) = \#(\Phi(\sigma'_1) \cap \dots \cap \Phi(\sigma'_t)). \qquad (*)$$

Indeed,

$$\begin{aligned}\#(\sigma'_1 \cap \cdots \cap \sigma'_t) &= \dim_k(M_{\sigma'_1} \cap \cdots \cap M_{\sigma'_t}) \\ &= \dim_k(f^*(M_{\sigma'_1}) \cap \cdots \cap f^*(M_{\sigma'_t})) \\ &= \dim_k(L_{\Phi(\sigma'_1)} \cap \cdots \cap L_{\Phi(\sigma'_t)}) \\ &= \#(\Phi(\sigma'_1) \cap \cdots \cap \Phi(\sigma'_t)).\end{aligned}$$

Now we introduce the following equivalence relations on the vertex sets V and U: for $v_{i_1}, v_{i_2} \in V$ ($u_{j_1}, u_{j_2} \in U$) we put $v_{i_1} \sim v_{i_2}$ if and only if the two sets of maximal faces of Δ containing v_{i_1} and v_{i_2} respectively coincide (and similarly for u_{j_1} and u_{j_2}). The equivalence classes in V will be the minimal (with respect to inclusion) nonempty intersections of maximal faces of Δ (and similarly for the vertex set U). Accordingly, these equivalence classes will be in one-to-one correspondence (via Φ) with the minimal nonzero intersections (w.r.t. inclusion) of the linear subspaces $L_\sigma \subset k^n$ (similarly for the equvalence classes in U and the linear subspaces $M_{\sigma'} \subset k^m$). Since we are given a global linear isomorphism F^*, using Φ we immediately see that the two systems of equivalence classes are in natural bijective correspondence. By $(*)$ the corresponding equivalence classes have the same quantities of elements. This gives rise in a natural way to the bijective mapping $\psi : U \to V$ which satisfies the condition that $u \in \sigma'$ if and only if $\psi(u) \in \Phi(\sigma')$, where $u \in U$ and $\sigma' \in \Delta'$ is a maximal face. Since any face in an abstract simplicial complex is contained in some maximal face, we finally arrive at the conclusion that $\Psi = (\psi)^{-1} : V \to U$ satisfies the desired condition.

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Authors' addresses:

Winfried Bruns
Universität Osnabrück
Fachbereich
Mathematik/Informatik
49069 Osnabrück, Germany

Joseph Gubeladze
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbibisi 380093
Republic of Georgia