GEORGIAN MATHEMATICAL JOURNAL: Vol. 3, No. 3, 1996, 275-292

ALMOST PERIODIC HARMONIZABLE PROCESSES

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ABSTRACT. The class of harmonizable processes is a natural extension of the class of stationary processes. This paper provides sufficient conditions for the sample paths of harmonizable processes to be almost periodic uniformly, Stepanov and Besicovitch.

1. INTRODUCTION

The concept of almost periodic (a.p.) stationary stochastic processes was first introduced and studied by Slutsky [1], who obtained sufficient conditions for the sample paths of a stationary process to be Besicovitch or $B^2a.p.$. Later, Udagawa [2], gave conditions for the sample paths to be Stepanov or $S^2a.p.$. Kawata [3], extended these results to a very general setting and also gave conditions for uniformly a.p. (u.a.p.) sample paths.

The class of harmonizable stochastic processes provide a natural extension to the class of stationary stochastic processes. This nonstationary class of processes was first introduced by Loéve, [4] and later independently, by Rozanov [5]. These processes have been extensively studied by Rao [6], [7], [8], along with his students: Chang and Rao [9], [10], Mehlman [11] and Swift [12], [13], [14], [15]. The sample path behavior of harmonizable processes has not yet been throughly investigated, Swift [12], [15] considered the analyticity of the sample paths of harmonizable processes. This paper provides sufficient conditions for harmonizable processes to be almost periodic, similar to Kawata's in the stationary case, of which these are extensions.

The basic background and structure of harmonizable processes is outlined in the next section. The remaining sections of the paper develop the sufficient conditions of harmonizable processes to be almost periodic.

1991 *Mathematics Subject Classification.* 60H10.

Key words and phrases. Harmonizable processes, uniform, Stepanov, Besicovitch, almost periodic processes.

1072-947X/96/0500-0275$09.50/0

2. Preliminaries

To introduce the desired class of random functions, recall that if a process $X : \mathbb{R} \to L_0^2(P)$ is stationary then it can be expressed as

$$X(t) = \int\limits_{\mathbb{R}} e^{i\lambda t} dZ(\lambda), \tag{1}$$

where $Z(\cdot)$ is a σ-additive stochastic measure on the Borel sets of $\mathbb{R}$, with orthogonal values in the complex Hilbert space, $L_0^2(P)$, of centered random variables. The covariance, $r(\cdot,\cdot)$, of the process is

$$r(s,t) = \int\limits_{\mathbb{R}} e^{i(s-t)\lambda} dF(\lambda), \tag{2}$$

where $E(Z(A)\overline{Z(B)}) = F(A \cap B)$, F a bounded Borel measure on $\mathbb{R}$.

A generalization of the concept of stationarity which retains the powerful tools of Fourier analysis is given by processes $X : \mathbb{R} \to L_0^2(P)$ with covariance $r(\cdot,\cdot)$ expressible as

$$r(s,t) = \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} e^{i\lambda s - i\lambda' t} dF(\lambda, \lambda'), \tag{3}$$

where $F(\cdot,\cdot)$ is a complex bimeasure, called the *spectral bimeasure* of the process, of bounded variation in Vitali's sense or more inclusively in Fréchet's sense; in which case the integrals are strict Morse–Transue (cf. [6] of Rao). The covariance as well as the process are termed *strongly* or *weakly harmonizable* respectively. Every weakly or strongly harmonizable process $X : \mathbb{R} \to L^2(P)$ has an integral representation given by (1), where $Z : \mathcal{B} \to L^2(P)$ is a stochastic measure (not necessarily with orthogonal values) and is called the *spectral measure* of the process. Both of these concepts reduce to the stationary case if F concentrates on the diagonal $\lambda = \lambda'$ of $\mathbb{R} \times \mathbb{R}$. The interested reader is encouraged to pursue the papers by Rao and the others cited earlier.

3. Almost Periodic Harmonizable Processes

For convenient reference the classical definitions of the classes u.a.p., $S^2a.p$ and $B^2a.p$ are recalled here. The standard classical theory will be used throughout and may be found in Besicovitch's book [16].

Let $\mathcal{A}$ be the class of all finite trigonometric polynomials

$$S(t) = \sum_{k=1}^{n} a_k e^{i\lambda_k t}. \tag{4}$$

The various forms of almost periodicity are obtained by considering the following distances between two functions $f(t)$ and $\phi(t)$ from the class $\mathcal{A}$.

(i) the uniform distance:

$$D_u[f(t), \phi(t)] = \sup_{-\infty<t<\infty} |f(t) - \phi(t)|\,; \tag{5}$$

(ii) the S^2 (Stepanov) distance:

$$D_{S^2}[f(t), \phi(t)] = \sup_{-\infty<t<\infty} \left[\int_t^{t+1} \mid f(x) - \phi(x) \mid^2 dx \right]^{\frac{1}{2}}; \tag{6}$$

(iii) the B^2 (Besicovitch) distance:

$$D_{B^2}[f(t), \phi(t)] = \left[\lim_{t\to\infty} \sup \frac{1}{t} \int_0^t \mid f(x) - \phi(x) \mid^2 dx \right]^{\frac{1}{2}}. \tag{7}$$

The classical results show that these are norms on $\mathcal{A}$, and that the class of all uniformly almost periodic functions are given by the closure of $\mathcal{A}$ under $D_u[\cdot,\cdot]$. Similarly, the class of all S^2 almost periodic ($S^2 a.p$) functions, (respectively B^2 a.p.) is the closure of $\mathcal{A}$ under $D_{S^2}[\cdot,\cdot]$ (respectively $D_{B^2}[\cdot,\cdot]$).

The generalization of almost periodicity introduced by H. Weyl (cf. Besicovitch [16]), as well as the further generalizations obtained by T. Hillmann [17], for classes of Besicovitch–Orlicz almost periodic functions, (B^{Φ}, where Φ is a Young's function) are not considered in the following work. These generalizations provide further extensions and await a serious investigation.

The results that will subsequently be developed follow from assumptions on the covariance $r(\cdot,\cdot)$ being a u.a.p. function of two variables. More precisely,

Definition 3.1. A stochastic process $X : \mathbb{R} \to L_0^2(P)$ is quadratic mean uniformly almost periodic (q.m.u.a.p.) if for each $\varepsilon > 0$,

$$\left\{ \tau : \sup_{-\infty<t<\infty} E \mid X(t+\tau) - X(t) \mid^2 < \varepsilon \right\} \tag{8}$$

is relatively dense on $\mathbb{R}$.

Simiarly, we may consider q.m. S^2 a.p. processes:

Definition 3.2. A stochastic process $X : \mathbb{R} \to L_0^2(P)$ is quadratic mean S^2 almost periodic (q.m. S^2 a.p.) if for each $\varepsilon > 0$,

$$\left\{ \tau : \sup_{-\infty < t < \infty} \int_t^{t+1} E \mid X(t+\tau) - X(t) \mid^2 dt < \varepsilon \right\} \tag{9}$$

is relatively dense on $\mathbb{R}$.

The concept of a relatively dense set may be found in Besicovitch [16], and is recalled here in the following definition.

Definition 3.3. A set $E \subset \mathbb{R}$ is relatively dense if there exists a number $l > 0$ such that any interval of length l contains at least one number of E.

The following observation shows that these concepts coincide for u.a.p. covariances $r(\cdot,\cdot)$ without any further assumptions on $r(\cdot,\cdot)$.

Proposition 3.1. *$X(\cdot)$ is q.m.u.a.p. iff $r(\cdot,\cdot)$ is u.a.p.. $X(\cdot)$ is q.m. S^2 a.p. iff $r(\cdot,\cdot)$ is S^2 a.p..*

Proof. The statement $X(\cdot)$ q.m.u.a.p. implies $r(\cdot,\cdot)$ u.a.p. was recently observed by Hurd [18]. For the converse, suppose $r(\cdot,\cdot)$ is u.a.p. as a function of two variables. So, sets of the form

$$\left\{ (u,v) : \sup_{u,v \in \mathbb{R}} \mid r(s+u, t+v) - r(s,t) \mid < \varepsilon \right\}$$

are relatively dense in the plane. Further, since a u.a.p. function of two variables is u.a.p. with respect to each of the variables, the following is true. Using

$$\begin{aligned} E \mid X(t+\tau) - X(t) \mid^2 \leq & \mid r(t+\tau, t+\tau) - r(t+\tau, t) \mid + \\ & + \mid r(t, t+\tau) - r(t,t) \mid, \end{aligned}$$

for each term on the right side a relatively dense set on $\mathbb{R}$ may be chosen appropriately so that

$$\left\{ \tau : \sup_{-\infty < t < \infty} E \mid X(t+\tau) - X(t) \mid^2 < \varepsilon \right\}$$

is relatively dense on $\mathbb{R}$. Thus $X(\cdot)$ is q.m.u.a.p..

The equivalence $X(\cdot)$ q.m. S^2 a.p. $\Longleftrightarrow r(\cdot,\cdot)$ S^2 a.p. is a consequence of a classical result of Bochner's, as was noted by Kawata [3]. This finishes the proof. $\square$

Combining the concepts of uniform almost periodicity with harmonizability, the following important characterization can be given.

Theorem 3.1. *The spectral bi-measure $F(\cdot,\cdot)$ of a strongly harmonizable process has countable support iff the covariance $r(\cdot,\cdot)$ is a uniformly almost periodic function of two variables.*

Proof. If $F(\cdot,\cdot)$ has countable support $\{(\lambda_j,\lambda'_k)\}_{j,k=1}^{\infty}$, letting

$$a(\lambda,\lambda') = F(\lambda+0,\lambda'+0) - F(\lambda+0,\lambda') - F(\lambda,\lambda'+0) + F(\lambda,\lambda')$$

with $X(\cdot)$ harmonizable implies

$$r(s,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\lambda s - i\lambda' t}\, dF(\lambda,\lambda') = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} a(\lambda_j,\lambda'_k) e^{i\lambda_j s - i\lambda'_k t}.$$

Hence, by a classical approximation theorem, $r(\cdot,\cdot)$ is u.a.p. in two variables.

For the converse, if $r(\cdot,\cdot)$ was u.a.p. then the approximation theorem gives

$$r(s,t) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} a(\lambda_j,\lambda'_k) e^{i\lambda_j s - i\lambda'_k t}.$$

But, the uniqueness of Fourier transforms implies $F(\cdot,\cdot)$ has countable support $\{(\lambda_j,\lambda'_k)\}_{j,k=1}^{\infty}$. $\square$

Since the support $\{(\lambda_j,\lambda'_k)\}_{j,k=1}^{\infty}$, of $F(\cdot,\cdot)$ may have limit points in the plane, some regularity conditions on these limit points are needed and they will now be stated. The following assumptions will be in force for all subsequent analysis.

Assumption 1: Let $\{(\mu_j,\mu'_k)\}_{j,k=1}^{\infty}$ be the set of limit points of the support $\{(\lambda_j,\lambda'_k)\}_{j,k=1}^{\infty}$ of $F(\cdot,\cdot)$. It is required that

$$\inf_{k\neq j}\{\,|\,\mu_k-\mu_j\,|, |\,\mu'_k-\mu'_j\,|\,\} > 1. \tag{10}$$

The constant 1 on the right side here is chosen for simplicity; it may be replaced by any positive constant. Condition (10) implies that each semi-open square $(n,n+1]\times(m,m+1], n,m\in\mathbb{Z}$, contains at most only one limit point. Assume further that the limit points are enumerated so that $\mu_k < \mu_{k+1}$ in each strip $(k,k+1]\times\{\mu'_j\}$ with the second coordinate μ'_j ordered $\mu'_j < \mu'_{j+1}$.

Assumption 2: When $\mu_{k_n}\neq n$, let $N_n(\alpha)$ be the number of discontinuities between n and $\mu_{k_n}-\alpha$, where $0<\alpha<\mu_{k_n}-n$, and when $\mu_{k_n}\neq n+1$, let $M_n(\alpha)$ be the number of discontinuities between $\mu_{k_n}+\alpha$ and $n+1$, where $0<\alpha<n+1-\mu_{k_n}$. Suppose there is a nondecreasing function $h(\cdot)$

on $\mathbb{R}^+$ such that $h(x) \nearrow \infty$ as $x \nearrow \infty$, with $h(x) \equiv C > 0$ for $0 \leq x < 2$ and

$$N_n(\alpha) \leq h(\frac{1}{\alpha}) \text{ for } 0 < \alpha < \mu_{k_n} - n,$$
$$M_n(\alpha) \leq h(\frac{1}{\alpha}) \text{ for } 0 < \alpha < n + 1 - \mu_{k_n}$$

if $[n, n+1] \times [m, m+1]$ had no limiting point, it will be assumed that the set of discontinuities of $F(\cdot, \cdot)$ is bounded.

The following integral plays a key role in the analysis. Let

$$\Phi_{n,m} = \int_n^{n+1} \phi(|\ \lambda - \mu_{k_n}\ |) dF(\lambda, \lambda'_m) \tag{11}$$

where

$$\phi(u) = \begin{cases} g(\frac{1}{u})h(\frac{2}{u}), & \text{if } u > 0, \\ 1, & \text{if } u = 0 \end{cases}$$

with $g(\cdot)$ a nondecreasing function on $\mathbb{R}^+$ such that $\int_1^\infty dx/(xg(x)) < \infty$ and $g(x) = 1$ for $0 \leq x \leq 1$. The uniform almost periodicity of the sample paths may now be given.

Theorem 3.2. *If assumptions* 1 *and* 2 *are satisfied and if*

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \Phi_{n,m}^{\frac{1}{2}} < \infty \tag{12}$$

then $X(\cdot)$ *has almost all sample paths uniformly almost periodic.*

Proof. Let $A(\lambda) = Z(\lambda + 0) - Z(\lambda)$ and $a(\lambda, \lambda') = F(\lambda + 0, \lambda' + 0) - F(\lambda + 0, \lambda') - F(\lambda, \lambda' + 0) + F(\lambda, \lambda')$ where $Z(\cdot)$ is the stochastic measure satisfying the condition $E(Z(B_1)\overline{Z(B_2)}) = F(B_1, B_2)$, $F(\cdot, \cdot)$ a function of bounded Vitali variation. ($Z(\cdot)$ and $F(\cdot, \cdot)$ are assumed to be left continuous.) Let $J_{n,k} = \{j : \mu_{l_n} - 2^{-k} \leq \lambda_{n,j} < \mu_{l_n} - 2^{-(k+1)}, n < \lambda_{n,j}\}$. Suppose that there were infinitely many $\lambda_{n,j}$ in $[n, \mu_{l_n})$. Then

$$\sum_{n < \lambda_{n,j} < \mu_{l_n}} |a|^{\frac{1}{2}}(\lambda_{n,j}, \lambda') = \sum_{k=0}^{\infty} \sum_{j \in J_{n,k}} |a|^{\frac{1}{2}}(\lambda_{n,j}, \lambda') \leq$$
$$\leq \sum_{k=0}^{\infty} \left(\sum_{j \in J_{n,j}} |a|(\lambda_{n,j}, \lambda') \right)^{\frac{1}{2}} \left(\sum_{j \in J_{n,k}} 1 \right)^{\frac{1}{2}} \leq$$
$$\leq \sum_{k=0}^{\infty} \left(\sum_{j \in J_{n,j}} |a|(\lambda_{n,j}, \lambda') N_{l_n}(2^{-(k+1)}) \right)^{\frac{1}{2}}. \tag{13}$$

Now since $N_{l_n}(2^{-(k+1)}) \leq h(2^{k+1})$ and $2^{k+1} \leq \frac{2}{\mu_{l_n}-\lambda}$, one has $N_{l_n}(2^{-(k+1)}) \leq h(\frac{2}{\mu_{l_n}-\lambda})$ since $h(\cdot)$ is nondecreasing. But this implies, since $F(\cdot,\cdot)$ has countable support, that

$$\sum_{j\in J_{n,j}} |a|(\lambda_{n,j},\lambda')N_{l_n}(2^{-(k+1)}) \leq \int_{\mu_{l_n}-2^{-k}}^{\mu_{l_n}-2^{-(k+1)}} h(\frac{2}{\mu_{l_n}-\lambda})d|F|(\lambda,\lambda')$$

so equation (13) becomes

$$\sum_{n<\lambda_{n,j}<\mu_{l_n}} |a|^{\frac{1}{2}}(\lambda_{n,j},\lambda') \leq \sum_{k=0}^{\infty}\left(\int_{\mu_{l_n}-2^{-k}}^{\mu_{l_n}-2^{-(k+1)}} h(\frac{2}{\mu_{l_n}-\lambda})d|F|(\lambda,\lambda')\right)^{\frac{1}{2}}. \quad (14)$$

Letting

$$\eta(k)=\begin{cases} 1 & \text{for } \mu_{l_n}-2^{-k}>n,\\ 0 & \text{for } \mu_{l_n}-2^{-k}\leq n,\end{cases}$$

the right side of equation (14) becomes

$$\sum_{k=0}^{\infty}\left(\frac{g(2^k)}{g(2^k)}\eta(k)\int_{\mu_{l_n}-2^{-k}}^{\mu_{l_n}-2^{-(k+1)}} h(\frac{2}{\mu_{l_n}-\lambda})d|F|(\lambda,\lambda')\right)^{\frac{1}{2}} \leq$$

$$\leq\left(\sum_{k=0}^{\infty}\frac{1}{g(2^k)}\right)^{\frac{1}{2}}\left(\sum_{k=0}^{\infty} g(2^k)\eta(k)\int_{\mu_{l_n}-2^{-k}}^{\mu_{l_n}-2^{-(k+1)}} h(\frac{2}{\mu_{l_n}-\lambda})d|F|(\lambda,\lambda')\right)^{\frac{1}{2}}.$$

But

$$\sum_{k=0}^{\infty}\frac{1}{g(2^k)} \leq 2\int_1^{\infty}\frac{dx}{xg(x)} = C^2, \text{ (say)}$$

so that

$$\sum_{n<\lambda_{n,j}<\mu_{l_n}} |a|^{\frac{1}{2}}(\lambda_{n,j},\lambda') \leq$$

$$\leq C\left(\sum_{k=0}^{\infty}\eta(k)\int_{\mu_{l_n}-2^{-k}}^{\mu_{l_n}-2^{-(k+1)}} g(2^{2k})h(\frac{2}{\mu_{l_n}-\lambda})d|F|(\lambda,\lambda')\right)^{\frac{1}{2}} \leq$$

$$\leq C\left(\sum_{k=0}^{\infty}\eta(k)\int_{\mu_{l_n}-2^{-k}}^{\mu_{l_n}-2^{-(k+1)}} g(\frac{1}{\mu_{l_n}-\lambda})h(\frac{2}{\mu_{l_n}-\lambda})d|F|(\lambda,\lambda')\right)^{\frac{1}{2}} =$$

$$= C \left(\int_{n}^{\mu_{l_n}-0} \phi(\mu_{l_n} - \lambda)|F|(\lambda, \lambda') \right)^{\frac{1}{2}} . \tag{15}$$

A similar calculation shows that, if there are infinitely many $\lambda_{n,j}$ in $(\mu_{l_n}, n+1]$,

$$\sum_{\mu_{l_n} < \lambda_{n,j} < n+1} |a|^{\frac{1}{2}}(\lambda_{n,j}, \lambda') \le C_1 \left(\int_{\mu_{l_n}+0}^{n+1} \phi(\lambda - \mu_{l_m}) d|F|(\lambda, \lambda') \right)^{\frac{1}{2}}$$

hence

$$\sum_{n < \lambda_{n,j} < n+1} |a|^{\frac{1}{2}}(\lambda_{n,j}, \lambda') \le C' \left(\int_{n}^{n+1} \phi(| \lambda - \mu_{ln} |) d|F|(\lambda, \lambda') \right)^{\frac{1}{2}} = C' \Phi_{n,m}^{\frac{1}{2}}.$$

Summing over all support points of $F(\lambda, \lambda')$ in $[n, n+1] \times [m, m+1]$, we have

$$\sum_{m < \lambda' < m+1} \sum_{n < \lambda < n+1} |a|^{\frac{1}{2}}(\lambda, \lambda') < K \Phi_{n,m}^{\frac{1}{2}} \tag{16}$$

(using the boundedness condition in assumption 2).

Thus considering the spectral representation of the harmonizable process, which has the form $X(t) = \sum_{j,k} A(\lambda_{k,j}) e^{i\lambda_{k,j} t}$ and since $F(\cdot, \cdot)$ has countable support, one sees that the convergence of this series is in $L^2(P)$.

For the uniform almost periodicity, $\sum_{j,k} | A(\lambda_{k,j}) | < \infty$ with probability one must be shown. By the first Borel–Cantelli lemma, it suffices for this to show that for a given $\varepsilon > 0$, $\sum_{k,j} P(| A(\lambda_{k,j}) | \ge \varepsilon) < \infty$. Now

$$\begin{aligned} P(| A(\lambda_{k,j}) | \ge \varepsilon) &\le \frac{1}{\varepsilon} E(| A(\lambda_{k,j}) |) = \quad \text{(by Markov's inequality)} \\ &\le [E \mid A(\lambda_{k,j}) \mid^2]^{\frac{1}{2}} = \frac{|a|^{\frac{1}{2}}(\lambda_k, \lambda_j')}{\varepsilon}. \end{aligned}$$

But, (16) implies $\sum_{k,j} P(| A(\lambda_{k,j}) | \ge \varepsilon) \le (1/\varepsilon) \sum_k \sum_j |a|^{\frac{1}{2}}(\lambda_k, \lambda_j') < \infty$. So

$$\sum_{k,j} | A(\lambda_{k,j}) | < \infty \text{ with probability one.} \tag{17}$$

Recalling that $A(\lambda) = Z(\lambda+0) - Z(\lambda)$, so that $E(A(\lambda)\overline{A(\lambda')}) = F(\lambda+0, \lambda'+0) - F(\lambda+0, \lambda') - F(\lambda, \lambda'+0) + F(\lambda, \lambda') = a(\lambda, \lambda')$, one sets $S_{n,m}(t) = \sum_{n<\lambda<n+1} A(\lambda) e^{i\lambda t}$. By the spectral representation, converging in $L^2(P)$, this series exists in $L^2(P)$ as a sum representing $X(\cdot)$. Hence, equation (17)

implies that $S_{n,m}(\cdot)$ converges absolutely and uniformly with probability one.

Thus $S_{n,m}(t) = \sum_{n<\lambda<n+1} A(\lambda)e^{i\lambda t}$ is u.a.p. with probability one. So there is a set $\Omega_{n,m}$ with $P(\Omega_{n,m}) = 1$ such that on $\Omega_{n,m}$, $S_{n,m}(t)$ is u.a.p. Now

$$\sum_{n=0}^{\infty} \sum_{n<\lambda<n+1} E \mid A(\lambda) \mid \leq \sum_{n=0}^{\infty} \sum_{n<\lambda<n+1} |a|^{\frac{1}{2}}(\lambda, \lambda') \leq$$
$$\leq C' \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Phi_{n,m}^{\frac{1}{2}} < \infty.$$

Thus $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} S_{n,m}(t)$ is uniformly and absolutely convergent with probability one.

A similar argument applies for $\sum_{n=-\infty}^{-1} \sum_{m=-\infty}^{-1} S_{n,m}(t)$. Hence $\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} S_{n,m}(t)$ is uniformly and absolutely convergent, but each term of this series is u.a.p. for $\omega \in \bigcup_{m=-\infty}^{\infty} \bigcup_{n=-\infty}^{\infty} \Omega_{n,m}$, where $P(\bigcup_{n=-\infty}^{\infty} \bigcup_{m=-\infty}^{\infty} \Omega_{n,m})$ is one. Thus $\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} S_{n,m}(t)$ is u.a.p. with probability one. But, this series converges to $X(t)$ in $L^2(P)$, so $X(t)$ is u.a.p. with probability one. $\square$

The proof of this theorem actually showed some further aspects, in particular; under the assumptions of the previous theorem, the uniformly almost periodic strongly harmonizable process $X(\cdot)$ has an absolutely convergent Fourier series and spectral measure satisfying

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d|F|(\lambda, \lambda') < \infty.$$

4. Stepanov Almost Periodic Processes

The result for S^2 a.p. sample paths will follow from analysis done in the u.a.p. case. The following classical result (see, Kawata [3, p. 389]) is needed.

Lemma 4.1. *For any Borel measureable set E of $\mathbb{R}$ and a positive interger r, let*

$$\sigma_r(E) = C_r \int_E \frac{\sin^{2r}(\frac{t}{2})}{(\frac{t}{2})^{2r}} dt,$$

where C_r is a constant such that $\sigma_r(R) = 1$ with $\sigma_r(t) = \sigma_r((-\infty, t))$, then

$$\int_{-\infty}^{\infty} e^{i\lambda t} d\sigma_r(t) = 0 \ \text{for} \ \mid \lambda \mid \geq r.$$

This lemma implies for any sequence $\{v_j\}$ of real numbers such that $\mid v_j - v_k \mid \geq r$, $j \neq k$, the sequence $\{e^{ivt}\}$ is orthonormal with respect to $\sigma_n(E)$. The S^2 a.p. result may now be given.

Theorem 4.1. *If assumptions* 1 *and* 2 *are satisfied and if*

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \Phi_{n,m} < \infty, \tag{18}$$

then $X(\cdot)$ *has almost all sample paths* S^2*-almost periodic.*

Proof. As was noted earlier, $S_{n,m}(t)$ is u.a.p. with probability one. Let $\Omega_{n,m}$ be a set such that $S_{n,m}(t,\omega)$ is u.a.p. for $\omega \in \Omega_{n,m}$ and let

$$\Omega' = \bigcap_{m=-\infty}^{\infty} \bigcap_{n=-\infty}^{\infty} \Omega_{n,m}.$$

Consider the sequence $\{S_{2n,2m}(t)\}$ since the limit points of discontinuity satisfy assumption 1, the absolute convergence of $S_{n,m}(\cdot)$ and the lemma imply that $\{S_{2n,2m}(t)\}$ forms an orthogonal sequence with respect to $\sigma_1(E)$.

Hence, for any $y, \{S_{2n,2m}(t+y)\}$ is also an orthonormal sequence with respect to $\sigma_1(E)$. Thus for any $\omega \,\varepsilon\, \Omega'$ and N, M integers,

$$\begin{aligned} &\int_{-\infty}^{\infty} \left| \sum_{n,m=N+1}^{M} S_{2n,2m}(t+y) \right|^2 d\sigma_1(t) \leq \\ &\leq \int_{-\infty}^{\infty} \sum_{n,m=N+1}^{M} \left(\sum_{2n<\lambda<2n+1} \mid A(\lambda) \mid \right)^2 d\sigma_1(t) = \\ &= \sum_{n=N+1}^{M} \left(\sum_{2n<\lambda<2n+1} \mid A(\lambda) \mid \right)^2 . \end{aligned} \tag{19}$$

(Note that the measure $\sigma_1(\cdot)$ provided a useful aid in computation.)
So,

$$\begin{aligned} &\sup_y \int_{-\infty}^{\infty} \left| \sum_{n,m=N+1}^{M} S_{2n}(t+y) \right|^2 d\sigma_1(t) \leq \\ &\leq \sum_{n,m=N+1}^{M} \left(\sum_{2n<\lambda<2n+1} \mid A(\lambda) \mid \right)^2 . \end{aligned} \tag{20}$$

Thus,

$$E\left(\sum_{n,m=-\infty}^{\infty}\left(\sum_{2n<\lambda<2n+1} \mid A(\lambda)\mid\right)^2\right) \leq$$

$$\leq \sum_{n,m=-\infty}^{\infty}\left(\sum_{2n<\lambda<2n+1}\left(E \mid A(\lambda)\mid^2\right)^{\frac{1}{2}}\right)^2 =$$

(since the inside sum is finite, Liaponouv's and Minkowski's inequalities apply)

$$= \sum_{n,m=-\infty}^{\infty}\left(\sum_{2n<\lambda<2n+1} |a|^{\frac{1}{2}}(\lambda,\lambda')\right)^2 \leq \sum_{n,m=-\infty}^{\infty} \Phi_{2n,2m} < \infty.$$

Hence by the first Borel–Cantelli lemma,

$$\sum_{n,m=-\infty}^{\infty}\left(\sum_{2n<\lambda<2n+1} \mid A(\lambda)\mid\right)^2 < \infty \text{ with probability one.}$$

So,

$$\lim_{M,N\to\pm\infty} \sup_y \int_{-\infty}^{\infty}\left|\sum_{n,m=N+1}^{M} S_{2n,2m}(t)\right|^2 d\sigma_1(t-y) = 0 \text{ a.e.} \tag{21}$$

Using Lemma 4.1,

$$\int_{-\infty}^{\infty}\left|\sum_{n,m=N+1}^{M} S_{2n,2m}(t)\right|^2 d\sigma_1(t-y) =$$

$$= \int_{-\infty}^{\infty}\left|\sum_{n,m=N+1}^{M} S_{2n,2m}(t)\right|^2 \frac{c_1 \sin^2\left(\frac{t-y}{2}\right)}{\left(\frac{t-y}{2}\right)} dt \tag{22}$$

and the elementary results, $\int_{-\infty}^{\infty} \sin^2\left(\frac{t}{2}\right) / \left(\frac{t}{2}\right) dt = \pi$ and $\sin^2 u/u^2 \geq 4/\pi^2$ for $\mid u \mid < \varepsilon$, imply

$$\int_{-\infty}^{\infty}\left|\sum_{n,m=N+1}^{M} S_{2n,2m}(t)\right|^2 \frac{\sin^2\left(\frac{t-y}{2}\right)}{\pi\left(\frac{t-y}{2}\right)} dt \geq$$

$$\geq \frac{4}{\pi^3}\int_{y-\pi}^{y+\pi}\left|\sum_{n,m=N+1}^{M} S_{2n,2m}(t)\right|^2 dt. \tag{23}$$

This gives that

$$\lim_{M,N\to\pm\infty} \sup_{y} \int_{-\infty}^{\infty} \left| \sum_{n,m=N+1}^{M} S_{2n,2m}(t) \right|^2 dt = 0 \text{ a.e.,} \tag{24}$$

that is, $\sum_{n,m=N+1}^{M} S_{2n,2m}(t)$ converges to an S^2 a.p. function in S^2 a.p.-norm with probability one. The same calculations are valid for $\sum_{n,m=N+1}^{M} S_{2n+1,2m+1}(t)$. Thus,

$$T_N(t) = \sum_{n,m=-N}^{N} S_{n,m}(t)$$

converges in L^2 to an S^2 a.p. function, say $X_1(t)$, with probability one.

Thus, using the spectral representation of $X(\cdot)$,

$$\int_{-T}^{T} E\left|T_N(t) - X(t)\right|^2 dt =$$

$$= \int_{-T}^{T} E \left| \sum_{-\infty}^{-N+1} A(\lambda_{k,j}) e^{i\lambda_{k,j}t} + \sum_{N+1}^{\infty} A(\lambda_{k,j}) e^{i\lambda_{k,j}t} \right|^2 dt. \tag{25}$$

Interchanging the sum and expectation here, equation (25) becomes

$$\int_{-T}^{T} E\left|T_N(t) - X(t)\right|^2 dt =$$

$$= 2T \left(\int_{N+1}^{\infty} \int_{N+1}^{\infty} |\, dF(\lambda, \lambda')\,| + \int_{-\infty}^{-N} \int_{-\infty}^{-N} |\, dF(\lambda, \lambda')\,| \right).$$

So for fixed T as $N \nearrow \infty$, $\int_{-\infty}^{\infty} E|T_N(t) - X(t)|^2 dt \to 0$. Hence, $T_n(t) - X(t) \to 0$ in $L^2(P)$, so some subsequence $\{T_{n_k}(t)\}$ converges in $\Omega \times (-T,T)$ almost everywhere. That is, for $\omega \in \Omega$ fixed, $T_{n_k}(t,\omega) \to X(t,\omega)$, but this implies $X(t) = X_1(t)$ for almost all t with probability one. Hence, $X(\cdot)$ is S^2 a.p. with probability one. $\square$

5. Besicovitch Almost Periodic Processes

B^2 a.p. sample paths will now be considered. This result extends Slutsky's theorem to harmonizable processes.

Theorem 5.1. *If assumptions* 1 *and* 2 *are satisfied and*

$$\Phi_{n,m} < \infty, \tag{26}$$

for every n, m, *then* $X(\cdot)$ *has almost all sample paths* B^2 *almost periodic.*

Proof. Using the same argument to derive equation (24) with $S_{3n,3m}(\cdot)$, $S_{3n+1,3m+1}(\cdot)$ and $S_{3n+2,3m+2}(\cdot)$ and $\sigma_2(\cdot)$ in place of $\sigma_1(\cdot)$ it follows that

$$\int\limits_{y-\pi}^{y+\pi} \left| \sum_{|n|,|m|=k}^{M} S_{n,m}(t) \right|^2 dt \leq$$

$$\leq K \int\limits_{-\infty}^{\infty} \sum_{l=0}^{2} \left| \sum_{\substack{k \leq |3n+l| \leq M \\ k \leq |3m+l| \leq M}} S_{3n+l,3m+l}(t) \right|^2 d\sigma_2(t-y). \tag{27}$$

Let $A > 0$ be a fixed, large real number, and let $N \in \mathbb{Z}$ be the smallest integer so that $2N + 1 \geq A$. Writing $y = 2\nu\pi$, for $\nu = -N, \dots, N$ in equation (27) and adding, one obtains

$$\frac{1}{2A} \int\limits_{-A}^{A} \left| \sum_{|n|,|m|=k}^{M} S_{n,m}(t) \right|^2 dt \leq$$

$$\leq \frac{C}{2A} \sum_{\nu=-N}^{N} \int\limits_{-\infty}^{\infty} \sum_{l=0}^{2} \left| \sum_{\substack{k \leq |3n+l| \leq M \\ k \leq |3m+l| \leq M}} S_{3n+l,3m+l}(t) \right|^2 d\sigma_2(t-2\nu\pi). \tag{28}$$

By Lemma 4.1, $\{S_{3n+l,3m+l}(\cdot)\}, n, m = 0, 1, 2, \dots$, forms an orthogonal sequence of functions of t, for each l, with respect to $\sigma_2(\cdot)$. Using the same computations as in the proof of the previous theorem, (see equation (24)), the right-hand side of equation (28) is bounded by

$$\frac{C}{2A} \sum_{\nu=-N}^{N} \int\limits_{-\infty}^{\infty} U_{k,m}(t) d\sigma_2(t-2\nu\pi) \tag{29}$$

where

$$U_{k,m}(t) = \sum_{l=0}^{2} \sum_{\substack{k \leq |3n+l| \leq M \\ k \leq |3m+l| \leq M}} \left| S_{3n+l,3m+l}(t) \right|^2 d\sigma_2(t-2\nu\pi).$$

But by the lemma, $\sigma_2(\cdot)$ has a specific form, which implies

$$\frac{C}{2A}\sum_{\nu=-N}^{N}\int_{-\infty}^{\infty} U_{k,m}(t)d\sigma_2(t-2\nu\pi) \leq$$

$$\leq C\int_{-\infty}^{\infty} U_{k,m}\frac{\sin^4\left(\frac{t}{2}\right)}{2N}\sum_{\nu=-N}^{N}\frac{2}{(t-2\nu\pi)^4}dt. \tag{30}$$

But, a result of Kawata [3, p.394] implies

$$\frac{\sin^2 t}{2N}\sum_{\nu=-N}^{N}\frac{2}{(t-2\nu\pi)^4} \leq \begin{cases} Ct^4 & \text{for } \mid t \mid > 3N\pi, \\ \frac{C}{N} & \text{for } \mid t \mid \leq 3N\pi, \end{cases}$$

so that

$$\frac{1}{2A}\int_{-A}^{A}\left|\sum_{|n|,|m|=k}^{M} S_{n,m}(t)\right|^2 dt \leq$$

$$\leq C\int_{|t|>3N\pi}\frac{U_{k,m}(t)}{t^4}dt + \frac{C}{N}\int_{|t|>3N\pi} U_{k,m}(t)dt. \tag{31}$$

Now it was shown in the proof of the previous theorem that for $t \in (-A, A)$ as $M \to \infty$,

$$\sum_{|n|,|m|=k}^{M} S_{n,m}(t) \to X(t) - \sum_{\substack{|n|\leq k\\|m|\leq k}} S_{n,m}(t) \text{ in } L^2(P).$$

Hence, there is a sequence $M_j \in Z$ and a set $\tilde{\Omega}, P(\tilde{\Omega}) = 1$ such that

$$\sum_{|n|,|m|=k}^{M_j} S_{n,m}(t) \to X(t) - \sum_{\substack{|n|\leq k\\|m|\leq k}} S_{n,m}(t), \text{ a.e.} \tag{32}$$

This implies

$$\frac{1}{2A}\int_{-A}^{A}\left|X(t) - \sum_{\substack{|n|\leq k\\|m|\leq k}} S_{n,m}(t)\right|^2 dt \leq$$

$$\leq \lim_{M_j\to\infty}\inf\frac{1}{2A}\int_{-A}^{A}\left|\sum_{|n|,|m|=k}^{M_j} S_{n,m}(t)\right|^2 dt \leq$$

$$\leq C \int\limits_{|t|>3N\pi} \frac{U_{k,m}(t)}{t^4} dt + \frac{C}{N} \int\limits_{|t|>3N\pi} U_{k,m}(t) dt. \tag{33}$$

Now since $U_{k,m}(t) \nearrow U_{k,\infty} = \lim_{m\to\infty} U_{k,m}(t)$ it follows that

$$\sum_N \int\limits_{|t|>3N\pi} E\left(\sum_{|n|,|m|=0}^{\infty} |S_{3n+l,3m+l}(t)|^2\right) \frac{dt}{t^4} \leq$$

$$\leq \sum_N \int\limits_{|t|>3N\pi} \frac{dt}{t^4} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} |\, dF(\lambda, \lambda') \,| \leq \tilde{K} \sum_N \frac{1}{N^4} < \infty.$$

So by the first Borel–Cantelli lemma

$$\int\limits_{|t|>3N\pi} |S_{3n+l,3m+l}(t)|^2 \frac{dt}{t^4} \to 0$$

with probability one for $l = 0, 1, 2$. But, this implies

$$\int\limits_{|t|>3N\pi} U_{k,\infty}(t) \frac{dt}{t^4} \to 0, \text{ as } N \to \infty.$$

However,

$$\lim_{N\to\infty} \sup \frac{1}{N} \int\limits_{|t|\leq 3N\pi} \sum_{\substack{k\leq|3n+l| \\ k\leq|3m+l|}} |S_{3n+l,3m+l}(t)|^2 \, dt =$$

$$= \lim_{N\to\infty} \sup \sum_{\substack{k\leq|3n+l| \\ k\leq|3m+l|}} \frac{1}{N} \int\limits_{|t|\leq 3N\pi} \left| \sum_{|3n+l|<\lambda<|3n+l|+1} A(\lambda) e^{i\lambda t} \right|^2 dt =$$

$$= \lim_{N\to\infty} \sup \left(3\pi \sum_{\substack{k\leq|3n+l| \\ k\leq|3m+l|}} \sum_{|3n+l|<\lambda<|3n+l|+1} |A(\lambda)|^2 + \right.$$

$$\left. + \frac{1}{N} \int\limits_{|t|\leq 3N\pi} \sum_{\lambda\neq\lambda'} A(\lambda)\overline{A(\lambda')} e^{i(\lambda-\lambda')t} dt \right),$$

and since

$$\lim_{N\to\infty} \frac{1}{N} \int\limits_{|t|\leq 3N\pi} e^{i\lambda t} dt = 0 \text{ for } \lambda \neq 0,$$

the second term of the last equation is zero. Thus

$$\lim_{N\to\infty}\sup \frac{1}{N}\int_{|t|\le 3N\pi} U_{k,\infty}(t)dt = 3\pi \sum_{\substack{k<|n|\\ k<|m|}} \sum_{n<\lambda\le n+1} |A(\lambda)|^2. \qquad (34)$$

Combining this inequality with (31) and (32) gives

$$\lim_{A\to\infty}\sup \int_{-A}^{A} \left| X(t) - \sum_{\substack{|n|<k\\ |m|<k}} S_{n,m}(t) \right|^2 dt \le 3\pi \sum_{\substack{|n|<k\\ |m|<k}} \sum_{n<\lambda\le n+1} |A(\lambda)|^2. (35)$$

Here,

$$E\left(\sum_{n<\lambda\le n+1} |A(\lambda)|^2\right) = \int_{|\lambda'|>k}\int_{|\lambda|>k} |dF(\lambda,\lambda')|.$$

Choosing $k = k_j$ so that

$$\sum_{j=1}^{\infty} \int_{|\lambda'|>k_j}\int_{|\lambda|>k_j} |dF(\lambda,\lambda')| < \infty$$

one gets

$$\sum_{j=1}^{\infty} E\left(\sum_{n<\lambda\le n+1} |A(\lambda)|^2\right) < \infty.$$

So by the first Borel–Cantelli lemma, $\lim_{j\to\infty}\sum_{n<\lambda\le n+1} |A(\lambda)|^2 = 0$ with probability one. This implies

$$\lim_{j\to\infty}\lim_{A\to\infty}\sup \int_{-A}^{A} \left| X(t) - \sum_{\substack{|n|<k\\ |m|<k}} S_{n,m}(t) \right|^2 dt = 0,$$

with probability one. Now since $S_n(\cdot)$ is u.a.p. with probability one, it follows that $X(\cdot)$ is B^2 a.p. with probability one. $\square$

Acknowledgements

The author expresses his thanks to Professor M.M. Rao for his advice and encouragement during the work of this project. The author also expresses his gratitude to the Mathematics Department at Western Kentucky University for release time during the Spring 1994 semester, during which this work was completed.

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(Received 28.12.1994)

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