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THE WEIGHTED BMO CONDITION AND A CONSTRUCTIVE DESCRIPTION OF CLASSES OF ANALYTIC FUNCTIONS SATISFYING THIS CONDITION

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Abstract. The problem of local polynomial approximation of analytic functions prescribed in finite domains with a quasiconformal boundary is investigated in weighted plane integral metrics; a constructive description of the class of analytic functions satisfying a weak version of the known BMO condition is obtained.

The First results of the investigation of the problem (formulated by V. I. Belyi) dealing with a local polynomial approximation of analytic functions prescribed in finite domains with quasiconformal boundary have been described in [1, 2] for weighted plane integral metrics. This problem is investigated in [3] for the nonweighted case, where a constructive description of Hölder classes as well as of some other classes of analytic functions has been obtained. In the present paper we continue the investigation of the above-mentioned problem for the weighted case; moreover, a constructive description of one more class of analytic functions in weighted plane integral metrics is obtained.

1. Notation and Definitions. The Basic Results

Let G be the domain with a quasiconformal boundary $\partial G = \Gamma$, and let $y = y(\zeta)-$ be a quasiconformal reflection across the curve Γ [4]. We will be concerned only with the special, so-called canonical, quasiconformal reflection (see relations (2.1) and (2.2)). Let w be some weight function (i.e., nonnegative and measurable) defined in the domain Γ. Let us introduce the

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notation

$$H'(G)=\{f: f \ \text{ holomorphic in } \ G\},$$
$$L_p(G,w)=\{f:|f|^p w\in L_1(G)\},\quad H'_p(G,w)=L_p(G,w)\cap H'(G)\quad (p\geq 1).$$

Furthermore, let σ be the plane Lebesgue measure, and let μ be the Borel measure defined by the equality

$$\mu(E)=\iint\limits_E w(z)d\sigma_z \quad (E\subset G). \tag{1.1}$$

The integral with respect to the measure μ of the function f will be denoted by the symbols

$$\iint\limits_E f(z)d\mu_z=\iint\limits_E fd\mu.$$

If $\mu=\sigma$, then (in cases where this does not cause misunderstanding) we shall use the brief notation

$$\iint\limits_E f(z)d\sigma_z=\iint\limits_E f.$$

Let $Q=Q(z,r)$ be an open square with center at the point z, whose sides are parallel to the coordinate axes and have the length r, and let

$$F(G)=\{Q=Q(z,r):z\in\Gamma,\quad r>0\}.$$

For a square Q denote $|Q|=\sigma(Q)$.

Let the function f and the weight function w be defined in the domain G, let the measure μ be defined by equality (1.1), and

$$f_{\mu,Q\cap G}=\frac{1}{\mu(Q\cap G)}\iint\limits_{Q\cap G} fd\mu. \tag{1.2}$$

We say that f satisfies the weighted BMO condition $\mathrm{BMO}_p(G,w)$ (briefly, $f\in\mathrm{BMO}_p(G,w)$), if

$$\sup_{Q\in F(G)}\left(\frac{1}{\mu(Q\cap G)}\iint\limits_{Q\cap G}|f-f_{\mu,Q\cap G}|^p d\mu\right)^{\frac{1}{p}}\overset{df}{=}\|f\|_{BMO_p(G,w)}<\infty. \tag{1.3}$$

When $p=1$ and $w(z)=1$ everywhere in G, we shall use the usual notation $f\in\mathrm{BMO}(G)$ and $\|f\|_{\mathrm{BMO}(G)}$ respectively.

The $\mathrm{BMO}(G)$ condition is a weaker analogue of the well-known BMO (bounded mean oscillation) condition (see, e.g., [5, Ch. VI]).

Next, we say that the weight function w given in the domain G satisfies the condition $A_p(F(G))$ $(1 < p < \infty)$ (briefly $w \in A_p(F(G))$), if (assuming $0 \cdot \infty = 0$)

$$\sup_{Q \in F(G)} \left(\frac{1}{|Q \cap G|} \iint\limits_{Q \cap G} w \right) \left(\frac{1}{|Q \cap G|} \iint\limits_{Q \cap G} w^{-\frac{1}{p-1}} \right)^{p-1} < \infty.$$

The condition $A_p(F(G))$, introduced for the first time in [6] (with the unit circle as G), is a weaker analogue of the well-known Muckenhoupt condition (A_p) [7].

Let $z_0 \in \Gamma$, $\rho_n(z_0)$ $(n \in N)$ be the distance from the point z_0 to the external level line $\Gamma_{1+1/n}$ of G, $u(z_0, r) = \{z : |z - z_0| < r\}$, $c_0 > 0$. Set

$$\begin{aligned} G(z_0, c_0) &= \big\{ z \in G : \ |\zeta - z| \ge c_0 |\zeta - z_0| \quad \forall \zeta \in CG \big\}, \\ G_n(z_0, c_0) &= G(z_0, c_0) \cup \big\{ u(z_0, \rho_n(z_0)) \cap G \big\}. \end{aligned}$$

The set $G(z_0, c_0)$ is a kind of a "nontangential" subset of G with the vertex at the point z_0.

In the sequel, for brevity we shall write

$$\Big(\iint\limits_{G_n(z_0, c_0)} |f(z) - P_n(z)|^p w(z) d\sigma_z \Big)^{\frac{1}{p}} \stackrel{df}{=} \|f - P_n\|_{z_0, p, w}.$$

Let us now formulate the basic results in which G denotes a finite domain with a quasiconformal boundary Γ, the weight function $w \in A_p(F(G))$ $(1 < p < \infty)$, and μ is the measure defined by equality (1.1).

Theorem 1. *For the function f to belong to the class* $\mathrm{BMO}_p(G, w) \cap H'(G)$ (*neglecting its values on the set of measure zero*), *it is necessary and sufficient that a sequence of algebraic polynomials P_n of order not higher than n exist such that for all $z_0 \in \Gamma$ and $n \in N$ the relation*

$$\|f - P_n\|_{z_0, p, w} \le c(c_0) \Big(\mu\{u(z_0, \rho_n(z_0)) \cap G\} \Big)^{\frac{1}{p}} \tag{1.4}$$

holda, where the constant $c(c_0)$ does not depend on z_0 and n.

Theorem 2. *Let $f \in H'(G)$. The following conditions are equivalent:*
(a) $f \in \mathrm{BMO}(G)$;
(b) $f \in \mathrm{BMO}_p(G, w)$.

This theorem is an analogue of the well-known John and Nirenberg theorem (see, e.g., [5]).

Obviously, Theorem 2 allows us to formulate Theorem 1 as follows:

Theorem 1*. *Theorem 1 remains valid when the class* $\mathrm{BMO}_p(G)$ *is replaced by the class* $\mathrm{BMO}(G)$.

Thus we have given the constructive description of the class of functions $\mathrm{BMO}(G) \cap H'(G)$ in the weighted plane integral metrics.

2. Auxiliary Results

Let G be the domain with a quasiconformal boundary Γ, and let $0 \in G$, $y = y(\zeta)$ be a quasiconformal reflection across the curve Γ [4]. As follows from the Ahlfors theorem [4] (see also [8]), the reflection $y = y(\zeta)$ can always be chosen to be canonical in the sense that it is differentiable for $\zeta \notin \Gamma$, and for any fixed sufficiently small $\delta > 0$ it will satisfy the relations

$$|y_{\bar{\zeta}}(\zeta)| \asymp M, \;\; |y_{\zeta}(\zeta)| \preccurlyeq M, \;\; \delta < |\zeta| < 1/\delta, \;\; \zeta \neq \Gamma, \tag{2.1}$$

$$|y_{\bar{\zeta}}(\zeta)| \preccurlyeq M|\zeta|^{-2}, \;\; |y(\zeta) \asymp |\; M|\zeta|^{-1}, \;\; |\zeta| \leq \delta, \;\; |\zeta| \geq \frac{1}{\delta}, \tag{2.2}$$

where $M = M(\delta, \Gamma)$ is a constant depending only on δ and Γ.

The symbol $A \preccurlyeq B$ for the numbers A and B depending on some parameters denotes that $A \leq cB$, where $c = \text{const} > 0$ does not depend on those parameters; the symbol $A \succcurlyeq B$ means that $B \preccurlyeq A$; $A \asymp B$ if simultaneously $A \preccurlyeq B$ and $A \succcurlyeq B$.

Let $w \in A_p(F(G))$ $(1 < p < \infty)$, and let $y = y(\zeta)$ be a canonical quasiconformal reflection across the curve $\Gamma = \partial G$. Let us introduce the notation

$$w^*(z) = \begin{cases} w(z) & \text{for } \; z \in G, \\ w(y(z)) & \text{for } \; z \notin G, \end{cases} \quad \mu^*(E) = \iint\limits_E w^* d\sigma. \tag{2.3}$$

It is clear that if $E \subset G$, then $\mu(E) = \mu^*(E)$. Suppose further that

$$\begin{gathered} \rho(E_1, E_2) = \inf \big\{ |z_1 - z_2| : \; z_1 \in E_1, \; z_2 \in E_2 \big\}, \\ \operatorname{diam} E = \sup \big\{ |z_1 - z_2| : \; z_1, \; z_2 \in E \big\}, \\ F(G,k) = \big\{ Q : \; \operatorname{diam} Q \geq k\rho(Q, \Gamma), \; Q \cap G \neq \varnothing \big\} \;\; (k > 0). \end{gathered} \tag{2.4}$$

Let $w \in A_p(F(G))$ $(1 < p < \infty)$, and let $w^*(z)$ be the function defined by equality (2.3). Then owing to relations (2.1) and (2.2), we can conclude that for all $Q \in F(G,k)$, $\operatorname{diam} Q < k_0$ $\;(k_0, k > 0$ are arbitrary fixed numbers) the inequality

$$\bigg(\frac{1}{|Q|} \iint\limits_Q w^* \bigg) \bigg(\frac{1}{|Q|} \iint\limits_Q w^{*-\frac{1}{p-1}} \bigg)^{p-1} \leq c(k, k_0) < \infty \tag{2.5}$$

holds, where $c(k, k_0)$ is a constant independent of Q.

Lemma 1 ([9], [10]). *Let $w \in A_p(F(G))$, and let $w^*(z)$ be the function defined by equality* (2.3). *There exist numbers $0 < \delta = \delta(k,k_0) < 1$, $0 < \varepsilon = \varepsilon(k,k_0) < 1$ such that for every $e \subset Q$ the inequality $|e| < \delta|Q|$ implies*

$$\iint_e w^* < \varepsilon \iint_Q w^*$$

for all $Q \in F(G,k)$, $\operatorname{diam} Q < k_0$.

Next, by virtue of the Hölder inequality and relation (2.5), we find that for all $Q(z,r)$, $Q(z,R) \in F(G,k)$ $(0 < r \le R \le k_0 < \infty)$ the inequality

$$\left(\frac{r}{R}\right)^2 \le c(k,k_0)\frac{\mu^*(Q(z,r))}{\mu^*(Q(z,R))} \tag{2.6}$$

holds, where μ^* is defined by equality (2.3).

In particular, it follows from (2.6) that μ^* satisfies the known "doubling" condition

$$\mu^*(2Q) \preccurlyeq \mu^*(Q), \quad (Q \in F(G,k), \quad \operatorname{diam} Q < k_0). \tag{2.7}$$

Let us now prove that for all squares $Q \in F(G)$ $(\operatorname{diam} Q < k_0)$ the relation

$$\mu(Q \cap G) \succcurlyeq \mu^*(Q) \tag{2.8}$$

holds, where μ and μ^* are defined respectively by equalities (1.1) and (2.3).

Indeed, let $Q = Q(z_0,r)$ $(z_0 \in \Gamma,\, r > 0)$, $\operatorname{diam} Q < k_0$. Owing to relations (2.1) and (2.2), we get

$$\mu^*(Q(z_0,(1/M)r) \cap CG) \preccurlyeq c(M)\mu^*(Q(z_0,r) \cap G) = c(M)\mu(Q \cap G),$$

where $M > 1$ is the constant from (2.1) and (2.2), and CG is the complement to the domain G. But then, using the "doubling" condition (2.7), we get

$$\begin{aligned}\mu^*(Q) &\preccurlyeq \mu^*(Q(z_0,(1/M)r) \cap G) + \mu^*(Q(z_0,(1/M)r) \cap CG) \preccurlyeq \\ &\preccurlyeq c(M)\mu^*(Q(z_0,r) \cap G) \preccurlyeq \mu(Q \cap G). \qquad \square\end{aligned}$$

Let w be a weight function, and let μ^* be the measure defined by equality (2.3). Let f be a function given in the domain G, and let Q be a square. Introduce the notation

$$f^*(z) = \begin{cases} f(z) & \text{for} \quad z \in G, \\ f(y(z)) & \text{for} \quad z \notin G, \end{cases} \qquad f^*_{\mu^*,Q} = \frac{1}{\mu^*(Q)} \iint_Q f^* d\mu^*. \tag{2.9}$$

In the case of the Lebesgue measure σ, we shall use f^*_Q instead of $f^*_{\sigma^*,Q}$.

Lemma 2. *Let w be some weight function, $p > 1$, $f \in \mathrm{BMO}_p(G, w)$. Then for all squares $Q \in F(G, k)$, $\operatorname{diam} Q < k_0$ ($k, k_0 > 0$ are fixed numbers) the relation*

$$\left(\frac{1}{\mu^*(Q)} \iint\limits_{Q} |f^* - f^*_{\mu^*,Q}|^p d\mu^*\right)^{\frac{1}{p}} \leq c^* \|f\|_{\mathrm{BMO}_p(G,w)} \tag{2.10}$$

holds, where c^* is a constant independent of Q, p, f, and w.

Proof. Assume first that $Q \in F(G)$, $\operatorname{diam} Q < k_0$. Let $M > 1$ be the number from relations (2.1) and (2.2), and let MQ be the square obtained by an M-fold increase of the square Q. It follows from the "doubling" condition (2.7) that

$$\mu^*(Q) \succcurlyeq c(M)\mu^*(MQ) \geq c(M)\mu(MQ \cap G). \tag{2.11}$$

On account of relations (2.1) and (2.2) we obtain

$$\left(\iint\limits_{Q \cap CG} |f^* - f_{\mu,MQ}|^p d\mu^*\right)^{\frac{1}{p}} \preccurlyeq M^2 \left(\iint\limits_{MQ \cap G} |f - f_{\mu,MQ}|^p d\mu\right)^{\frac{1}{p}}. \tag{2.12}$$

Hence, using the Minkowsky inequality and relations (2.11), (2.12), and (1.3), we obtain

$$\left(\frac{1}{\mu^*(Q)} \iint\limits_{Q} |f^* - f^*_{\mu^*,Q}|^p d\mu^*\right)^{\frac{1}{p}} \preccurlyeq \left(\frac{2}{\mu^*(Q)} \iint\limits_{Q} |f^* - f_{\mu,MQ\cap G}|^p d\mu^*\right)^{\frac{1}{p}} \preccurlyeq$$

$$\preccurlyeq \left(\frac{M^2+1}{c(M)} \frac{1}{\mu(MQ \cap G)} \iint\limits_{MQ \cap G} |f - f_{\mu,MQ\cap G}|^p d\mu\right)^{\frac{1}{p}} \preccurlyeq c^* \|f\|_{\mathrm{BMO}_p(G,w)}.$$

Thus we have proved that inequality (2.10) is true for all $Q \in F(G)$, $\operatorname{diam} Q < k_0$. Using the "doubling" condition (2.7), it is not difficult to show that (2.10) holds for all $Q \in F(G, k)$ as well. $\square$

Lemma 3. *Let $f \in \mathrm{BMO}(G)$ be an analytic function in the domain G, $\zeta \in G$, $Q = Q(\zeta, a) \notin F(G, k)$ $(k \leq 1)$, $Q \subset G$. Then*

$$|f(\zeta) - f(z)| \leq 4\|f\|_{\mathrm{BMO}(G)} \quad \forall (z \in Q(\zeta, a)).$$

Proof. Since $Q = Q(\zeta, a) \notin F(G, k)$ $(k \leq 1)$, it follows from the definition of the set $F(G, k)$ (see (2.4)) that $\operatorname{diam} Q < k\rho(Q, \Gamma)$ $(k \leq 1)$. Hence

$$|\zeta - z| \leq \frac{1}{2} \operatorname{diam} Q < \frac{1}{2} k\rho(Q, \Gamma) < \frac{1}{2} k\rho(\zeta, \Gamma) < \frac{1}{2}\rho(\zeta, \Gamma)$$

for all $z \in Q$. Then assuming for brevity that $\rho(\zeta, \Gamma) = \rho$, owing to the mean value theorem and the condition $f \in \mathrm{BMO}(G)$, we obtain

$$|f(z) - f(\zeta)| \leq \frac{1}{|u(z,\rho/2)|} \iint\limits_{u(z,\rho/2)} |f(\xi) - f(\zeta)| d\sigma_\xi \leq$$

$$\leq \frac{4}{|u(\zeta,\rho)}| \iint\limits_{u(\zeta,\rho)} |f(\xi) - f_{u(\zeta,\rho)}| d\sigma_\xi \leq 4\|f\|_{\mathrm{BMO}(G)}.$$

for all $z \in Q(\zeta, a)$. $\square$

Lemma 4. *Let $f \in \mathrm{BMO}(G)$ be an analytic function in the domain G, and let f^* and f_Q^* be defined by equalities* (2.9) (*the case $\mu = \sigma$*), *$Q \in F(G,k)$ ($k \leq 1$), $\alpha > c^*\|f\|_{\mathrm{BMO}(G)}$ (c^* is a constant from* (2.10)). *Then there exists at most a countable set of nonintersecting squares $A = \{Q_j\}$ such that $Q_j \in F(G,k)$, and*

(1) $|f^*(z) - f_Q^*| \leq 12\alpha \quad \forall (z \in Q \backslash Q_j,\ Q_j \in A);$ (2.13)

(2) $\alpha \leq \frac{1}{|Q_j|} \iint_{Q_j} |f^*(z) - f_Q^*| d\sigma_z < 4\alpha \quad \forall (Q_j \in A);$ (2.14)

(3) $\sum\limits_{Q_j \in A} |Q_j| \leq \frac{1}{\alpha} \|f\|_{\mathrm{BMO}(G)} |Q|.$ (2.15)

Proof. It is obvious that the conditions (2.10) and $\alpha > c^*\|f\|_{\mathrm{BMO}(G)}$ yield

$$\frac{1}{|Q|} \iint\limits_{Q} |f^*(z) - f_Q^*| d\sigma_z < \alpha.$$

Let Q^* be the square obtained by partitioning Q into four equal squares.

In the case $Q^* \in F(G,k)$ we insert Q^* in A if $\iint_{Q^*} |f(z) - f_Q| d\sigma_z \geq \alpha |Q^*|$, while when the opposite inequality holds we again partition Q^* in four equal squares and argue as above.

Let us show that the squares $Q_j \in A$ obtained in such a way satisfy all the requirements of Lemma 2.

Let $\zeta \in \{Q \cap G\} \backslash \cup \{Q_j : Q_j \in A\}$. Then, obviously, there exist the squares Q_1 and Q_2 from the above-mentioned partitioning such that $\zeta \in Q_1 \subset Q_2$, $Q_1 \notin F(G,k)$, $Q_2 \in F(G,k)$, and

$$\frac{1}{|Q_2|} \iint\limits_{Q_2} |f^*(z) - f_Q^*| < \alpha,$$

whence it follows that

$$\frac{1}{|Q_1|}\iint\limits_{Q_1}|f^*(z)-f^*_Q|d\sigma_z<\frac{4}{|Q_2|}\iint\limits_{Q_2}|f^*(z)-f^*_Q|d\sigma_z<4\alpha.$$

Then, denoting by z_1 the center of the square Q_1 and using Lemma 3 and the mean value theorem, we obtain

$$\begin{aligned}&|f(\zeta)-f_{Q_1}|\le|f(\zeta)-f(z_1)|+|f(z_1)-f_{Q_1}|\le\\&\le 4\|f\|_{\mathrm{BMO}(G)}+\frac{2}{|Q_1|}\iint\limits_{Q_1}|f(z)-f_{Q_1}|d\sigma_z\le 12\alpha.\end{aligned}$$

Thus the validity of the first requirement of Lemma 2 is proved.

Further, it is evident that the left-hand side of the "double" inequality (2.14) holds for all $Q_j\in A$. Let us show that the right-side of that inequality is also valid.

Let Q^*_j be a square whose partitioning into four equal squares gives the square $Q_j\in A$. Clearly, $Q^*_j\supset Q_j$, and

$$\frac{1}{|Q^*_j|}\iint\limits_{Q^*_j}|f^*(z)-f^*_Q|d\sigma_z<\alpha.$$

Taking into account the above inequality, we obtain

$$\frac{1}{|Q_j|}\iint\limits_{Q_j}|f^*(z)-f^*_Q|d\sigma_z<\frac{4}{|Q^*_j|}\iint\limits_{Q^*_j}|f^*(z)-f^*_Q|d\sigma_z<4\alpha.$$

Thus relation (2.14) is proved.

Finally, using the already proven relation (2.14) and inequality (2.10) (the case where $\mu=\sigma$ is the Lebesgue measure), we get

$$\begin{aligned}&|\bigcup_A Q_j|=\sum_{Q_j\in A}|Q_j|\le\frac{1}{\alpha}\sum_{Q_j\in A}\iint\limits_{Q_j}|f^*(z)-f^*_Q|d\sigma_z\le\\&\le\frac{1}{\alpha}\iint\limits_{Q}|f^*(z)-f^*_Q|d\sigma_z\le\frac{1}{\alpha}|Q|\,\|f\|_{\mathrm{BMO}(G)}.\qquad\square\end{aligned}$$

The proof of the following lemma can be found in [11]. Let us formulate it in a way convenient for us.

Lemma 5. *Let G be a finite domain with a quasiconformal boundary Γ, $z_0\in\Gamma$, $n,m\in N$, $n>m$. Then*

$$\left(\frac{m}{n}\right)^2\preccurlyeq\frac{\rho_n(z_0)}{\rho_m(z_0)}\preccurlyeq\left(\frac{m}{n}\right)^\beta,\tag{2.16}$$

where $\beta = \beta(G) > 0$ is a constant depending only on G.

In particular, from relation (2.16) we obtain the known inequality

$$\rho_n(z_0) \succcurlyeq \Big(\frac{1}{n}\Big)^2. \tag{2.17}$$

Lemma 6. *Let G be a finite domain with a quasi-conformal boundary Γ, $p > 1$, $w \in A_p(F(G))$, $z_0 \in \Gamma$, $u(z_0, r) = \{z : |z - z_0| < r\}$, let μ be a measure defined by equality* (1.1), *and let $\{\Pi_n(z)\}_{n=1}^{\infty}$ be a sequence of algebraic polynomials of order not higher than n such that*

$$\|\Pi_n\|_{z_0,p,w} \le c_1 \big(\mu\{u(z_0, \rho_n(z_0))\}\big)^{\frac{1}{p}},$$

where c_1 is a constant not depending on z_0 and n.

Then for all $z \in u(z_0, \rho_n(z_0))$ we have the inequality

$$|\Pi_n'(z)| \le c_2 |\rho_n(z_0)|^{-1},$$

where c_2 is a constant not depending on z_0 and n.

This lemma is the analogue of the well-known theorem on the derivatives of algebraic polynomials [12, p.420], [13] which can be proved analogously to the result of [3, p.14].

3. Proofs of the Basic Results

Proof of Theorem 1. For brevity we shall use the notation $\mu\{u(z_0, t) \cap G\} = \mu(z_0, t)$ $(t \ge 0)$.

Let us prove first the necessity. Assume that $f \in \mathrm{BMO}_p(G, w) \cap H'(G)$ and let us show that relation (1.4) holds.

Let $n \in N$, $z_0 \in \Gamma$, $Q = Q(z_0, \rho_n(z_0))$, and μ and μ^* be the measures defined by the equalities (1.1) and (2.3). Relations (2.8), (2.6), and (2.17) yield

$$\mu(Q \cap G) \succcurlyeq \mu(Q^*) \succcurlyeq [\rho_n(z_0)]^2 \succcurlyeq \big(\frac{1}{n}\big)^4,$$

but then, obviously, we shall have

$$f_{\mu, Q\cap G} \preccurlyeq \frac{1}{\mu(Q \cap G)} \iint\limits_G |f| d\mu \preccurlyeq c(f, \mu) \big(\frac{1}{n}\big)^{-4}. \tag{3.1}$$

Clearly, $f \in H_p'(G, w)$. But then, repeating the arguments (and taking into account (3.1)) cited in [2, pp. 174, 182], we can see that there exists a sequence of algebraic polynomials P_n of order not higher than n, such that

$$\|f-P_n\|_{z_0,p,w}\le c\rho_n(z_0)\Big(\mu(z_0,\rho_n(z_0))\Big)^{\frac{1}{p}}\times$$
$$\times\int\limits_{\rho_n(z_0)}^{\infty}\frac{\sigma_p(f-f_{\mu,Q\cap G},w,z_0,t)}{t^2\mu^{1/p}(z_0,t)}dt. \tag{3.2}$$

Now let us estimate the value $\sigma_p(f-f_{\mu,Q\cap G},w,z_0,t)$ for $t\ge\rho_n(z_0)$.

Let $Q_m=Q(z_0,2^m\rho_n(z_0))$ $(m\in N)$, $Q_0=Q$. Using relations (2.8) and (2.3) for all $m\ge 1$, we get

$$\mu(Q_{m-1}\cap G)\succcurlyeq\mu^*(Q_{m-1})\succcurlyeq\mu^*(2Q_{m-1})=\mu^*(Q_m)\ge\mu(Q_m\cap G).$$

Then, on account of (1.3), we have

$$|f_{\mu,Q_m\cap G}-f_{\mu,Q_{m-1}\cap G}|\preccurlyeq\frac{1}{\mu(Q_m\cap G)}\iint\limits_{Q_m\cap G}|f-f_{\mu,Q-m\cap G}|d\mu\le\|f\|_{\mathrm{BMO}_p(G,w)}.$$

Thus, using the Minkowsky inequality and relation (1.3) for all $k\ge 1$, we obtain

$$\sigma_p\Big(f-f_{\mu,Q\cap G},w,z_0,2^k\rho_n(z_0)\Big)\le\bigg(\iint\limits_{Q_k\cap G}|f-f_{\mu,Q_k\cap G}|^pd\mu\bigg)^{\frac{1}{p}}+$$
$$+\sum_{m=1}^{k}|f_{\mu,Q_m\cap G}-f_{\mu,Q_{m-1}\cap G}|\bigg(\iint\limits_{Q_k\cap G}d\mu\bigg)^{\frac{1}{p}}\le$$
$$\le\|f\|_{\mathrm{BMO}_p(G,w)}(1+k)\Big(\iint\limits_{Q_k\cap G}d\mu\Big)^{\frac{1}{p}}=$$
$$=\|f\|_{\mathrm{BMO}_p(G,w)}\bigg(1+\log_2\frac{2^k\rho_n(z_0)}{\rho_n(z_0)}\bigg)\cdot\mu^{\frac{1}{p}}(z_0,2^k\rho_n(z_0)),$$

whence, obviously,

$$\sigma_p(f-f_{\mu,Q\cap G},w,z_0,t)\le\|f\|_{\mathrm{BMO}_p(G,w)}\Big(1+\log_2\frac{t}{\rho_n(z_0)}\Big)\cdot\mu^{\frac{1}{p}}(z_0,t)$$

for all $t\ge\rho_n(z_0)$.

Consequently, owing to (3.2), we have

$$\|f-P_n\|_{z_0,p,\omega}\preccurlyeq$$
$$\preccurlyeq\|f\|_{\mathrm{BMO}_p(G,w)}\rho_n(z_0)\Big(\mu(z_0,\rho_n(z_0))\Big)^{\frac{1}{p}}\int\limits_{\rho_n(z_0)}^{\infty}\frac{\big(1+\log_2\frac{t}{\rho_n(z_0)}\big)}{t^2}dt=$$

$$= \|f\|_{\mathrm{BMO}_p(G,w)} \Big(\mu(z_0, \rho_n(z_0))\Big)^{\frac{1}{p}} \int\limits_1^\infty \frac{\big(1+\log_2 \tau\big)}{\tau^2} d\tau \preccurlyeq$$

$$\preccurlyeq \|f\|_{\mathrm{BMO}_p(G,w)} \big(\mu\{u(z_0, \rho_n(z_0)) \cap G\}\big)^{\frac{1}{p}}. \qquad \square$$

Assume now that relation (1.4) is fulfilled for some function f given in G. Then, obviously, $f \in L_p(G, w)$. Moreover, $f \in H'(G)$ if we neglect the values of the function f on the set of measure zero. Indeed, if $z \in G$ is an arbitrary point and $z^* \in \Gamma$ is a point such that $|z - z^*| = \rho(z, \Gamma)$, then it is easy to check that $u(z, \rho(z, \Gamma)) \subset G_n(z^*, c_0)$ ($c_0 \le \frac{1}{2}$).Taking into account relation (1.4) it is not difficult to prove that the polynomials P_n converge uniformly on $u(z, \frac{1}{2}\rho(z, \Gamma))$. Cleary, the limiting analytic function coincides with the functions f a.e.

Further, let $z_0 \in \Gamma$, $r > 0$, $Q = Q(z_0, r)$, and let $n \in N$ be a number such that

$$\rho_{2^{n+1}}(z_0) < r \le \rho_{2^n}(z_0). \tag{3.3}$$

Using the Minkowsky inequality, we can see that

$$\Big(\iint\limits_{Q\cap G} |f(z) - f_{\mu, Q\cap G}|^p d\mu_z\Big)^{\frac{1}{p}} \le \Big(\iint\limits_{Q\cap G} |f(z) - P_{2^n}(z)|^p d\mu_z\Big)^{\frac{1}{p}} +$$

$$\Big(\iint\limits_{Q\cap G} |P_{2^n}(z) - f_{\mu, Q\cap G}|^p d\mu_z\Big)^{\frac{1}{p}} \overset{df}{=} I_1 + I_2.$$

By virtue of (1.4),

$$I_1 \le \mathrm{const}\, \mu^{1/p}(z_0, \rho_n(z_0)). \tag{3.4}$$

It remains to estimate I_2. Evidently,

$$I_2 \preccurlyeq \mu^{1/p}(Q \cap G) \cdot \max_{z \in Q\cap G} |P_{2^n}(z) - f_{\mu, Q\cap G}|. \tag{3.5}$$

Then it is obvious that for all $z \in Q \cap G$

$$|P_{2^n}(z) - f_{\mu, Q\cap G}| \le \frac{1}{\mu(Q\cap G)} \iint\limits_{Q\cap G} |f(\zeta) - P_{2^n}(z)| d\mu_\zeta \le$$

$$\le \frac{1}{\mu(Q\cap G)} \iint\limits_{Q\cap G} |f(\zeta) - P_{2^n}(\zeta)| d\mu_\zeta +$$

$$+ \max_{\zeta \in Q\cap G} |P_{2^n}(\zeta) - P_{2^n}(z)| \overset{df}{=} I_2' + I_2''.$$

Using the Hölder inequality and relation (1.4), we obtain

$$I_2' \leq \frac{1}{\mu(Q\cap G)}\Big(\iint\limits_{Q\cap G}|f-P_{2^n}|^p d\mu\Big)^{\frac{1}{p}}\Big(\iint\limits_{Q\cap G}d\mu\Big)^{1-\frac{1}{p}} \preccurlyeq \text{const}\,. \tag{3.6}$$

To estimate I_1'', let us consider the polynomial

$$\Pi_{2^k}(z)=P_{2^k}(z)-P_{2^{k-1}}(z) \quad (k\geq 1).$$

By (2.15), the Minkowsky inequality and relation (1.4) imply

$$\|\Pi_{2^k}\|_{z_0,p,w}\leq \|f-P_{2^k}\|_{z_0,p,w}+\|f-P_{2^{k-1}}\|_{z_0,p,w}\preccurlyeq \mu^{1/p}(z_0,\rho_n(z_0)).$$

But then, according to Lemma 6, we have

$$|\Pi_{2^k}'(z)|\preccurlyeq|\rho_{2^k}(z_0)|^{-1} \quad \forall (z\in u(z_0,\rho_{2^k}(z_0)), \quad k\geq 1).$$

Hence, taking into account (2.15), we obtain

$$\Big|P_{2^n}(\zeta)-P_{2^n}(z)\Big| = \Big|\big(P_1(\zeta)-P_1(z)\big)+$$
$$+\sum_{k=1}^{n}\Big(\Pi_{2^k}(\zeta)-\Pi_{2^k}(z)\Big)\Big| \preccurlyeq |\zeta-z|+\sum_{k=1}^{n}\int\limits_{[\zeta,z]}\Big|\Pi_{2^k}'(\xi)\Big||d\xi|\preccurlyeq$$
$$\preccurlyeq |\zeta-z|\Big(1+\sum_{k=1}^{n}\big|\rho_{2^k}(z_0)\big|^{-1}\Big)\preccurlyeq \Big(\rho_{2^n}(z_0)+\sum_{k=1}^{n}\frac{\rho_{2^n}(z_0)}{\rho_{2^k}(z_0)}\Big)\preccurlyeq$$
$$\preccurlyeq \Big(\rho_{2^n}(z_0)+\sum_{k=1}^{n}\big(\frac{1}{2^{n-k}}\big)^{\beta}\Big)\leq \text{const}$$

for all $z,\zeta\in Q\cap G$.

This means that $I_2''\leq$ const.

But then, taking into account (3.6) and (3.5), we get

$$I_2\leq \text{const}\,\mu^{1/p}(z_0,\rho_n(z_0)),$$

which, with regard to (3.4), completes the proof of Theorem 1. □

Proof of Theorem 2. Let us prove first that (b)⇒(a). Let $\omega\in A_p(F(G))$ $(1<p<\infty)$, $f\in \mathrm{BMO}_p(G,\omega)$ and let us show that $f\in \mathrm{BMO}(G)$.

Indeed,

$$\frac{1}{|Q\cap G|}\iint\limits_{Q\cap G}|f-f_{Q\cap G}|d\sigma\leq$$
$$\leq \frac{2}{|Q\cap G|}\Big(\iint\limits_{Q\cap G}|f-f_{\mu,Q\cap G}|^p w d\sigma\Big)^{\frac{1}{p}}\Big(\iint\limits_{Q\cap G}w^{-\frac{1}{p-1}}d\sigma\Big)^{\frac{p-1}{p}}\preccurlyeq$$

$$\preccurlyeq \Big(\frac{1}{\mu(Q\cap G)}\iint\limits_{Q\cap G}|f-f_{\mu,Q\cap G}|^p d\mu\Big)^{\frac{1}{p}}\times$$

$$\times\Big(\frac{1}{|Q\cap G|}\iint\limits_{Q\cap G} w\Big)^{\frac{1}{p}}\Big(\frac{1}{|Q\cap G|}\iint\limits_{Q\cap G} w^{-\frac{1}{p-1}}\Big)^{\frac{p-1}{p}} \le \mathrm{const}\,\|f\|_{\mathrm{BMO}(G)}.$$

It remains to prove that (a)⇒(b).

Let $f\in \mathrm{BMO}(G)$ be an analytic function in the domain G, let f^* be a function defined by the equality (2.9), $\omega\in A_P(F(G))$ $(1<p<\infty)$, and let μ^* be the measure defined by (2.3). We prove first that for all $Q\in F(G)$ and $\lambda>0$ the relation

$$\frac{1}{\mu^*(Q)}\mu^*\{z\in Q:|f^*(z)-f^*_Q|>\lambda\}\le C\exp\big(\frac{-c\lambda}{\|f\|_{\mathrm{BMO}(G)}}\big) \tag{3.7}$$

holds, where C and c are the constants independent of f^*, Q, and λ.

Choose a square $Q\in F(G)$. Let δ be the number from Lemma 1, and let c^* be the constant from (2.10). Without loss of generality we can assume that $\delta<1/c^*$. In this case we can apply Lemma 4 to the function f^* and $\alpha=(1/\delta)\|f\|_{\mathrm{BMO}(G)}$. Hence we get a family of disjoint squares $A_1=\{Q_j^1 : Q_j^1\in F(G,k)\}$ such that

$$|f^*(z)-f^*_Q|\le 12\alpha$$

for all $z\in Q\setminus \underset{A_1}{\cup} Q_j^1$,

$$|f^*_{Q_j^1}-f^*_Q|<4\alpha \tag{3.8}$$

according to (2.14), and by (2.15) we have

$$|\underset{A_1}{\cup} Q_j^1|=\sum_{A_1}|Q_j^1|\le\frac{1}{\alpha}\|f\|_{\mathrm{BMO}(G)}\cdot|Q|.$$

Since $(1/\alpha)\|f\|_{\mathrm{BMO}(G)}=\delta$, by virtue of Lemma 1 we obtain

$$\mu^*\{\underset{A_1}{\cup} Q_j^1\}\le\varepsilon\mu^*(Q). \tag{3.9}$$

Applying again Lemma 4 to the function f^* and $\alpha=(1/\delta)\|f\|_{\mathrm{BMO}(G)}$, for every Q_j^1 we obtain a family of nonintersecting squares $A_2=\{Q_j^2 : Q_j^2\in F(G,k)\}$ such that each of these squares is contained in one of the Q_j^1. Thus, by (3.8) and (2.13) the relation

$$|f^*-f^*_Q|\le|f^*-f^*_{Q_j^1}|+|f^*_{Q_j^1}-f^*_Q|<12\alpha+4\alpha<2\cdot 12\alpha$$

is fulfilled on $Q\setminus \bigcup\limits_{A_2} Q_j^2$, while owing to (2.14) and (3.8) we have that

$$|f^*_{Q_j^2} - f^*_Q| \le |f^*_{Q_j^2} - f^*_{Q_j^1}| + |f^*_{Q_j^1} - f^*_Q| < 4\alpha + 4\alpha < 2 \cdot 12\alpha.$$

Finally, according to (2.15), we have

$$\Big|\bigcup_{Q_j^2\subset Q_j^1} Q_j^2\Big| = \sum_{Q_j^2\subset Q_j^1} |Q_j^2| \le \frac{1}{\alpha}\|f\|_{\mathrm{BMO}(G)} \cdot |Q_j^1|$$

for every Q_j^1.

Then again, by virtue of Lemma 1 and (3.9), we obtain

$$\mu^*\{\bigcup_{A_2} Q_j^2\} = \sum_{Q_j^1\in A_1} \mu^*\{\bigcup_{Q_j^2\subset Q_j^1} Q_j^2\} \le \sum_{Q_j^1\in A_1} \varepsilon\mu^*(Q_j^1) \le \varepsilon^2\mu^*(Q).$$

Continuing this process ad infinitum, we obtain at the step n a family of intersecting squares $A_=\{Q_j^n\}$ such that

$$|f^* - f^*_Q| \le 12\alpha \cdot n \quad \text{a.e. in} \quad Q\setminus \bigcup_{A_n} Q_j^n \quad \text{and} \quad \mu^*\{\bigcup_{A_n} Q_j^n\} \le \varepsilon^n\mu^*(Q).$$

Assume now that $\lambda > 12\alpha$. Let $n \ge 1$ be a natural number such that $12\alpha n < \lambda \le 12\alpha n + 12\alpha$. Then, obviously, we shall have

$$\mu^*\big\{z \in Q : |f^*(z) - f^*_Q| > \lambda\big\} \le \mu^*\big\{z \in Q : |f^*(z) - f^*_Q| > 12\alpha n\big\} \le$$
$$\le \mu^*\{\bigcup_{A_n} Q_j^n\} \le \varepsilon^n\mu^*(Q) \le \varepsilon^{\frac{\lambda}{12\alpha}-1}\mu^*(Q) = \frac{1}{\varepsilon}\exp\Big(\frac{-c\lambda}{\|f\|_{\mathrm{BMO}(G)}}\Big)\mu^*(Q)$$

for $c = (1/12)\delta \cdot \ln(1/\varepsilon)$.

Hence, estimate (3.7) is valid for all $\lambda > 12\alpha$. But for all $0 < \lambda \le 12\alpha$ we, obviously, have

$$\mu^*\{z \in Q : |f^*(z) - f^*_Q| > \lambda\} \le \mu^*(Q) = \exp\Big(\frac{c\lambda}{\|f\|_{\mathrm{BMO}(G)}}\Big) \times$$
$$\times \exp\Big(\frac{-c\lambda}{\|f\|_{\mathrm{BMO}(G)}}\Big)\mu^*(Q) \le \exp\Big(\frac{12c}{\delta}\Big)\exp\Big(\frac{-c\lambda}{\|f\|_{\mathrm{BMO}(G)}}\Big)\mu^*(Q).$$

Consequently, assuming $C = \max\Big\{\frac{1}{\varepsilon}, \exp\Big(\frac{12c}{\delta}\Big)\Big\}$, we get estimate (3.7) for all $\lambda > 0$.

Relation (3.7) with regard to (2.8) implies that

$$\frac{1}{\mu(Q\cap G)}\mu\{z \in Q\cap G : |f(z) - f^*_Q| > \lambda\} \preccurlyeq \exp\Big(\frac{-c\lambda}{\|f\|_{\mathrm{BMO}(G)}}\Big). \quad (3.10)$$

The latter relation allows us to complete the proof of Theorem 2. Indeed, using first the Minkowsky inequality and then writing the corresponding integral in terms of a distribution function, applying estimate (3.10), we obtain

$$\left(\frac{1}{\mu(Q\cap G)}\iint\limits_{Q\cap G}|f-f_{\mu,Q\cap G}|^p d\mu\right)\le\left(\frac{2}{\mu(Q\cap G)}\iint\limits_{Q\cap G}|f-f_Q^*|^p d\mu\right)^{\frac{1}{p}}=$$

$$=\left(2p\int\limits_0^\infty\lambda^{p-1}\frac{1}{\mu(Q\cap G)}\mu\{z\in Q\cap G:|f(z)-f_Q^*|>\lambda\}d\lambda\right)^{\frac{1}{p}}\preccurlyeq$$

$$\preccurlyeq\left(2p\int\limits_0^\infty\lambda^{p-1}\exp\left(\frac{-c\lambda}{\|f\|_{\mathrm{BMO}(G)}}\right)d\lambda\right)^{\frac{1}{p}}\preccurlyeq c(p)\big(\|f\|_{\mathrm{BMO}(G)}\big)^{\frac{1}{p}},$$

which implies that $f\in\mathrm{BMO}_p(G,w)$. $\square$

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