

ON KRULL DIMENSION OF ORE EXTENSIONS

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ABSTRACT. The Krull dimension of rings of skew polynomials is studied. Earlier the problem of Krull dimension was investigated only for some particular cases, namely, for Weyl algebras [2], a ring of differential operators [7,8], as well as for rings of Laurent skew polynomials [9–10].

Let R be a ring with unity and let $R[x]$ be a ring of left polynomials (i.e., polynomials with coefficients from the left to the powers of x) over R . Suppose that α is an endomorphism of R and δ is an α -differentiation of R , i.e., $\delta(a+b) = \delta(a) + \delta(b)$ and $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$ for any $a, b \in R$. Let $R[x; \alpha; \delta]$ denote the ring of left skew polynomials over R [1] (the additive group of this ring coincides with the one of $R[x]$ and the multiplication in it is defined by means of operators in $R[x]$ and the following commutation formula:

$$x \cdot a = \alpha(a)x + \delta(a), \quad a \in R. \quad (1)$$

If δ is the zero mapping of R , we use the notation $R[x; \alpha]$ for $R[x; \alpha; \delta]$.

Denote by $\text{K. dim}(A)$ the Krull dimension of a ring A in the sense of Gabriel and Rentschler (i.e., the deviation of the set of left ideals of A) [2].

Theorem 1. *Let R be a ring with unity, let α be its automorphism, and let δ be a nilpotent ($\delta^d = 0$) α -differentiation of R . Suppose that $\delta^{-i}(1_R) \neq \emptyset$ for $i = 1, 2, \dots, d - 1$. Then*

$$\text{K. dim}(R[x; \alpha; \delta]) = \text{K. dim}(R[x; \alpha]) = \text{K. dim}(R[x]).$$

Theorem 1 is a trivial consequence of the following propositions:

1991 *Mathematics Subject Classification.* 16S36.

Key words and phrases. Skew polynomial ring, α -differentiation.

Proposition 1. *Let R be a ring with unity, let α be its automorphism, and let δ be a nilpotent ($\delta^d = 0$) α -differentiation of R . If $\delta^{-i}(1_R) \neq \emptyset$ for $i = 1, 2, \dots, d - 1$, then*

$$\text{K. dim}(R[x; \alpha]) \leq \text{K. dim}(R[x; \alpha; \delta]).$$

Proposition 2. *Let α be an injective endomorphism of a ring R with unity satisfying*

$$\alpha(\mathfrak{a}) < R\alpha(\mathfrak{m}) \Rightarrow \mathfrak{a} < \mathfrak{m}$$

for any left ideals \mathfrak{a} and \mathfrak{m} of R , where

$$R\alpha(\mathfrak{m}) = \left\{ \sum_{p=1}^n \lambda_p \alpha(m_p); m_p \in \mathfrak{m}; \lambda_p \in R \right\}.$$

Then

$$\text{K. dim}(R[x]) \leq \text{K. dim}(R[x; \alpha]).$$

Proposition 3. *Let α be an automorphism of a ring R with unity. Then*

$$\text{K. dim}(R[x; \alpha; \delta]) \leq \text{K. dim}(R[x]).$$

The proofs of these propositions as well as of the other ones given in this paper are based on

Lemma 1 [2]. *Let E and F be partially ordered sets. If there exists a strictly isotonic mapping $\Phi : E \rightarrow F$, then $\text{dev } E \leq \text{dev } F$.*

To prove Proposition 1, we shall also need

Lemma 2. *Let α be an automorphism of a ring R with unity, and let δ be an α -differentiation of R . Then the condition*

$$f_1 c_1 + f_2 c_2 + \dots + f_n c_n = c, \tag{2}$$

where $f_1, f_2, \dots, f_n \in R[x; \alpha; \delta]$ and $c_1, c_2, \dots, c_n, c \in R$, implies

$$c = b_1 c_1 + b_2 c_2 + \dots + b_n c_n$$

with $b_1, b_2, \dots, b_n \in R$.

Proof. Suppose that k is the maximum of degrees of polynomials f_1, f_2, \dots, f_n . We can write these polynomials as

$$\begin{aligned} f_1 &= a_{1k}x^k + a_{1,k-1}x^{k-1} + \dots + a_{10}, \\ f_2 &= a_{2k}x^k + a_{2,k-1}x^{k-1} + \dots + a_{20}, \\ &\dots\dots\dots \\ f_n &= a_{nk}x^k + a_{n,k-1}x^{k-1} + \dots + a_{n0}, \\ a_{ip} &\in R, \quad i = 1, 2, \dots, n, \quad p = 0, 1, \dots, k. \end{aligned}$$

Here some of the a_{ip} 's may be equal to zero.

Calculate the left side of (2) (using a^α and a^δ instead of $\alpha(a)$ and $\delta(a)$).

$$\begin{aligned} f_1c_1 + f_2c_2 + \dots + f_nc_n &= \sum_{i=1}^n a_{ik}c_i^{\alpha^k} x^k + \\ &+ \sum_{i=1}^n \left(a_{ik}(c_i^{\delta\alpha^{k-1}} + c_i^{\alpha\delta\alpha^{k-2}} + \dots + c_i^{\alpha^{k-1}\delta}) + a_{i,k-1}c_i^{\alpha^{k-1}} \right) x^{k-1} + \dots + \\ &+ \sum_{i=1}^n \left(a_{ik}(c_i^{\delta^d\alpha^{k-d}} + c_i^{\delta^{d-1}\alpha\delta\alpha^{k-d-1}} + \dots + c_i^{\delta^{d-1}\alpha^{k-d}\delta} + c_i^{\delta^{d-2}\alpha\delta^2\alpha^{k-d-1}} + \right. \\ &\quad \left. + \dots + c_i^{\alpha^{k-d}\delta^d} \right) + a_{i,k-1}(c_i^{\delta^{d-1}\alpha^{k-d}} + \dots + c_i^{\alpha^{k-d}\delta^{d-1}}) + \dots + \\ &\quad + a_{id}c_i^{\alpha^d} x^d + \dots + \sum_{i=1}^n (a_{ik}c_i^{\delta^k} + a_{i,k-1}c_i^{\delta^{k-1}} + \dots + a_{i0}c_i). \end{aligned}$$

Taking into consideration (2), we have

$$\begin{aligned} \sum_{i=1}^n a_{ik}c_i^{\alpha^k} &= 0, \\ \sum_{i=1}^n \left(a_{ik}(c_i^{\delta\alpha^{k-1}} + c_i^{\alpha\delta\alpha^{k-2}} + \dots + c_i^{\alpha^{k-1}\delta} + a_{i,k-1}c_i^{\alpha^{k-1}}) \right) &= 0, \\ \dots & \\ \sum_{i=1}^n \left(a_{ik}(c_i^{\delta^{k-1}\alpha} + c_i^{\delta^{k-2}\alpha\delta} + \dots + c_i^{\alpha\delta^{k-1}}) + \right. \\ &\quad \left. + a_{i,k-1}(c_i^{\delta^{k-2}\alpha} + \dots + c_i^{\alpha\delta^{k-2}}) + \dots + a_{i1}c_i^\alpha \right) = 0, \\ \sum_{i=1}^n (a_{ik}c_i^{\delta^k} + a_{i,k-1}c_i^{\delta^{k-2}} + \dots + a_{i0}c_i) &= c. \end{aligned}$$

Using these equalities and calculating $(\sum_{i=1}^n a_{ik}c_i^{\alpha^k})^{(\alpha^{-1}\delta)^k}$, we obtain

$$\begin{aligned} 0 &= \left(\sum_{i=1}^n a_{ik}^{\alpha^{-1}} c_i^{\alpha^{k-1}} \right)^{\delta(\alpha^{-1}\delta)^{k-1}} = \left(\sum_{i=1}^n (a_{ik}^{\alpha^{-1}\delta} c_i^{\alpha^{k-1}} + a_{ik}c_i^{\alpha^{k-1}\delta}) \right)^{(\alpha^{-1}\delta)^{k-1}} = \\ &= \left(\sum_{i=1}^n (a_{ik}^{\alpha^{-1}\delta} c_i^{\alpha^{k-1}} - a_{ik}(c_i^{\delta\alpha^{k-1}} + \dots + c_i^{\alpha^{k-2}\delta\alpha}) - a_{i,k-1}c_i^{\alpha^{k-1}}) \right)^{(\alpha^{-1}\delta)^{k-1}} = \\ &= \left(\sum_{i=1}^n (a_{ik}^{\alpha^{-1}\delta\alpha^{-1}} c_i^{\alpha^{k-2}} + a_{ik}^{\alpha^{-1}\delta} c_i^{\alpha^{k-2}\delta} - a_{ik}^{\alpha^{-1}\delta} (c_i^{\delta\alpha^{k-2}} + \dots + c_i^{\alpha^{k-2}\delta}) - \right. \end{aligned}$$

$$\begin{aligned}
 & -a_{ik}(c_i^{\delta\alpha^{k-2}\delta} + \dots + c_i^{\alpha^{k-2}\delta^2}) - a_{i,k-1}c_i^{\alpha^{k-2}} - a_{i,k-1}c_i^{\alpha^{k-2}\delta})^{(\alpha^{-1}\delta)^{k-2}} = \\
 & = \left(\sum_{i=1}^n (a_{ik}^{(\alpha^{-1}\delta)^2} c_i^{\alpha^{k-2}} - a_{ik}^{\alpha^{-1}\delta} (c_i^{\delta\alpha^{k-2}} + \dots + c_i^{\alpha^{k-3}\delta\alpha}) + a_{ik}(c_i^{\delta^2\alpha^{k-2}} + \right. \\
 & \quad \left. + c_i^{\delta\alpha\delta\alpha^{k-3}} + \dots + c_i^{\alpha^{k-3}\delta^2\alpha}) + a_{i,k-1}(c_i^{\delta\alpha^{k-2}} + \dots + c_i^{\alpha^{k-3}\delta\alpha}) - \right. \\
 & \quad \left. - a_{i,k-1}c_i^{\alpha^{k-2}} + a_{i,k-2}c_i^{\alpha^{k-2}}) \right)^{(\alpha^{-1}\delta)^{k-2}} = \dots = \left(\sum_{i=1}^n (a_{ik}^{(\alpha^{-1}\delta)^{k-1}} c_i^{\alpha} - \right. \\
 & \quad \left. - a_{ik}^{(\alpha^{-1}\delta)^{k-2}} c_i^{\delta\alpha} + \dots + (-1)^{k-1} a_{ik} c_i^{\delta^{k-2}\alpha} - a_{i,k-1}^{(\alpha^{-1}\delta)^{k-2}} c_i^{\alpha} + \right. \\
 & \quad \left. + a_{i,k-1}^{(\alpha^{-1}\delta)^{k-3}} c_i^{\delta\alpha} + \dots + (-1)^{k-1} a_{i,k-1} c_i^{\delta^{k-3}\alpha} + \dots + (-1)^{k-1} a_{i1} c_i^{\alpha} \right)^{\alpha^{-1}\delta} = \\
 & = \sum_{i=1}^n (a_{ik}^{(\alpha^{-1}\delta)^k} c_i - a_{i,k-1}^{(\alpha^{-1}\delta)^{k-1}} c_i + \dots + \\
 & \quad + (-1)^{k-1} a_{i1}^{\alpha^{-1}\delta} c_i + (-1)^{k-1} (a_{ik} c_i^{\delta^k} + a_{i,k-1} c_i^{\delta^{k-1}} + \dots + a_{i1} c_i^{\delta})),
 \end{aligned}$$

and therefore

$$c = \sum_{i=1}^n \left((-1)^k (a_{ik}^{(\alpha^{-1}\delta)^k} - a_{i,k-1}^{(\alpha^{-1}\delta)^{k-1}} + \dots + (-1)^{k-1} a_{i1}^{\alpha^{-1}\delta}) + a_{i0} \right) c_i. \quad \square$$

Proof of Proposition 1. Taking into account that the Krull dimension of the ring A is equal to $\text{dev}(\text{Id } A)$, where $\text{Id } A$ is the set of left ideals of A , by Lemma 1 it suffices to construct a strictly isotonic mapping F from the set of left ideals of $R[x; \alpha]$ into the set of left ideals of $R[x; \alpha; \delta]$.

First of all, let us show by induction on k that for any $a, b \in R$ we can write $\delta^k(a \cdot b)$ as

$$\delta^k(ab) = a_k \delta^k(b) + a_{k-1} \delta^{k-1}(b) + \dots + a_1 \delta(b) + a_0 b, \quad (3)$$

where $a_0, a_1, \dots, a_k \in R$ are the coefficients found from the representation

$$x^k a = a_k x^k + a_{k-1} x^{k-1} + \dots + a_0.$$

Indeed, if $k = 1$, then

$$\delta(ab) = \alpha(a)\delta(b) + \delta(a)b \text{ and } xa = \alpha(a)x + \delta(a).$$

Suppose the validity of (3) for some natural k . Then

$$\begin{aligned}
 \delta^{k+1}(ab) &= \delta(\delta^k(ab)) = \delta(a_k \delta^k(b) + a_{k-1} \delta^{k-1}(b) + \dots + a_0 b) = \\
 &= a'_{k+1} \delta^{k+1}(b) + a'_k \delta^k(b) + \dots + a'_0 b,
 \end{aligned}$$

where $a'_{k+1} = \alpha(a_k)$, $a'_0 = \delta(a_0)$, and $a'_n = \delta(a_n) + \alpha(a_{n-1})$ for $n = 1, 2, \dots, k$.

On the other hand,

$$\begin{aligned} x^{k+1}a &= x(x^k a) = x(a_k x^k + a_{k-1} x^{k-1} + \cdots + a_0) = \\ &= \alpha(a_k) x^{k+1} + (\delta(a_k) + \alpha(a_{k-1})) x^k + \cdots + (\delta(a_1) + \alpha(a_0)) x + \delta(a_0) = \\ &= a'_{k+1} x^{k+1} + a'_k x^k + \cdots + a'_0. \end{aligned}$$

The induction is thus completed.

Now calculate $x^d a$. Suppose that

$$x^d a = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_0, \quad a_0, a_1, \dots, a_d \in R.$$

We have

$$\delta^d(ab) = a_d \delta^d(b) + a_{d-1} \delta^{d-1}(b) + \cdots + a_0 b, \quad b \in R,$$

where $a_0 = \delta^d(a)$. Since $\delta^d = 0$, we obtain

$$a_{d-1} \delta^{d-1}(b) + \cdots + a_2 \delta^2(b) + a_1 \delta(b) = 0. \quad (4)$$

Since b is an arbitrary element of R and $\delta^{-1}(1_R) \neq \emptyset$, we can assume in (4) that $b \in \delta^{-1}(R)$. Taking into account that $\delta(0) = 0$ and $\delta(1_R) = \delta(1_R \cdot 1_R) = \delta(1_R) + \delta(1_R) = 0$, we have $a_1 = 0$. Further, assuming that $b \in \delta^{-2}(1_R)$, we obtain $a_2 = 0$, and so on. Finally, we have

$$a_0 = a_1 = \cdots = a_{d-1} = 0.$$

Therefore $x^d a = a_d x^d$. Using the commutation formula (1), we easily obtain that $a_d = \alpha^d(a)$, and hence

$$x^d a = \alpha^d(a) x^d.$$

Moreover, if $p = md + q$, we can write

$$x^p a = x^q \alpha^{md}(a) x^{md}, \quad p, q, m \in \mathbb{N} \cup \{0\}. \quad (5)$$

Now we begin the construction of the mapping F .

Let I be an arbitrary left ideal of $R[x; \alpha]$. Let n be the minimum of degrees of nonzero polynomials from I . For any $k \geq n$ denote by \mathfrak{a}_k the set of highest-degree coefficients of k th polynomials from I . Thus we obtain the sequence of left ideals of R :

$$\mathfrak{a}_n, \mathfrak{a}_{n+1}, \dots$$

satisfying

$$\alpha(\mathfrak{a}_{n+i}) \subseteq \mathfrak{a}_{n+i+1} \quad \text{for } i \in \mathbb{N} \cup \{0\}. \quad (6)$$

(This means that this sequence is α -nondecreasing.)

Consider all monomials of the form

$$\alpha^{(n+i)(d-1)}(a_{n+i}) x^{(n+i)d}, \quad a_{n+i} \in \mathfrak{a}_{n+i}, \quad i \in \mathbb{N} \cup \{0\}, \quad (7)$$

and let J be the left ideal of $R[x; \alpha; \delta]$ generated by them. Define the mapping F by

$$F(I) = J,$$

and show that it is strictly isotonic.

Let us first study the structure of J . Taking into account (5) and the set of generators of J , we easily check that any polynomial from J with degree divisible by d is a monomial. Let

$$g = \alpha^{r(d-1)}(a_r)x^{dr}, \quad a_r \in \mathfrak{a}_r,$$

be any of them, and suppose that

$$g = \sum_{\lambda=1}^m f_\lambda g_\lambda, \quad f_\lambda \in R[x; \alpha; \delta],$$

where g_λ is of the form (7).

Take any $\lambda \in \{1, 2, \dots, m\}$ and let the degree of g_λ be $d \cdot k$. By (5) we can assume: if $k > r$, then $f_\lambda = 0$; if $k = r$, then the degree of f_λ is less than d ; if $k < r$, then f_λ contains only terms of the degree from the interval $[dr - dk; d(r+1) - dk]$ of natural numbers.

Suppose that

$$g_\lambda = \alpha^{k(d-1)}(a_k)x^{dk}, \quad k < r, \quad a_k \in \mathfrak{a}_k.$$

We can assume that

$$f_\lambda = c_1 x^{d(r+1)-1-dk} + \dots + c_d x^{dr-dk}, \quad c_1, c_2, \dots, c_d \in R.$$

Then using (5), we obtain

$$f_\lambda \cdot g_\lambda = (c_1 x^{d-1} + c_2 x^{d-2} + \dots + c_d) \alpha^{dr-dk} (\alpha^{k(d-1)}(a_k)) x^{dr}.$$

But

$$\alpha^{dr-dk} (\alpha^{k(d-1)}(a_k)) = \alpha^{r(d-k)}(a_k) = \alpha^{r(d-1)}(\alpha^{r-k}(a_k)).$$

By (6), $\alpha^{r-k}(a_k) \in \mathfrak{a}_r$. Therefore any product $f_\lambda \cdot g_\lambda$ such that the degree of g_λ is less than the degree of g can be replaced by the product $f'_\lambda \cdot g'_\lambda$, where the degree of f'_λ is less than d , the degree of g'_λ is equal to the degree of g , and $g'_\lambda \in J$. Hence by Lemma 2 the coefficient of g can be rewritten as

$$\sum_{\lambda=1}^m b_\lambda \alpha^{r(d-1)}(a_{r_\lambda}), \quad b_\lambda \in R.$$

Since \mathfrak{a}_r is an ideal, we conclude that

$$a = \alpha^{r(d-1)}(a) \quad \text{with} \quad a \in \mathfrak{a}_r. \quad (8)$$

Now we can show that the mapping F constructed above is strictly isotonic.

Let I_1 be any left ideal of $R[x; \alpha]$ such that $I \subset I_1$, and let $F(I_1) = J_1$. We have to show that $J \subset J_1$. By the construction of F it is clear that $J \subseteq J_1$.

Since $I_1 \supset I$, there exists a polynomial

$$h = b_k x^k + b_{k-1} x^{k-1} + \dots + b_0, \quad b_k, b_{k-1}, \dots, b_0 \in R,$$

in I_1 such that it is not an element of I . More than that we can assume that b_k cannot be obtained as a left linear combination of the highest coefficients of polynomials of k th degree from I . Indeed, if it is not the case, we can take the appropriate difference $h - \sum_{\mu} c_{\mu} f_{\mu}$ which will be an element of $I_1 \setminus I$ whose degree will be less than that of h , and so on.

Obviously, $\alpha^{k(d-1)}(b_k)x^{dk} \in J_1$. Let us show that this monomial does not belong to J . Suppose the contrary. Then by (8),

$$b_k x^k \in I$$

which contradicts the choice of h . \square

Before proving Proposition 2 note that for the first time an endomorphism of the type considered in this proposition has been studied by Lesieur [3]. He proved that the condition

$$\alpha(a) \subset R\alpha(\mathfrak{m}) \Rightarrow \mathfrak{a} \subset \mathfrak{m}$$

is equivalent to

$$\alpha(a) = \sum_{j=1}^n \lambda_j \alpha(b_j) \Rightarrow a = \sum_{j=1}^n \mu_j b_j, \quad a, \lambda_j, \mu_j, b_j \in R. \quad (9)$$

It was also shown by him that elements of the left ideal

$$\underbrace{R\alpha(R\alpha(\dots(R\alpha(\mathfrak{a})\dots))}_{n \text{ times}}$$

of R generally have the form

$$\sum_{i=1}^d \mu_i \alpha^n(a_i), \quad \mu_i \in R; \quad a_i \in \mathfrak{a}. \quad \square$$

Proof of Proposition 2. By Lemma 1 it suffices to construct a strictly isotonic mapping F from the set of left ideals of $R[x]$ into the set of left ideals of $R[x; \alpha]$.

Let I be any left ideal of $R[x]$. Let n be the minimum of the degrees of nonzero polynomials from I . Consider the nondecreasing sequence

$$\mathfrak{a}_n \subseteq \mathfrak{a}_{n+1} \subseteq \dots, \quad n \in \mathbb{N} \cup \{0\}, \quad (10)$$

of left ideals of R , where \mathfrak{a}_{n+i} ($i = 0, 1, 2, \dots$) is the set (in fact, the left ideal) of highest coefficients of all polynomials of degree $n+i$ from I . With the help of this sequence we can construct the α -nondecreasing sequence of left ideals of R :

$$\mathfrak{a}'_n \xrightarrow{\alpha} R\alpha(\mathfrak{a}'_{n+1}) \xrightarrow{\alpha} R\alpha(R\alpha(\mathfrak{a}'_{n+2})) \xrightarrow{\alpha} \dots, \quad (11)$$

where $\mathfrak{a}'_{n+i} = \underbrace{R\alpha(R\alpha(\dots(R\alpha(\mathfrak{a}_{n+i}))\dots))}_{n \text{ times}}$, $i = 0, 1, 2, \dots$

Consider all monomials of the form

$$a_{n+i}x^{n+i}, \quad a_{n+i} \in \underbrace{R\alpha(R\alpha(\dots(R\alpha(\mathfrak{a}'_{n+i}))\dots))}_{i \text{ times}}, \quad i = 0, 1, \dots, \quad (12)$$

and let J be the left ideal of $R[x; \alpha]$ generated by them.

Define the mapping F by

$$F(I) = J.$$

Let $I_1 \supset I$ be any left ideal, and let $J_1 = F(I_1)$. It follows from the construction of F that $J \subseteq J_1$.

As in proving Proposition 1, choose a polynomial

$$g = a_d x^d + \dots + a_0, \quad a_0, \dots, a_d \in R, \quad d \in \mathbb{N} \cup \{0\},$$

from I_1 which does not belong to I . Here we can also assume that the highest term of g cannot be obtained as a left linear combination of highest terms of the polynomials from g having the same degree. Obviously, $\alpha^d(a_d)x^d \in J_1$. Show that this monomial is not in J .

Suppose the contrary. Then the monomial $\alpha^d(a_d)x^d$ can be represented as the sum of the products of monomials of the form

$$c_{d-p}x^{d-p} \cdot a_p x^p, \quad c_{d-p} \in R, \quad a_p \in \underbrace{R\alpha(R\alpha(\dots(R\alpha(\mathfrak{a}_p))\dots))}_{p \text{ times}}, \quad (13)$$

where $p \leq d$ and $a_p = \sum_{i=1}^k \mu_i \alpha^p(b_i)$, $b_i \in \mathfrak{a}_p$.

If we carry out the multiplication in (13), we obtain

$$c_{d-p}x^{d-p}a_p x^p = \sum_{i=1}^k \lambda_i \alpha^d(b_i)x^d,$$

where $\lambda_i = c_{d-p}\alpha^{d-p}(\mu_i)$, $\lambda_i \in R$. Moreover, by (11) we can assume that $b_i \in \mathfrak{a}_d$ ($i = 1, 2, \dots, k$).

Thus we have

$$\alpha^d(a_d) = \sum_{j=1}^n \sum_{i=1}^k \lambda_{ij} \alpha^d(b_{ij}), \quad \lambda_{ij} \in R; \quad b_{ij} \in \mathfrak{a}_d.$$

This equality can be rewritten as

$$\alpha^d(a_d) = \sum_{l=1}^q \nu_l \alpha^d(b_l), \quad \nu_l \in R, \quad b_l \in \mathfrak{a}_d, \quad q \leq nk.$$

Using here equality (9) d times, we get

$$a_d = \sum_{l=1}^q \beta_l b_l, \quad \beta_l \in R, \quad b_l \in \mathfrak{a}_d.$$

But this contradicts the choice of g .

Thus we have proved that F is strictly isotonic. \square

Proof of Proposition 3. Let I be any left ideal of $R[x; \alpha; \delta]$. Consider all the monomials of the form

$$\alpha^{-n}(a_n)x^n, \quad a_n \in \mathfrak{a}_n, \tag{14}$$

where \mathfrak{a}_n is the left ideal of all polynomials of degree n from I .

Let J be the left ideal of $R[x]$ generated by monomials (14). Define the mapping F from the set of left ideals of $R[x; \alpha; \delta]$ into the set of left ideals of $R[x]$ by

$$F(I) = J.$$

Since α is the automorphism of R and $\alpha^{-n}(\mathfrak{a}_n) \subseteq \alpha^{-(n+1)}(\mathfrak{a}_{n+1}), \forall n \in \mathbb{N}$, it can be proved quite analogously to the proofs of Propositions 1 and 2 that F is strictly isotonic. \square

Remark 1. As an example let us show that the equality given in Proposition 3 can be strict.

Let $R = K[y]$, where K is a field of characteristic zero. Let α be the identical automorphism of R , and let δ be the partial differentiation by the variable y . Then δ is the α -differentiation of R , and thus we obtain the ring of skew polynomials $R[x; \alpha; \delta]$ which can be considered as the Weyl algebra over K . Its Krull dimension is equal to 1 [2]. On the other hand, $R[x] = K[y][x] = K[y; x]$. Therefore, the Krull dimension of $R[x]$ is 2.

Remark 2. Let R be any division ring and let α be its automorphism. Then the definition of the Krull dimension and Proposition 3 imply that

$$\text{K. dim}(R[x; \alpha; \delta]) = 1.$$

Taking into consideration the already known results concerning the Krull dimension of polynomial rings, from Theorem 1 we obtain

Corollary 1. *Let R be a left Noetherian ring with the Krull dimension n . Let α be an automorphism of R and let δ be a nilpotent ($\delta^d = 0$) α -differentiation of R . Moreover, let $\delta^{-i}(1) \neq \emptyset$, $i = 1, 2, \dots, d - 1$. Then*

$$\text{K. dim}(R[x; \alpha; \delta]) = \text{K. dim}(R[x; \alpha]) = n + 1.$$

Let R be a ring and let α and δ satisfy the conditions of Theorem 1. Denote by $R[[x; \alpha; \Delta]]$ the ring of left skew formal power series over $R[s]$ for which

$$xa = a^{\delta_0}x + a^{\delta_1}x^2 + \dots + a^{\delta_{d-1}}x^{d-1}, \quad a \in R,$$

where $\delta_0 = \alpha$, $\delta_1 = \delta\alpha, \dots, \delta_{d-1} = \delta^{d-1}\alpha$.

Proposition 4. $\text{K. dim}(R[x; \alpha; \delta]) \leq \text{K. dim}(R[[x; \alpha; \Delta]])$.

Proof. In order to construct a strictly isotonic mapping from the set of left ideals of $R[x; \alpha; \delta]$ into the set of left ideals of $R[[x; \alpha; \Delta]]$, it suffices to associate with any left ideal I of $R[x; \alpha; \delta]$ the left ideal J of $R[[x; \alpha; \Delta]]$ generated by the monomials of the form

$$\alpha^{dr}(a_d) \cdot x^{dr}; \quad a_d \in R,$$

where a_dx^d is the highest-degree term of the d th degree polynomial from I . The fact that this mapping is strictly isotonic, can be proved as above. \square

Consider now the ring $R[x_1, \dots, x_n; \alpha_1, \dots, \alpha_n; \delta_1, \dots, \delta_n]$ of left skew polynomials in n variables over R [6], where

$$\begin{aligned} a^{\alpha_i\alpha_j} &= a^{\alpha_j\alpha_i}, \quad a^{\delta_i\delta_j} = a^{\delta_j\delta_i}, \quad i, j = 1, 2, \dots, n; \\ a^{\alpha_i\delta_j} &= a^{\delta_j\alpha_i}, \quad i \neq j; \\ x_i x_j &= x_j x_i, \quad x_i a = \alpha_i(a)x_i + \delta_i(a), \quad a \in R. \end{aligned} \tag{15}$$

It is easy to show that if the endomorphisms α_i ($i = 1, 2, \dots, n$) of R and the corresponding α_i -differentiations δ_i ($i = 1, 2, \dots, n$) of R satisfy (15), then $R[x_1, \dots, x_n; \alpha_1, \dots, \alpha_n; \delta_1, \dots, \delta_n]$ can be represented as the ring of left skew polynomials in one variable $A_{n-1}[x_n; \bar{\alpha}_n; \bar{\delta}_n]$ over $A_{n-1} = R[x_1, \dots, x_{n-1}; \alpha_1, \dots, \alpha_{n-1}; \delta_1, \dots, \delta_{n-1}]$, where the mappings $\bar{\alpha}_n$ and $\bar{\delta}_n$ are defined as follows: if

$$f = \sum_{\nu} a_{\nu} x_1^{\nu_1} \dots x_{n-1}^{\nu_{n-1}} \in A_{n-1}, \quad a_{\nu} \in R,$$

then

$$\bar{\alpha}_n(f) = \sum_{\nu} \alpha_n(a_{\nu}) x_1^{\nu_1} \dots x_{n-1}^{\nu_{n-1}}$$

and

$$\bar{\delta}_n(f) = \sum_{\nu} \delta_n(a_{\nu}) x_1^{\nu_1} \dots x_{n-1}^{\nu_{n-1}}$$

(the fact that $\bar{\delta}_n$ is an $\bar{\alpha}_n$ -differentiation of A_{n-1} can be checked by direct calculation). If α_n is an automorphism of R , then $\bar{\alpha}_n$ is an automorphism of A_{n-1} . This enables us to generalize Proposition 3. \square

Proposition 5. *Let α_i ($i = 1, 2, \dots, n$) be automorphisms of R . Then*

$$\text{K. dim}(R[x_1, \dots, x_n; \alpha_1, \dots, \alpha_n; \delta_1, \dots, \delta_n]) \leq \text{K. dim}(R[x_1, \dots, x_n]).$$

Taking into account that if δ_n is a nilpotent α_n -differentiation of R , then $\bar{\delta}_n$ is a nilpotent $\bar{\alpha}_n$ -differentiation of A_{n-1} from Theorem 1 we obtain

Theorem 2. *Let α_i ($i = 1, 2, \dots, n$) be automorphisms of R , and let δ_i ($i = 1, 2, \dots, n$) be nilpotent ($\delta_i^{d_i} = 0$) α_i -differentiations of R such that $\delta_i^{-k_i}(1) \neq \emptyset$ for $k_i = 1, 2, \dots, d_i - 1$. Then*

$$\begin{aligned} \text{K. dim}(R[x_1, \dots, x_n; \alpha_1, \dots, \alpha_n; \delta_1, \dots, \delta_n]) = \\ \text{K. dim}(R[x_1, \dots, x_n; \alpha_1, \dots, \alpha_n]) = \text{K. dim}(R[x_1, \dots, x_n]). \end{aligned}$$

If, in addition, R is a left Noetherian ring with finite Krull dimension, then

$$\text{K. dim}(R[x_1, \dots, x_n; \alpha_1, \dots, \alpha_n; \delta_1, \dots, \delta_n]) = \text{K. dim } R + n.$$

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(Received 02.09.1994)

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