GEORGIAN MATHEMATICAL JOURNAL: Vol. 3, No. 3, 1996, 293-300

SEMIDIRECT PRODUCTS AND WREATH PRODUCTS OF STRONGLY π-INVERSE MONOIDS

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ABSTRACT. In this paper we determine the necessary and sufficient conditions for the semidirect products and the wreath products of two monoids to be strongly π-inverse. Furthermore, we determine the least group congruence on a strongly π-inverse monoid, and we give some important isomorphism theorems.

1. INTRODUCTION

Our terminology and notation will follow [1] and [2].

Let S and T be two monoids, and let $\mathrm{End}(T)$ be the endomorphism monoid of T, and write endomorphisms as exponents to the right of arguments. If $\alpha : S \to \mathrm{End}(T)$ is a homomorphism, and if $s \in S$ and $t \in T$, write t^s for $t^{\alpha(s)}$, since $\alpha(s) \in \mathrm{End}(T)$ for $s \in S$, then for $t_1, t_2 \in T$, $(t_1, t_2)^s = t_1^s t_2^s$. Since α is a homomorphism, $(t^{s_1})^{s_2} = t_{s_1 s_2}$ for every $t \in T$ and $s_1, s_2 \in S$.

The semidirect product $S \times_\alpha T$ is the monoid with elements $\{(s,t) : s \in S, t \in T\}$ and multiplication $(s_1, t_1)(s_2, t_2) = (s_1 s_2, t_1^{s_2} t_2)$.

In [3], [4] the authors have determined the necessary and sufficient conditions for $S \times_\alpha T$ to be regular, inverse, and orthodox. In this paper we determine the necessary and sufficient conditions for $S \times_\alpha T$ to be strongly π-inverse and give their applications to the wreath product.

For a monoid S, $E(S)$ and $\mathrm{Reg}\, S$ denote the set of idempotents of S and the set of regular elements of S, respectively.

A semigroup is π-regular if for every $s \in S$ there is an $m \in \mathbb{N}$ such that $s^m \in \mathrm{Reg}\, S$. If S is π-regular and $E(S)$ is a commutative subsemigroup, then we call S a strongly π-inverse semigroup. It is easy to see that $\mathrm{Reg}\, S$ is an inverse subsemigroup of a strongly π-inverse semigroup S.

1991 *Mathematics Subject Classification.* 20M18.

Key words and phrases. Wreath product, π-inverse monoid, regular monoid, group congruence.

1072-947X/96/0500-0293\$09.50/0

2. Semidirect Products

Let S and T be two monoids and let $S \times_\alpha T$ be the semidirect product of S and T, where $\alpha : S \to \mathrm{End}(T)$ is a given homomorphism.

Lemma 1. *Let $S \times_\alpha T$ be a strongly π-inverse monoid; then*
(1) *both S and T are strongly π-inverse monoids;*
(2) *$u^e = u$ for every $e \in E(S)$ and every $u \in E(T)$;*
(3) *if $t^e t = t$ for $t \in T$ and $e \in E(S)$, then $t^e = t$;*
(4) *$t^e = t$ for every $t \in \mathrm{Reg}\, T$ and every $e \in E(S)$;*
(5) *for every $s \in S$ and $t \in T$, there exists $m \in \mathbb{N}$ such that $s^m \in \mathrm{Reg}\, S$ and $t^{s(m)} \in \mathrm{Reg}\, T$, where $t^{s(m)} = t^{s^{m-1}} t^{s^{m-2}} \cdots t^s t$.*

Proof. (1) For arbitrary $s \in S$, there exist $m \in \mathbb{N}$ and $(s_1, t_1) \in T$ such that $(s,1)^m (s_1, t_1)(s,1)^m = (s,1)^m$. Hence $(s^m s_1 s^m, t_1^{s^m}) = (s^m, 1)$, $s^m s_1 s^m = s^m$ and then S is π-regular.

Since $(e,1)$, $(f,a) \in E(S \times_\alpha T)$ for $e, f \in E(S)$, we have $(e,1)(f,1) = (f,1)(e,1)$, and then $ef = fe$. Hence S is strongly π-inverse monoid.

For arbitrary $t \in T$, there exist $m \in \mathbb{N}$ and $(s_2, t_2) \in S \times_\alpha T$ such that $(1,t)^m (s_2, t_2)(1,t)^m = (1,t)^m$, that is, $(s_2, (t^m)^{s_2} t_2 t^m) = (1, t^m)$. Then $s_2 = 1$ and $t^m t_2 t^m = (t^m)^{s_2} t_2 t^m = t^m$. Thus T is π-regular.

Since $(1,u)$, $(1,v) \in E(S \times_\alpha T)$ for $u, v \in E(T)$ and $S \times_\alpha T$ is strongly π-inverse, we have $(1,u)(1,v) = (1,v)(1,u)$, so that $uv = vu$, which implies that T is strongly π-inverse.

(2) Let $e \in E(S)$ and $u \in E(T)$. Then $(e,1)$, $(1,u) \in E(S \times_\alpha T)$ and $(e,1)(1,u) = (1,u)(e,1)$, which implies $u^e = u$.

(3) If $t^e t = t$, then $(e,t) \in E(S \times_\alpha T)$ and $(e,t)(e,1) = (e,1)(e,t)$ since $(e,1) \in E(S \times_\alpha T)$. Hence $t^e = t$.

(4) From (1), for every $t \in \mathrm{Reg}\, T$, there exists a unique $t_1 \in T$ such that $tt_1 t = t$, $t_1 t t_1 = t_1$. Then $t^e t_i^e t^e = t^e$, further, $(t^e t_1)^e t^e t_1 = t^e t_1$. From (3) we have $(t^e t_1)^e = t^e t_1$, that is, $(tt_1)^e = t^e t_1$. Since $tt_1 \in E(T)$, from (2) we have $(tt_1)^e = tt_1 = t^e t_1$, and then $t_1 t t_1 = t_1 t^e t_1 = t_1$, $t^e t_1 t^e = t^e$. Thus both t and t^e are inverses of t_1, and then $t^e = t$.

(5) Since $S \times_\alpha T$ is a strongly π-inverse monoid, for every $(s,t) \in S \times_\alpha T$ there exist $m \in \mathbb{N}$ and $(s_1, t_1) \in S \times_\alpha T$ such that $(s,t)^m (s_1, t_1)(s,t)^m = (s,t)^m$. Then

$$\big(s^m s_1 s^m, (t^{s(m)})^{s_1 s^m} t_1^{s^m} t^{s(m)}\big) = \big(s^m, t^{s(m)}\big),$$

so that $s^m s_1 s^m = s^m$, $(t^{s(m)})^{s_1 s^m} t_1^{s^m} t^{s(m)} = t^{s(m)}$. Then

$$(t^{s(m)} t_1^{s^m})^{s_1 s^m} t^{s(m)} t_1^{s^m} = t^{s(m)} t_1^{s^m}.$$

From (3) we have $(t^{s(m)} t_1^{s^m})^{s_1 s^m} = t^{s(m)} t_1^{s^m}$. Thus $t^{s(m)} t_1^{s^m} t^{s(m)} = t^{s(m)}$, and then $s^m \in \mathrm{Reg}\, S$ and $t^{s(m)} \in \mathrm{Reg}\, T$. $\square$

Theorem 2. *Let S and T be two monoids and let $\alpha : S \to \mathrm{End}(T)$ be the given homomorphism, and let $S \times_\alpha T$ be the semidirect product of S and T. Then $S \times_\alpha T$ is a strongly π-inverse monoid iff*

(1) *both S and T are strongly π-inverse monoids,*

(2) *$t^e = t$ for every $t \in \mathrm{Reg}\, T$ and every $e \in E(S)$, and*

(3) *for every $s \in S$ and $t \in T$ there exists $m \in \mathbb{N}$ such that $s^m \in \mathrm{Reg}\, S$ and $t^{s(m)} \in \mathrm{Reg}\, T$, where $t^{s(m)} = t^{s^{m-1}} t^{s^{m-2}} \cdots t^s t$.*

Proof. The necessity of the assertion is obvious by Lemma 1. We only prove the sufficient part.

For every $(s,t) \in S \times_\alpha T$, from (3) there exist $m \in \mathbb{N}$, $s_1 \in S$ and $t_1 \in T$ such that

$$s^m s_1 s^m = s^m,$$
$$t^{s(m)} t_1 t^{s(m)} = t^{s(m)}.$$

From (2) we have $(t^{s(m)} t_1)^{s_1 s^m} = t^{s(m)} t_1$. Hence $(t^{s(m)})^{s_1 s^m} t_1^{s_1 s^m} t^{s(m)} = t^{s(m)}$, and then $(s,t)^m (s_1, t_1^{s_1})(s,t)^m = (s,t)^m$. This means that $S \times_\alpha T$ is π-regular.

For arbitrary $(e,u) \in E(S \times_\alpha T)$ we prove that $e \in E(S)$ and $u \in E(T)$. In fact, if $(e,u)^2 = (e,u)$, then $e^2 = e$, $u^e u = u$. Thus $u^e \in E(T)$, and then, from (3) there exists $m \in \mathbb{N}$ such that $u^{e^{m-1}} \cdots u^e u = u^e u \in \mathrm{Reg}\, T$. From (2), $(u^e u)^e = u^e u = u^e u^e = u^e = u$. So that $u^2 = u$.

Now, for $(e,u), (f,v) \in E(S \times_\alpha T)$, we have $e, f \in E(S)$ and $u, v \in E(T)$. By (1) and (2) we have

$$(e,u)(f,v) = (ef, u^f v) = fe, v^e u) = (f,v)(e,u).$$

Therefore $S \times_\alpha T$ is strongly π-inverse. $\square$

Theorem 3. *Let S and T be two monoids and let $S \times_\alpha T$ be a strongly π-inverse monoid.*

(1) *$(e,u) \in E(S \times_\alpha T)$ iff $e \in E(S)$ and $u \in E(T)$.*

(2) *For every $e \in E(S)$, let $\alpha^*(e)$ be the restriction of $\alpha(e)$ on $E(T)$; then $\alpha^*(e) \in \mathrm{End}(E(T))$.*

(3) *Let $\alpha^* : E(S) \to \mathrm{End}(E(T))$ such that $e \to \alpha^*(e)$; then α^* is a homomorphism from $E(S)$ to $\mathrm{End}(T)$.*

(4) *$E(S \times_\alpha T) \cong E(S) \times E(T) \cong E(S) \times_{\alpha^*} E(T)$.*

Proof. It is a immediate consequence of Lemma 1 and Theorem 2. $\square$

Theorem 4. *Let S and T be two monoids and let $S \times_\alpha T$ be a strongly π-inverse monoid.*

(1) *$(s,t) \in \mathrm{Reg}(S \times_\alpha T)$ iff $s \in \mathrm{Reg}\, S$ and $t \in \mathrm{Reg}\, T$.*

(2) *For every $s \in \mathrm{Reg}\, S$, let $\alpha^*(s)$ be the restriction of $\alpha(s)$ on $\mathrm{Reg}\, T$; then $\alpha^*(s) \in \mathrm{End}(\mathrm{Reg}\, T)$.*

(3) *Define* $\alpha^* : \operatorname{Reg} S \to \operatorname{End}(\operatorname{Reg} T)$ *by* $s \to \alpha^*(s)$*; then* α^* *is a homomorphism from* $\operatorname{Reg} S$ *to* $\operatorname{End}(\operatorname{Reg} T)$*.*

(4) $\operatorname{Reg}(S \times_\alpha T) \cong \operatorname{Reg}(S) \times_{\alpha^*} \operatorname{Reg}(T)$.

Proof. (1) Let $(s,t) \in \operatorname{Reg}(S \times_\alpha T)$; then there exists $(s_1, t_1) \in S \times_\alpha T$ such that $(s,t)(s_1,t_1)(s,t) = (s,t)$, and then $ss_1s = s$, $t^{s_1s}t_1^s t = t$. From the latter equation we have $(tt_1^s)^{s_1s}tt_1^s = tt_1^s$, and from Lemma 1 (3), $(tt_1^s)^{s_1s} = tt_1^s$, that is, $tt_1^s t = t$. So that $s \in \operatorname{Reg} S$ and $t \in \operatorname{Reg} T$.

Conversely, let $s \in \operatorname{Reg} S$ and $t \in \operatorname{Reg} T$; then there exist $s_1 \in S$ and $t_1 \in T$ such that $ss_1s = s$, $tt_1t = t$. Hence

$$(s,t)(s_1, t_1^{s_1})(s,t) = (ss_1s, t_1^{s_1s}t_1^{s_1s}t) = (s, tt_1t) - (s,t).$$

Therefore $(s,t) \in \operatorname{Reg}(S \times_\alpha T)$.

(2) For every $s \in \operatorname{Reg} S$ and $t \in \operatorname{Reg} T$, $t^s \in \operatorname{Reg} T$; then $\alpha^*(s) \in \operatorname{End}(\operatorname{Reg} T)$.

(3) and (4) are obvious. □

3. Least Group Congruence on a Strongly π-Inverse Monoid

Theorem 5. *Let* S *be a strongly* π*-inverse monoid; then the relation*

$$\delta = \{(s_1, s_2) \in S \times S : s_1 e = s_2 e \ \textit{for some} \ e \in E(S)\}$$

is the least group congruence on S*.*

Proof. It is obvious that δ is a left compatible equivalent relation on S. Let $xe = ye$, for $x, y \in S$ and $e \in E(S)$. For any $z \in S$, since S is a strongly π-inverse monoid, there exist $m \in \mathbb{N}$ and $s \in S$ such that $z^m s z^m = z^m$, $sz^m s = s$; and we have

$$xz(z^{m-1}sez) = xez^m sez = yez^m sez = yz(z^{m-1}sez)$$

and

$$(z^{m-1}sez)^2 = z^{m-1}sez^m sez = z^{m-1}sz^m sez = z^{m-1}sz^m sez = z^{m-1}sez.$$

Thus $(xz, yz) \in \delta$.

It is obvious that $e\delta = f\delta = 1$ is the identity of S/δ for every $e, f \in E(S)$. Now, for $s\delta \in S/\delta$ there exist $m \in \mathbb{N}$, $s_1 \in S$ such that $s^m s_1$, $s_1 s^m \in E(S)$. Thus $(s^m s_1)\delta = s\delta(s^{m-1}s_1)\delta = 1$ and $(s_1 s^m)\delta = (s_1 s^{m-1})\delta(s\delta) = 1$, so that $s\delta$ has an inverse element. This means that S/δ is a group.

Let ρ be an arbitrary group congruence on S. If $(x,y) \in \delta$, then there exists $e \in E(S)$ such that $xe = ye$, so that $(xe)\rho = (ye)\rho$. Since $e\rho = 1 \in S/\rho$, we have $x\rho = y\rho$. Hence $\delta \subset \rho$. □

Theorem 6. *Let S and T be two monoids, let $S \times_\alpha T$ be a strongly π-inverse monoid, and let $\delta_{S\times_\alpha T}$, δ_S and δ_T be the least group congruences on $S \times_\alpha T$, S and T, respectively. Then*

(1) *for every $s\delta_S \in S/\delta_S$ define $\alpha^*(s\delta_S) : T/\delta_T \to T/\delta_T$ by $t\delta_T \to t^s\delta_T$; then $\alpha^*(s\delta_S) \in \mathrm{End}(T/\delta_T)$;*

(2) *define $\alpha^* : S/\delta_S \to \mathrm{End}(T/\delta_T)$ by $s\delta_S \to \alpha^*(s\delta_S)$; then α^* is a homomorphism;*

(3) *$S/\delta_S \times_{\alpha^*} T/\delta_T \cong (\times_\alpha T)/\delta_{S\times_\alpha T}$.*

Proof. (1) If $t_1\delta_T = t_1\delta_T$ for $t_1, t_2 \in T$, then there exists $u \in E(T)$ such that $t_1u = t_2u$ and then $t_1^s u^s = t_2^s u^s$ for $s \in S$. Since $u^s \in E(T)$, we have $t_1^s\delta_T = t_2^s\delta_T$. Thus $\alpha^*(s\delta_S)$ is well defined for every $s\delta_S \in S/\delta_S$. It is easy to see that $\alpha^*(s\delta_S)$ is a homomorphism.

(2) If $s_1\delta_S = s_2\delta_S$ for $s_1, s_2 \in S$, then $s_1e = s_2e$ for $e \in E(S)$. For arbitrary $t\delta_T \in T/\delta_T$, there exists $t_1\delta_T \in T/\delta_T$ such that $(t_1t)\delta_T = 1 \in T/\delta_T$, and then $t_1tu = u$ for some $u \in E(T)$. Thus, $tut_1tu = tu$, hence $tu \in \mathrm{Reg}\, T$. From Lemma 1, $t^eu = tu$ for every $e \in E(S)$ and then $t^e\delta_T = t\delta_T$, so $\alpha^*(e\delta_S)$ is an identity mapping on T/δ_T. Thus $t^{s_1}\delta_T = t^{s_1e}\delta_T = t^{s_2e}\delta_T = t^{s_2}\delta_T$, and then $\alpha^*(s_1\delta_S) = \alpha^*(s_2\delta_S)$, so that α^* is well defined.

For arbitrary $s_1\delta_S$, $s_2\delta_S \in S/\delta_S$ and arbitrary $t\delta_T \in T/\delta_T$, we have $t^{(s_1s_2)}\delta_T = (t^{s_1})^{s_2}\delta_T$, that is, $\alpha^*(s_1s_2) = \alpha^*(s_1\delta_S)\alpha^*(s_2\delta_S)$. Thus α^* is a homomorphism from S/δ_S to $\mathrm{End}(T/\delta_T)$.

(3) Define $\varphi : (S \times_\alpha T)/\delta_{S\times_\alpha T} \longrightarrow S/\delta_S \times T/\delta_T$ by $(s,t)\delta_{S\times_\alpha T} \longrightarrow (s\delta_S, t\delta_T)$. Suppose $(s_1,t_1)\delta_{S\times_\alpha T} = (s_1,t_2)\delta_{S\times_\alpha T}$. Then there exist $(e,u) \in E(S\times_\alpha T)$ such that $(s_1,t_1)(e,u) = (s_2,t_2)(e,u)$; then $s_1e = s_2e$, $t_1^eu = t_2^eu$. So $s_1\delta_S = s_2\delta_S$, $t_1\delta_T = t_2\delta_T$, and then $(s_1\delta_S, t_1\delta_T) = (s_2\delta_S, t_2\delta_T)$. Thus φ is well defined. φ is obviously surjective.

If $(s_1\delta_S, t_1\delta_T) = (s_2\delta_S, t_1\delta_T)$, then $s_1\delta_S = s_2\delta_S$, $t_1\delta_T = t_2\delta_T$, and then there exist $e \in E(S)$ and $u \in E(T)$ such that $s_1e = s_2e$, $t_1u = t_2u$. From Lemma 1 (2), $t_1^eu = t_1^eu^e = t_2^eu^e = t_2^eu$. Hence $(s_1,t_1)(e,u) = (s_2,t_2)(e,u)$, and then $(s_1,t_1)\delta_{S\times_\alpha T} = (s_2,t_2)\delta_{S\times_\alpha T}$. Thus φ is one-to-one.

It is easy to see that φ is a homomorphism. Thus $(S \times_\alpha T)/\delta_{S\times_{\alpha^*} T} \cong \delta_S \times_{\alpha^*} T/\delta_T$. $\square$

Corollary 7. *Let S be a strongly π-inverse monoid. Then for every $s \in S$ there exist $e, f \in E(S)$ such that $se, fs \in \mathrm{Reg}\, S$.*

Proof. From the proof of Theorem 6 we know that $se \in \mathrm{Reg}\, S$ for some $e \in E(S)$. Using a similar way, we can prove that the binary relation defined on S by

$$\sigma = \big\{(s_1,s_2) : fs_1 = fs_2 \ \text{ for some } \ f \in E(S)\big\}$$

is also the least group congruence on S. Then there exists $f \in E(S)$ such that $fs \in \mathrm{Reg}\, S$, using the same method as in the proof of Theorem 6. $\square$

4. Wreath Products

Let S be a monoid, S acts on a set X from the left, that is, $sx \in X$, $1x = x$ and $s(rx) = (sr)x$ for every $s, r \in S$ and $x \in X$. Let T also be a monoid, then the wreath product $Sw_xT = S \times_\alpha T^X$, where $T^X = \{f : X \longrightarrow T$ is a function$\}$ is the Cartesian power of T, that is, $fg(x) = f(x)g(x)$ for every $f, g \in T^X$ and every $x \in X$, and where the homomorphism $\alpha : S \longrightarrow \mathrm{End}(T^X)$ is defined by $(f^s)(x) = f(sx)$ for every $s \in S$, $f \in T^X$ and $x \in X$.

Lemma 8. *Let T be a monoid and let $R = \{T' \subset T \,\big|\, |T'| \leq |X|\}$. Then T^X is a strongly π-inverse monoid iff*

(1) *T is a strongly π-inverse monoid, and*

(2) *for every $T' \in R$ there exists $m \in \mathbb{N}$ such that $(t')^m \in \mathrm{Reg}\, T$ for all $t' \in T'$.*

Proof. Suppose that T^X is strongly π-inverse and $T' \in R$. Then there exists $g \in T^X$ such that $g(X) = T'$. Let $m \in \mathbb{N}$ such that $g^m \in \mathrm{Reg}\, T^X$; then $(t')^m = (g(x))^m = g^m(x) \in \mathrm{Reg}\, T$ for all $t' \in T'$.

Now, for each $t \in T$, let $T' = \{t\}$; then there exists $m \in \mathbb{N}$ such that $t^m \in \mathrm{Reg}\, T$. Thus T is π-regular.

For every $u, v \in E(T)$, define $g : X \longrightarrow T$ by $g(x) = u$ and $h : X \longrightarrow T$ by $h(x) = v$ for all $x \in X$. Then, $g, h \in E(T^X)$ and

$$uv = g(x)h(x) = gh(x) = h(x)g(x) = vu.$$

Thus T is a strongly π-inverse monoid.

Conversely, for any $g \in T^X$, we have $g(x) \in R$, and then there exists $m \in \mathbb{N}$ such that $g^m(x) = (g(x))^m \in \mathrm{Reg}\, T$ for all $x \in X$, so that $g^m \in \mathrm{Reg}\, T^X$. Thus T^X is π-regular.

For each $g, h \in E(T^X)$ we have $g(x)$, $h(x) \in E(T)$ for all $x \in X$. Then

$$gh(x) = g(x)h(x) = h(x)g(x) = (hg)(x)$$

for all $x \in X$, and then $gh = hg$. Thus T^X is strongly π-inverse. □

Lemma 9. *Let S and T be two monoids; S acts on a set X from the left. Then the following conditions are equivalent:*

(1) *for each $e \in E(S)$ and $g \in \mathrm{Reg}\, T^X$, $g^e = g$;*

(2) *$|T| = 1$ or $ex = x$ for each $e \in E(S)$ and $x \in X$.*

Proof. Suppose that (1) holds. If there exist $e \in E(S)$ and $x \in X$ such that $ex \neq x$, then for $t' \in \mathrm{Reg}\, T$ define $g : X \longrightarrow T$ by

$$g(y) = \begin{cases} 1, & \text{if } y = ex, \\ t', & \text{if } y \neq ex. \end{cases}$$

We have $g \in \operatorname{Reg} T^X$. Hence $g^e = g$, and then $t' = g(x) = g^e(x) = g(ex) = 1$. Thus $\operatorname{Reg} T = \{1\}$. But for each $t \in T$ there exists $m \in \mathbb{N}$ such that $t^m \in \operatorname{Reg} T$. So $t^m = 1$, and then $t \in \operatorname{Reg} T$. Hence $t = 1$, therefore $|T| = 1$.

Conversely, assume (2) holds. Let $e \in E(S)$ and $g \in \operatorname{Reg} T^X$. If $|T| = 1$, then (1) holds. If $|T| \neq 1$, then $ex = x$ for $e \in E(S)$ and all $x \in X$. So $g^e(x) = g(x)$ for all $x \in X$, which means that $g^e = g$. □

Theorem 10. *Let S and T be two monoids; S acts on a set X from the left. Then the wreath product Sw_xT is a strongly π-inverse monoid iff*

(1) *S and T are strongly π-inverse monoids,*

(2) *for each subset T' of T with $|T'| \leq |X|$ there exists $m \in \mathbb{N}$ such that $(t')^m \in \operatorname{Reg} T'$ for all $t' \in T'$,*

(3) *$|T| = 1$ or $ex = x$ for every $e \in E(S)$ and all $x \in X$, and*

(4) *for each $x \in S$ and $g \in T^X$ there exists $m \in \mathbb{N}$ such that $s^m \in \operatorname{Reg} S$ and $g^{s(m)}(x) \in \operatorname{Reg} T$ for all $x \in X$, where $g^{s(m)} = g^{s^{m-1}} \cdots g^s g \in T^X$.*

Proof. It is easy to see that $g^{s(m)} \in \operatorname{Reg} T^X$ iff $g^{s(m)}(x) \in \operatorname{Reg} T$ for all $x \in X$. Thus Theorem 10 is an immediate consequence of Lemma 8, Lemma 9, and Theorem 2. □

Recall that the standard wreath product SwT of two monoids is formed by the left regular representation of S on itself, and we have

Theorem 11. *The standard wreath product SwT of two monoids S and T is a strongly π-inverse monoid iff*

(1) *both S and T are strongly π-inverse monoids,*

(2) *for each subset T' of T with $|T'| \leq |S|$ there exists $m \in \mathbb{N}$ such that $(t')^m \in \operatorname{Reg} T$ for all $t' \in T'$,*

(3) *S is a group or $|T| = 1$, and*

(4) *for every $s \in S$ and $g \in T^S$ there exists $m \in \mathbb{N}$ such that $g^{s(m)}(x) \in \operatorname{Reg} T$ for all $x \in S$.*

Acknowledgement

The authors would like to thank Professor T. Saito for his careful reading of the paper and his constructive suggestions.

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(Received 17.05.1994)

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